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BILINEAR OPERATORS WITH HOMOGENEOUS SYMBOLS, SMOOTH MOLECULES, AND KATO-PONCE INEQUALITIES

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Abstract. We present a unifying approach to establish mapping properties for bilinear pseudodifferential operators with homogeneous symbols in the settings of function spaces that admit a discrete transform and molecular decompositions in the sense of Frazier and Jawerth. As an application, we obtain related Kato-Ponce inequalities.

1. Introduction and main results

As the main purpose of this note we present a unifying approach towards establishing mapping properties of the form

$$\|T_\sigma(f,g)\|_Y \lesssim \|f\|_X \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_X \,, \tag{1.1}$$

where X and Y are function spaces admitting a molecular decomposition and a φ-transform in the sense of Frazier-Jawerth as introduced in [10, 11], and T_σ is a bilinear pseudodifferential operator given by

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \quad \forall x \in \mathbb{R}^n,$$

with a bilinear symbol σ in the class $\dot{BS}^m_{1,1}$ for some $m \in \mathbb{R}$, that is, σ is such that for all multiindices $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, it holds

$$\|\sigma\|_{\gamma,\alpha,\beta} := \sup_{(x,\xi,\eta)\in\mathbb{R}^{3n}\setminus\{0\}} |\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi,\eta)|(|\xi|+|\eta|)^{-m-|\gamma|+|\alpha+\beta|} < \infty. \tag{1.2}$$

When $m = 0$, the x-independent symbols in $\dot{BS}^0_{1,1}$ constitute the well-known class of Coifman-Meyer bilinear multipliers. The bilinear forbidden class $BS^0_{1,1}$ is defined as the family of symbols satisfying (1.2) with $m = 0$ and with $|\xi| + |\eta|$ replaced by $1 + |\xi| + |\eta|$. Note that if σ belongs to $BS^0_{1,1}$, then $\sigma = \sigma_1 + \sigma_2$ where σ_1 is in $\dot{BS}^0_{1,1}$ and σ_2 is a smoothing symbol supported in $\{(x,\xi,\eta) : |\xi| + |\eta| \leq 1\}$. We refer the reader to the work of Coifman and Meyer in [7] and the references it contains for pioneering work related to such symbols. As we will describe next, these two classes of symbols possess distinct essential features, and, as a noteworthy consequence of our Theorem 1.1 below, it will follow that they share various mapping properties of the form (1.1).

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Coifman-Meyer bilinear multipliers can be realized as bilinear Calderón-Zygmund operators. As such, they inherit their mapping properties; for instance, Calderón-Zygmund operators are bounded in the settings of Lebesgue spaces, BMO, the Hardy space H^1 (Grafakos-Torres [15]), and in weighted Lebesgue spaces (Lerner et al. [21]).

On the other hand, the bilinear forbidden class $BS^0_{1,1}$ is known to produce bilinear pseudodifferential operators with a bilinear Calderón–Zygmund kernel, but, in general, they are not bilinear Calderón–Zygmund operators (Bényi-Torres [4]). In particular, they do not always possess mapping properties of the form $L^{p_1} \times L^{p_2} \to L^p$ with $1 < p_1, p_2 \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. Mapping properties for bilinear pseudodifferential operators with symbols in $BS^0_{1,1}$ have been studied in Bényi [2] in the setting of Besov spaces, in Bényi-Torres [4] and Bényi-Nahmod-Torres [3] in the scale of Lebesgue-Sobolev spaces, and in Naibo [23] and Koezuka-Tomita [20] in the context of Besov and Triebel-Lizorkin spaces.

In our main result, Theorem 1.1 below, we prove molecular estimates on T_σ, with $\sigma \in B\dot{S}^m_{1,1}$, when one of its arguments is a fixed function and its other argument is a smooth molecule.

Theorem 1.1. *Given $m \in \mathbb{R}$ and $\sigma \in B\dot{S}^m_{1,1}$, there exist $\sigma^1, \sigma^2 \in B\dot{S}^m_{1,1}$ with $T_\sigma = T_{\sigma^1} + T_{\sigma^2}$ and such that if $1 \le r \le \infty$, $0 < M < \infty$, $\psi \in \mathcal{S}(\mathbb{R}^n)$, with $\widehat{\psi}$ supported in $\{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$, and $\gamma \in \mathbb{N}_0^n$, it holds that*

$$|\partial^\gamma T_{\sigma^1}(\psi_{\nu,k}, g)(x)| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x - k|)^M} \|g\|_{L^r} \quad \forall x \in \mathbb{R}^n$$

and

$$|\partial^\gamma T_{\sigma^2}(f, \psi_{\nu,k})(x)| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x - k|)^M} \|f\|_{L^r} \quad \forall x \in \mathbb{R}^n,$$

for every $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$, and where

$$\psi_{\nu,k}(x) = 2^{\frac{\nu n}{2}} \psi(2^\nu x - k).$$

Here $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of smooth rapidly decreasing functions defined on $\mathbb{R}^n$; the notation $\lesssim$ means $\le C$, where C is a constant that may depend on some of the parameters used but not on the functions or variables involved.

1.1. **A sample of applications of Theorem 1.1.**

In the case $r = \infty$, Theorem 1.1 implies that, up to uniform multiplicative constants, the functions

$$2^{-\nu m} T_{\sigma^1}(\psi_{\nu,k}, g)/\|g\|_{L^\infty} \quad \text{and} \quad 2^{-\nu m} T_{\sigma^2}(f, \psi_{\nu,k})/\|f\|_{L^\infty}$$

can be regarded as smooth molecules, as introduced in [10, 11] in the settings of Besov and Triebel-Lizorkin spaces. Since smooth molecules also serve as building blocks for a variety of other function spaces, Theorem 1.1 will apply to such spaces as well.

As a concrete application, we will implement Theorem 1.1 in the scales of homogeneous Besov-type and Triebel-Lizorkin-type spaces. These spaces were introduced and studied in Sawano-Yang-Yuan [25] and Yang-Yuan [28, 29] as natural spaces that extend and unify the scales of homogeneous Besov spaces, homogeneous Triebel-Lizorkin spaces, and Q-spaces. The latter were introduced in Essén et al. [9] as a refinement of BMO functions. In addition, as proved in [25], the Besov-type and Triebel-Lizorkin-type spaces also contain or coincide with Besov-Morrey and Triebel-Lizorkin-Morrey spaces.

We refer the reader to Section 3 for detailed notation and precise definitions. In the following, $\mathcal{S}_0(\mathbb{R}^n)$ denotes the closed subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ that have vanishing moments of all orders; that is, $f \in \mathcal{S}_0(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0$ for all $\alpha \in \mathbb{N}_0^n$. For $0 < p, q \le \infty$, set

$$s_{p,q} := n\left(\frac{1}{\min\{1,p,q\}} - 1\right) \quad \text{and} \quad s_p := n\left(\frac{1}{\min\{1,p\}} - 1\right). \tag{1.3}$$

By means of Theorem 1.1 and molecular techniques, we obtain the following mapping properties in the scales of homogeneous Besov-type and Triebel-Lizorkin-type spaces.

Theorem 1.2. *Let $m \in \mathbb{R}$ and $\sigma \in \dot{BS}^m_{1,1}$. If $0 < p, q \le \infty$, $s_p < s < \infty$ and $0 \le \tau < \frac{1}{p} + \frac{s-s_p}{n}$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^{s+m,\tau}_{p,q}} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

If $0 < p < \infty$, $0 < q \le \infty$, $s_{p,q} < s < \infty$ and $0 \le \tau < \frac{1}{p} + \frac{s-s_{p,q}}{n}$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{F}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m,\tau}_{p,q}} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

Theorem 1.2 can be considered as a bilinear counterpart to Grafakos-Torres [14, Theorems 1.1 and 1.2] (see also Torres [27]), where boundedness properties in homogeneous Besov and Triebel-Lizorkin spaces were addressed for linear pseudodifferential operators with symbols in the class $\dot{S}^m_{1,1}$, the linear analog to $\dot{BS}^m_{1,1}$. In turn, the (linear) results in [14] were extended to the setting of Besov-type and Triebel-Lizorkin-type spaces in [25, Theorem 1.5]. We refer the reader to Hart-Torres-Wu [17] where very different techniques are used to obtain estimates in the spirit of those in Theorem 1.2 in the setting of Sobolev spaces for operators with x-independent symbols and a limited amount of regularity.

In Remark 4.1 we address Theorem 1.2 in the cases corresponding to $s \le s_p$ and $s \le s_{p,q}$ and show that analogous estimates are obtained, with a slightly different range for the parameter τ, if a number of cancellation conditions are imposed on the first adjoint of T_{σ^1} and on the second adjoint of T_{σ^2}, where σ^1 and σ^2 are as in Theorem 1.1. In Remark 4.2 we give a version of Theorem 1.2 involving the L^r norms of f and g instead of their L^∞ norms.

The next corollary of Theorem 1.2 follows from the realization of Q-spaces as special cases of Triebel-Lizorkin-type spaces (see Section 3.1.1).

Corollary 1.3. *Let $s, s+m \in (0,1)$ and $\sigma \in \dot{BS}^m_{1,1}$. If $1 \le q \le p \le \infty$ and $q \ne \infty$, it holds that*

$$\|T_\sigma(f,g)\|_{Q^{s,q}_p} \lesssim \|f\|_{Q^{s+m,q}_p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{Q^{s+m,q}_p} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

1.2. **Applications to Kato-Ponce inequalities.** As a consequence of Theorem 1.1 in the case $\sigma \equiv 1$, given a function space X that admits a molecular representation and a φ-transform, we obtain the following fractional Leibniz rule or Kato-Ponce inequality:

$$\|fg\|_X \lesssim \|f\|_X \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_X. \tag{1.4}$$

Inequalities of the form (1.4) were proved by Kato-Ponce [18] in the case where X is the Sobolev space $W^{s,p}(\mathbb{R}^n)$, with $1 < p < \infty$ and $0 < s < \infty$, in relation to Cauchy problems for the Euler and Navier-Stokes equations; prior work due to Strichartz [26] treats the range $n/p < s < 1$, while the case of $s \in \mathbb{N}$ can be obtained from the Leibniz rule and the Gagliardo-Nirenberg inequality. Later on, Gulisashvili-Kon [16] showed (1.4) for the homogeneous space $X = \dot{W}^{s,p}(\mathbb{R}^n)$, for the same range of parameters, in connection with the study of smoothing properties of Schrödinger semigroups. The estimates (1.4) also hold true in the settings of Besov and Triebel-Lizorkin spaces and have applications to partial differential equations (see, for instance, Bahouri-Chemin-Danchin [1], Chae [5], Runst-Sickel [24] and the references they contain). In particular, all such estimates imply that $X \cap L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication. Closely related versions to (1.4) were given by Christ-Weinstein [6] and Kenig-Ponce-Vega [19], in the contexts of Korteweg-de Vries equations, and by Gulisashvili-Kon [16]. Extensions to the cases of indices below 1 appear in Grafakos-Oh [13] and Muscalu-Schlag [22], and versions in weighted and variable exponent space settings were proved in Cruz-Uribe-Naibo [8].

In particular, in the scales of Besov-type and Triebel-Lizorkin-type spaces, Theorem 1.2 yields the following new Kato-Ponce inequalities.

Corollary 1.4. *If $0 < p, q \leq \infty$, $s_p < s < \infty$ and $0 \leq \tau < \frac{1}{p} + \frac{s - s_p}{n}$, it holds that*
$$\|fg\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^{s,\tau}_{p,q}} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$
If $0 < p < \infty$, $0 < q \leq \infty$, $s_{p,q} < s < \infty$ and $0 \leq \tau < \frac{1}{p} + \frac{s - s_{p,q}}{n}$, it holds that
$$\|fg\|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{F}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s,\tau}_{p,q}} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$
If $0 < s < 1$, $1 \leq q \leq p \leq \infty$ and $q \neq \infty$, it holds that
$$\|fg\|_{Q^{s,q}_p} \lesssim \|f\|_{Q^{s,q}_p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{Q^{s,q}_p} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

The article is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 contains the definitions of Besov-type and Triebel-Lizorkin-type spaces, smooth molecules, and the φ-transform. The proof of Theorem 1.2 and several closing remarks are given in Section 4.

2. Proof of Theorem 1.1

Our first step towards the proof of Theorem 1.1 will be obtaining a representation of a bilinear pseudodifferential operator with a symbol in $B\dot{S}^m_{1,1}$ as a superposition of paraproduct-like operators. Such representations can be traced back to the pioneering work of Coifman and Meyer; Lemma 2.1 gives a version of a decomposition suited for our purposes, and its proof follows ideas inspired from [7, pp. 154-155]. We then state and prove Lemma 2.2, which procures a formula for the derivatives of the building blocks, appropriately evaluated, given by Lemma 2.1. We close this section with the proof of Theorem 1.1.

The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ will be denoted by $\widehat{f}$; in particular, we use the formula $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

Let θ be a real-valued infinitely differentiable function supported on $(-2, 2)$ and such that $\theta(t) + \theta(1/t) = 1$ for every $t > 0$. For $\sigma \in B\dot{S}^m_{1,1}$, $m \in \mathbb{R}$, define
$$\sigma^1(x, \xi, \eta) := \sigma(x, \xi, \eta)\theta\left(\tfrac{|\eta|}{|\xi|}\right) \quad \text{and} \quad \sigma^2(x, \xi, \eta) := \sigma(x, \xi, \eta)\theta\left(\tfrac{|\xi|}{|\eta|}\right) \quad \forall x, \xi, \eta \in \mathbb{R}^n.$$
Simple computations show that $\sigma^1, \sigma^2 \in B\dot{S}^m_{1,1}$ with
$$\left\|\sigma^d\right\|_{\gamma,\alpha,\beta} \lesssim \sup_{\bar{\alpha} \leq \alpha, \bar{\beta} \leq \beta} \|\sigma\|_{\gamma,\bar{\alpha},\bar{\beta}} \quad \text{for } \gamma, \alpha, \beta \in \mathbb{N}^n_0 \text{ and } d = 1, 2,$$

where the implicit constant depends only on γ, α, β and θ, and we have

$$T_\sigma(f,g) = T_{\sigma^1}(f,g) + T_{\sigma^2}(f,g), \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n).$$

Endowing $\mathcal{S}_0(\mathbb{R}^n)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^n)$, a standard argument using integration by parts allows one to conclude that T_{σ^1} is continuous from $\mathcal{S}_0(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and T_{σ^2} is continuous from $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Let $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\Psi}$ and $\widehat{\Phi}$ are real-valued, $\operatorname{supp}(\widehat{\Psi}) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$, $\sum_{j\in\mathbb{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 = 1$ for every $\xi \neq 0$, $\widehat{\Phi} \equiv 1$ for $|\xi| \le 4$ and $\widehat{\Phi} \equiv 0$ for $|\xi| > 10$.

Lemma 2.1. *Let $\sigma \in \dot{BS}^m_{1,1}$. With the notation introduced above and given $N > n$, there exist sequences of functions $\{m_j^1(x,u,v)\}_{j\in\mathbb{Z}}$ and $\{m_j^2(x,u,v)\}_{j\in\mathbb{Z}}$ defined for $x,u,v \in \mathbb{R}^n$ such that if $\gamma \in \mathbb{N}_0^n$, then*

$$\sup_{x,u,v\in\mathbb{R}^n} |\partial_x^\gamma m_j^d(x,u,v)| \lesssim 2^{j(m+|\gamma|)}, \quad \forall j \in \mathbb{Z}, d = 1,2, \tag{2.5}$$

and, if $f \in \mathcal{S}_0(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, it holds that

$$T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} m_j^1(x,u,v)\, \Delta_j^u f(x)\, S_j^v g(x) \frac{du dv}{(1+|u|^2+|v|^2)^N} \tag{2.6}$$

and

$$T_{\sigma^2}(g,f)(x) = \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} m_j^2(x,u,v)\, S_j^u g(x)\, \Delta_j^v f(x) \frac{du dv}{(1+|u|^2+|v|^2)^N}, \tag{2.7}$$

where $\widehat{\Delta_j^u f}(\xi) = \widehat{\Psi^u}(2^{-j}\xi)\widehat{f}(\xi)$ with $\Psi^u(x) := \Psi(x+u)$ and $\widehat{S_j^v g}(\xi) = \widehat{\Phi^v}(2^{-j}\xi)\widehat{g}(\xi)$ with $\Phi^v(x) := \Phi(x+v)$.

Proof. We will prove (2.6), with the proof of (2.7) following analogously. Since the support of $|\widehat{\Psi}(2^{-j}\xi)|^2\sigma^1(x,\xi,\eta)$ is contained in $\{(x,\xi,\eta) : |\eta| \le 2|\xi| \text{ and } 2^{j-1} < |\xi| < 2^{j+1}\} \subset \{(x,\xi,\eta) : |\eta| \le 2^{j+2}\}$ and $\widehat{\Phi}(2^{-j}\eta) \equiv 1$ for $|\eta| \le 2^{j+2}$, we have

$$|\widehat{\Psi}(2^{-j}\xi)|^2\sigma^1(x,\xi,\eta) = |\widehat{\Phi}(2^{-j}\eta)|^2|\widehat{\Psi}(2^{-j}\xi)|^2\sigma^1(x,\xi,\eta) \quad \forall x,\xi,\eta \in \mathbb{R}^n, j \in \mathbb{Z}.$$

From this, the fact that $\sum_{j\in\mathbb{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 = 1$ for $\xi \neq 0$ and Fubini's theorem, it follows that if $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$, then

$$T_{\sigma^1}(f,g)(x) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma_j^1(x, 2^{-j}\xi, 2^{-j}\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\hat{f}(\xi)\hat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi d\eta, \tag{2.8}$$

where $\sigma_j^1(x,\xi,\eta) := \widehat{\Psi}(\xi)\widehat{\Phi}(\eta)\sigma^1(x,2^j\xi,2^j\eta)$.

Given multiindices $\gamma,\alpha,\beta \in \mathbb{N}_0^n$, the Leibniz rule implies that $\partial_x^\gamma\partial_\xi^\alpha\partial_\eta^\beta\sigma_j^1$ can be written as a linear combination of terms of the form

$$\partial^{\alpha_1}\widehat{\Psi}(\xi)\partial^{\beta_1}\widehat{\Phi}(\eta)(\partial_x^\gamma\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\sigma^1)(x,2^j\xi,2^j\eta)2^{j|\alpha_2+\beta_2|}, \quad \alpha_1+\alpha_2 = \alpha,\ \beta_1+\beta_2 = \beta. \tag{2.9}$$

Since $\sigma^1 \in \dot{BS}^m_{1,1}$, the absolute value of each term (2.9) can be bounded by a multiple of

$$|\partial^{\alpha_1}\widehat{\Psi}(\xi)\partial^{\beta_1}\widehat{\Phi}(\eta)|2^{j|\alpha_2+\beta_2|}(|2^j\xi|+|2^j\eta|)^{m+|\gamma|-|\alpha_2+\beta_2|} \lesssim 2^{j(m+|\gamma|)} \quad \forall x,\xi,\eta \in \mathbb{R}^n,$$

where we have used that $\partial^{\alpha_1}\widehat{\Psi}(\xi)\partial^{\beta_1}\widehat{\Phi}(\eta)$ is supported in $\{(\xi,\eta) : \frac{1}{2} < |\xi|+|\eta| < 12\}$, and the implicit constant is independent of j.

Define $m_j^1(x,u,v) := (1+|u|^2+|v|^2)^N \widehat{\sigma_j^1(x,\cdot,\cdot)}(u,v)$; by the above we have

$$\begin{aligned}
&|\partial_x^\gamma m_j^1(x,u,v)| \\
&\quad = (1+|u|^2+|v|^2)^N \left| \int_{\mathbb{R}^{2n}} \partial_x^\gamma \sigma_j^1(x,\xi,\eta) \frac{(1-\Delta_{\xi,\eta})^N e^{-2\pi i(u\cdot\xi+v\cdot\eta)}}{(1+4\pi^2|u|^2+4\pi^2|v|^2)^N}\, d\xi d\eta \right| \\
&\quad \sim \left| \int_{\frac{1}{2}<|\xi|+|\eta|<12} (1-\Delta_{\xi,\eta})^N (\partial_x^\gamma \sigma_j^1)(x,\xi,\eta) e^{-2\pi i(u\cdot\xi+v\cdot\eta)}\, d\xi d\eta \right| \lesssim 2^{j(m+|\gamma|)}.
\end{aligned}$$

Finally, using that

$$\sigma_j^1(x,2^{-j}\xi,2^{-j}\eta) = \int_{\mathbb{R}^{2n}} m_j^1(x,u,v) e^{2\pi i(u\cdot 2^{-j}\xi+v\cdot 2^{-j}\eta)} \frac{dudv}{(1+|u|^2+|v|^2)^N}$$

in (2.8), after interchanging summation and integral signs justified by Fubini's theorem, we get (2.6). $\square$

For each $u,v\in\mathbb{R}^n$, set

$$\sigma_{u,v}^1(x,\xi,\eta) := \sum_{j\in\mathbb{Z}} m_j^1(x,u,v)\widehat{\Psi^u}(2^{-j}\xi)\widehat{\Phi^v}(2^{-j}\eta);$$

then $T_{\sigma_{u,v}^1}(f,g)(x) = \sum_{j\in\mathbb{Z}} m_j^1(x,u,v)\, \Delta_j^u f(x)\, S_j^v g(x)$. Similarly define $\sigma_{u,v}^2$. In our next lemma we look at derivatives of $T_{\sigma_{u,v}^1}(\psi_{\nu,k},g)$ and $T_{\sigma_{u,v}^2}(f,\psi_{\nu,k})$.

Lemma 2.2. *If $\gamma\in\mathbb{N}_0^n$, $\nu\in\mathbb{Z}$, $k\in\mathbb{Z}^n$, $u,v\in\mathbb{R}^n$, $g\in\mathcal{S}(\mathbb{R}^n)$ and $\psi\in\mathcal{S}(\mathbb{R}^n)$ is such that* $\operatorname{supp}(\widehat{\psi})\subset\{\xi\in\mathbb{R}^n : \frac{1}{2}<|\xi|<2\}$*, then*

$$\begin{aligned}
&\partial^\gamma T_{\sigma_{u,v}^1}(\psi_{\nu,k},g)(x) \\
&\quad = 2^{\frac{\nu n}{2}} \sum_{\substack{j=\nu-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{\nu+1} C_{\gamma_1,\gamma_2,\gamma_3}\, 2^{\nu|\gamma-\gamma_1|}\, \partial_x^{\gamma_1} m_j^1(x,u,v) \\
&\qquad \times (\Phi_{\nu-j}^{\gamma_2} * g(2^{-\nu}\cdot))(2^\nu x + 2^{\nu-j}v)\, \Psi_{\nu-j}^{\gamma_3}(2^\nu x - k + 2^{\nu-j}u),
\end{aligned}$$

where $\Phi_{\nu-j}^{\gamma_2}, \Psi_{\nu-j}^{\gamma_3}\in\mathcal{S}(\mathbb{R}^n)$ are independent of g and $\psi_{\nu,k}(x) = 2^{\frac{\nu n}{2}}\psi(2^\nu x-k)$. An analogous formula holds for $\partial^\gamma T_{\sigma_{u,v}^2}(f,\psi_{\nu,k})$ with $f\in\mathcal{S}(\mathbb{R}^n)$.

Proof. In view of the supports of $\widehat{\psi}$ and $\widehat{\Psi}$, the supports of $\widehat{\psi}(2^{-\nu}\cdot)$ and $\widehat{\Psi}(2^{-j}\cdot)$ only intersect if $\nu-1\le j\le \nu+1$. We then have

$$\begin{aligned}
&T_{\sigma^1_{u,v}}(\psi_{\nu,k},g)(x)\\
&=\sum_{j=\nu-1}^{\nu+1} m^1_j(x,u,v)\int_{\mathbb{R}^{2n}} \widehat{\Psi^u}(2^{-j}\xi)\widehat{\Phi^v}(2^{-j}\eta)\widehat{\psi_{\nu,k}}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta\\
&=\sum_{j=\nu-1}^{\nu+1} m^1_j(x,u,v)2^{-\frac{\nu n}{2}}\\
&\quad\times\int_{\mathbb{R}^{2n}} \widehat{\Psi^u}(2^{-j}\xi)\widehat{\Phi^v}(2^{-j}\eta)e^{-2\pi i 2^{-\nu}k\cdot\xi}\widehat{\psi}(2^{-\nu}\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta\\
&=\sum_{j=\nu-1}^{\nu+1} 2^{\frac{\nu n}{2}} m^1_j(x,u,v)\\
&\quad\times\left(\int_{\mathbb{R}^n} 2^{\nu n}\widehat{g}(2^\nu\eta)\widehat{\Phi^v}(2^{\nu-j}\eta)e^{2\pi i 2^\nu x\cdot\eta}\,d\eta\right)\left(\int_{\mathbb{R}^n}\widehat{\Psi^u}(2^{\nu-j}\xi)\widehat{\psi}(\xi)e^{2\pi i(2^\nu x-k)\cdot\xi}\,d\xi\right).
\end{aligned}$$

Denoting

$$\begin{aligned}
&F_j(x)\\
&:=m^1_j(x,u,v)\left(\int_{\mathbb{R}^n} 2^{\nu n}\widehat{g}(2^\nu\eta)\widehat{\Phi^v}(2^{\nu-j}\eta)e^{2\pi i 2^\nu x\cdot\eta}\,d\eta\right)\left(\int_{\mathbb{R}^n}\widehat{\Psi^u}(2^{\nu-j}\xi)\widehat{\psi}(\xi)e^{2\pi i(2^\nu x-k)\cdot\xi}\,d\xi\right)
\end{aligned}$$

and given a multiindex $\gamma\in\mathbb{N}_0^n$, we have

$$\begin{aligned}
\partial^\gamma F_j(x) &= \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} C_{\gamma_1,\gamma_2,\gamma_3}\,\partial_x^{\gamma_1} m^1_j(x,u,v)\\
&\quad\times\left(\int_{\mathbb{R}^n} 2^{\nu n}\widehat{g}(2^\nu\eta)2^{\nu|\gamma_2|}\eta^{\gamma_2}\widehat{\Phi^v}(2^{\nu-j}\eta)e^{2\pi i 2^\nu x\cdot\eta}\,d\eta\right)\\
&\quad\times\left(\int_{\mathbb{R}^n} 2^{\nu|\gamma_3|}\xi^{\gamma_3}\widehat{\Psi^u}(2^{\nu-j}\xi)\widehat{\psi}(\xi)e^{2\pi i(2^\nu x-k)\cdot\xi}\,d\xi\right)\\
&=\sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} C_{\gamma_1,\gamma_2,\gamma_3}\,2^{\nu|\gamma-\gamma_1|}\,\partial_x^{\gamma_1} m^1_j(x,u,v)\\
&\quad\times\left(\int_{\mathbb{R}^n} 2^{\nu n}\widehat{g}(2^\nu\eta)\eta^{\gamma_2}\widehat{\Phi}(2^{\nu-j}\eta)e^{2\pi i(2^\nu x+2^{\nu-j}v)\cdot\eta}\,d\eta\right)\\
&\quad\times\left(\int_{\mathbb{R}^n} \xi^{\gamma_3}\widehat{\Psi}(2^{\nu-j}\xi)\widehat{\psi}(\xi)e^{2\pi i(2^\nu x-k+2^{\nu-j}u)\cdot\xi}\,d\xi\right)\\
&=\sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} C_{\gamma_1,\gamma_2,\gamma_3}\,2^{\nu|\gamma-\gamma_1|}\,\partial_x^{\gamma_1} m^1_j(x,u,v)(\Phi^{\gamma_2}_{\nu-j}*g(2^{-\nu}\cdot))(2^\nu x+2^{\nu-j}v)\\
&\quad\times\Psi^{\gamma_3}_{\nu-j}(2^\nu x-k+2^{\nu-j}u),
\end{aligned}$$

where $\widehat{\Phi^{\gamma_2}_{\nu-j}}(\eta):=\eta^{\gamma_2}\widehat{\Phi}(2^{\nu-j}\eta)$ and $\widehat{\Psi^{\gamma_3}_{\nu-j}}(\xi):=\xi^{\gamma_3}\widehat{\Psi}(2^{\nu-j}\xi)\widehat{\psi}(\xi)$. Since

$$\partial_x^\gamma T_{\sigma^1_{u,v}}(\psi_{\nu,k},g)(x)=\sum_{j=\nu-1}^{\nu+1}2^{\frac{\nu n}{2}}\partial^\gamma F_j(x),$$

we get the desired result. $\square$

Proof of Theorem 1.1. Let $\sigma \in \dot{BS}^{m}_{1,1}$, $1 \le r \le \infty$, $0 < M < \infty$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi}$ is supported in $\{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$ and $g \in \mathcal{S}(\mathbb{R}^n)$. With the notation used above, Lemma 2.2 and (2.5) imply

$$
\begin{aligned}
&\left|\partial^\gamma T_{\sigma^1_{u,v}}(\psi_{\nu,k}, g)(x)\right| \\
&\quad\lesssim 2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} \sum_{\substack{j=\nu-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{\nu+1} \left\|\Phi^{\gamma_2}_{\nu-j} * g(2^{-\nu}\cdot)\right\|_{L^\infty} \left|\Psi^{\gamma_3}_{\nu-j}(2^\nu x - k + 2^{\nu-j}u)\right| \\
&\quad\lesssim 2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} \sum_{\substack{j=\nu-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{\nu+1} \left\|\Phi^{\gamma_2}_{\nu-j}\right\|_{L^{r'}} \left\|g(2^{-\nu}\cdot)\right\|_{L^r} \frac{(1+|2^{\nu-j}u|)^M}{(1+|2^\nu x - k|)^M} \\
&\quad\lesssim 2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}} \frac{(1+|u|)^M}{(1+|2^\nu x-k|)^M} \left\|g\right\|_{L^r},
\end{aligned}
$$

where in the second inequality we have used that $\Psi^{\gamma_3}_{\nu-j} \in \mathcal{S}(\mathbb{R}^n)$. Since

$$
T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} T_{\sigma^1_{u,v}}(f,g)(x) \frac{du dv}{(1+|u|^2+|v|^2)^N},
$$

by choosing N sufficiently large so that $\int_{\mathbb{R}^{2n}} \frac{(1+|u|)^M}{(1+|u|^2+|v|^2)^N}\, du dv < \infty$, we obtain the desired estimate for $\partial^\gamma T_{\sigma^1}(\psi_{\nu,k}, g)(x)$. Analogous reasoning leads to the estimate for $\partial^\gamma T_{\sigma^2}(f, \psi_{\nu,k})(x)$. □

3. Function spaces

We recall that $\mathcal{S}_0(\mathbb{R}^n)$ denotes the closed subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ that have vanishing moments of all orders and we endow $\mathcal{S}_0(\mathbb{R}^n)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^n)$. The dual space of $\mathcal{S}_0(\mathbb{R}^n)$, $\mathcal{S}_0'(\mathbb{R}^n)$, can be identified with the space of tempered distributions modulo polynomials, $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$. That is, $\mathcal{D} := \{Q_{\nu,k}\}_{\nu\in\mathbb{Z}, k\in\mathbb{Z}^n}$ where

$$
Q_{\nu,k} := \{x \in \mathbb{R}^n : k_j \le 2^\nu x_j < k_j + 1, j = 1, \dots, n\}.
$$

We denote the edge length of $Q_{\nu,k}$ by $l(Q_{\nu,k})$ and set $x_Q = x_{\nu,k} := 2^{-\nu}k$ where $Q = Q_{\nu,k}$.

We will consider functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$
\operatorname{supp}(\widehat{\varphi}), \operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}, \tag{3.10}
$$

$$
|\widehat{\varphi}(\xi)|, |\widehat{\psi}(\xi)| > c \quad \text{for all } \xi \text{ such that } \tfrac{3}{5} < |\xi| < \tfrac{5}{3} \text{ and some } c > 0, \tag{3.11}
$$

$$
\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}(2^{-j}\xi)}\, \widehat{\psi}(2^{-j}\xi) = 1 \quad \text{for } \xi \ne 0. \tag{3.12}
$$

See [12, Lemma 6.9] for a construction of ψ given that φ satisfies (3.10) and (3.11).

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.10) and (3.11), $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we recall that $\varphi_{\nu,k}$ denotes the L^2-normalized function $\varphi_{\nu,k}(x) = 2^{\frac{\nu n}{2}}\varphi(2^\nu x - k) = 2^{\frac{\nu n}{2}}\varphi(2^\nu(x - x_{\nu,k}))$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ verifies (3.10), (3.11) and (3.12), then it follows that

$$
f = \sum_{\nu\in\mathbb{Z}, k\in\mathbb{Z}^n} \langle f, \varphi_{\nu,k}\rangle \psi_{\nu,k},
$$

where the series converges for $f \in L^2(\mathbb{R}^n)$ in the topology of $L^2(\mathbb{R}^n)$, for $f \in \mathcal{S}_0(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^n)$ and for $f \in \mathcal{S}'(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials (see [10, 11] for details).

3.1. **Homogeneous Besov-type and Triebel-Lizorkin-type spaces.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy conditions (3.10) and (3.11), and set $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Fix $s, \tau \in \mathbb{R}$ and $0 < q \le \infty$. For $0 < p \le \infty$, the Besov-type space $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \sum_{j=-\log_2(\ell(P))}^{\infty} \left[\int_P (2^{js}|\varphi_j * f(x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty.$$

For $0 < p < \infty$, the Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=-\log_2(\ell(P))}^{\infty} (2^{js}|\varphi_j * f(x)|)^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

These spaces are independent of the choice of φ (see [29, Corollary 3.1]). As in [29], we will use $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, excluding $p = \infty$ in the latter case.

3.1.1. *Special cases of* $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We refer the reader to [28, Section 3] and [29, Proposition 3.1] regarding the following statements:

(i) If $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $-\infty < \tau < 0$, then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ equals the equivalence class of all polynomials on $\mathbb{R}^n$; if $0 \le \tau < \infty$, they are quasi-Banach spaces and contain $\mathcal{S}_0(\mathbb{R}^n)$.

(ii) If $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $\tau = 0$, then $\dot{B}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Besov space $\dot{B}^{s}_{p,q}(\mathbb{R}^n)$, with equivalent norms.

(iii) If $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $\tau = 0$, then $\dot{F}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, with equivalent norms. In turn, $\dot{F}^{s}_{p,2}(\mathbb{R}^n)$ coincides with the Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < s < \infty$, with equivalent norms.

(iv) If $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$, then $\dot{F}^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{\infty,q}(\mathbb{R}^n)$, with equivalent norms. In particular, $\dot{F}^{0,\frac{1}{p}}_{p,2}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, with equivalent norms.

(v) If $0 < p \le \infty$, $1 \le q < \infty$ and $0 < s < 1$, then $\dot{F}^{s,\frac{1}{q}-\frac{1}{p}}_{q,q}(\mathbb{R}^n)$ coincides with the Q-space $Q^{s,q}_p(\mathbb{R}^n)$, with equivalent norms. Here $f \in Q^{s,q}_p(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}_0'(\mathbb{R}^n)$ with $f(x) - f(y)$ measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\|f\|_{Q^{s,q}_p(\mathbb{R}^n)} := \sup_I |I|^{1/p-1/q} \left\{ \int_I \int_I \frac{|f(x)-f(y)|^q}{|x-y|^{n+qs}} \, dy \, dx \right\}^{1/q} < \infty,$$

where I ranges over all cubes of $\mathbb{R}^n$ with dyadic edge lengths. In particular, $Q_s(\mathbb{R}^n) := Q^{s,2}_{n/s}(\mathbb{R}^n) = \dot{F}^{s,\frac{1}{2}-\frac{s}{n}}_{2,2}(\mathbb{R}^n)$. For $0 < s < 1$ if $n \ge 2$, or for $0 < s \le \frac{1}{2}$ if $n = 1$, the spaces $Q_s(\mathbb{R}^n)$ constitute a decreasing family of nontrivial subspaces of BMO; see [9].

(vi) Further special cases of the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ involving homogeneous Besov-Morrey and Triebel-Lizorkin-Morrey spaces can be found in [25, Theorem 1.1].

3.1.2. *Molecules.* Based on the pioneering work from [10, 11], it was proved in [29, Theorem 3.1] that the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ can be characterized in terms of the so-called φ-transform defined by $S_\varphi(f) = \{\langle f, \varphi_{\nu,k}\rangle\}_{\nu,k}$ for $f \in \mathcal{S}_0'(\mathbb{R}^n)$, where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.10) and (3.11). More precisely, if $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $0 \le \tau < \infty$, then

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} \sim \|\{\langle f, \varphi_{\nu,k}\rangle\}_{\nu,k}\|_{\dot{b}^{s,\tau}_{p,q}} \quad \text{and} \quad \|f\|_{\dot{F}^{s,\tau}_{p,q}} \sim \|\{\langle f, \varphi_{\nu,k}\rangle\}_{\nu,k}\|_{\dot{f}^{s,\tau}_{p,q}}, \tag{3.13}$$

where $\dot{b}^{s,\tau}_{p,q}$ and $\dot{f}^{s,\tau}_{p,q}$ refer to the following spaces of sequences: For $0 < p \le \infty$, the space $\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the collection of all sequences $t = \{t_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}$, indexed by the dyadic cubes, such that

$$\begin{aligned} &\|t\|_{\dot{b}^{s,\tau}_{p,q}} \\ &:= \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \sum_{j=-\log_2(\ell(P))}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} |Q|^{-s/n-1/2} |t_Q| \chi_Q(x) \right)^p dx \right]^{q/p} \right\}^{1/q} \\ &< \infty. \end{aligned}$$

For $0 < p < \infty$, the space $\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the collection of all sequences $t = \{t_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}$, indexed by the dyadic cubes, such that

$$\|t\|_{\dot{f}^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q\subset P} (|Q|^{-s/n-1/2} |t_Q| \chi_Q(x))^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

As before, we will use $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$, excluding the case $p = \infty$ in the latter case.

Let $0 < p, q \le \infty$, $s \in \mathbb{R}$, $0 \le \tau < \infty$ and $s^* := s - [s]$, where $[s]$ denotes the largest integer smaller than or equal to s. Set

$$J := \begin{cases} s_p + n & \text{if } \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n), \\ s_{p,q} + n & \text{if } \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n), \end{cases}$$

where s_p and $s_{p,q}$ are as in (1.3). We say that $\{m_Q\}_{Q\in\mathcal{D}}$, where $m_Q : \mathbb{R}^n \to \mathbb{C}$, is a family of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ if there exist δ and M with $\max\{s^*, (s+n\tau)^*\} < \delta \le 1$ and $J < M < \infty$ such that

$$\int_{\mathbb{R}^n} m_Q(x) x^\gamma \, dx = 0 \quad \text{if } |\gamma| \le \max\{[J-n-s], -1\},$$

$$|m_Q(x)| \le \frac{|Q|^{-1/2}}{(1 + l(Q)^{-1}|x - x_Q|)^{\max\{M, M-s\}}} \quad \forall x \in \mathbb{R}^n,$$

$$|\partial^\gamma m_Q(x)| \le \frac{|Q|^{-1/2-|\gamma|/n}}{(1 + l(Q)^{-1}|x - x_Q|)^{M}} \quad \forall x \in \mathbb{R}^n \text{ and } |\gamma| \le [s + n\tau],$$

$$
\begin{aligned}
&|\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \\
&\quad \le |Q|^{-1/2-|\gamma|/n-\delta/n} |x-y|^\delta \\
&\qquad \times \sup_{|z|\le|x-y|} \frac{1}{(1+l(Q)^{-1}|x-z-x_Q|)^M} \quad \forall x,y \in \mathbb{R}^n \text{ and } |\gamma| = [s+n\tau].
\end{aligned}
$$

It easily follows that $\{\varphi_{\nu,k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{\psi_{\nu,k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$ are families of smooth synthesis molecules for any $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with parameters $\delta = 1$ and any $M > J$.

Through analogous ideas on almost-diagonal operators used to prove [11, Theorem 3.5] it follows that if $0 < p,q \le \infty$, $s \in \mathbb{R}$, $\max\{s^*, (s+n\tau)^*\} < \delta \le 1$, $J < M < \infty$, $0 \le \tau < \min\{\frac{1}{p} + \frac{M-J}{2n}, \frac{1}{p} + \frac{1-(J-s)^*}{n}\}$ if $\max\{[J-n-s], -1\} \ge 0$, $0 \le \tau < \min\{\frac{1}{p} + \frac{M-J}{2n}, \frac{1}{p} + \frac{s+n-J}{n}\}$ if $\max\{[J-n-s],-1\} < 0$, and $\{m_Q\}_{Q\in\mathcal{D}}$ is a family of synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with parameters δ and M, then

$$
\left\| \sum_{Q\in\mathcal{D}} t_Q m_Q \right\|_{\dot{A}^{s,\tau}_{p,q}} \lesssim \|t\|_{\dot{a}^{s,\tau}_{p,q}} \quad \forall t = \{t_Q\}_{Q\in\mathcal{D}} \in \dot{a}^{s,\tau}_{p,q}, \tag{3.14}
$$

where the implicit constant does not depend on the family of molecules ([29, Theorem 4.2]).

4. Proof of Theorem 1.2 and closing remarks

Proof of Theorem 1.2. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.10), (3.11) and (3.12). Since T_{σ^1} and T_{σ^2}, as given by Theorem 1.1, are continuous from $\mathcal{S}_0(\mathbb{R}^n)\times\mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and $h = \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle h, \varphi_{\nu,k}\rangle \psi_{\nu,k}$ for $h \in \mathcal{S}_0(\mathbb{R}^n)$ with convergence in $\mathcal{S}_0(\mathbb{R}^n)$ (see Section 3), we have

$$
T_{\sigma^1}(f,g) = \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \varphi_{\nu,k}\rangle T_{\sigma^1}(\psi_{\nu,k}, g) \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n),
$$

$$
T_{\sigma^2}(f,g) = \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle g, \varphi_{\nu,k}\rangle T_{\sigma^2}(f, \psi_{\nu,k}) \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n),
$$

where the convergence is in $\mathcal{S}(\mathbb{R}^n)$.

Theorem 1.1 implies that there are constants c_1 and c_2 such that if $f,g \in \mathcal{S}_0(\mathbb{R}^n)$, then

$$
\left\{ \frac{c_1 2^{-\nu m} T_{\sigma^1}(\psi_{\nu,k}, g)}{\|g\|_{L^\infty}} \right\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \quad \text{and} \quad \left\{ \frac{c_2 2^{-\nu m} T_{\sigma^2}(f, \psi_{\nu,k})}{\|f\|_{L^\infty}} \right\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}
$$

are families of smooth synthesis molecules for any $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ if $0 < p,q \le \infty$, $s > J-n$ and $0 \le \tau < \infty$ (with $\delta = 1$ and any $M > J$; note that the zero moment condition is void since $J-n-s < 0$). If, in addition, $0 \le \tau < \frac{1}{p} + \frac{s+n-J}{n}$, we can apply (3.14) and (3.13) to get

$$
\begin{aligned}
\|T_{\sigma^1}(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} &\lesssim \|\{2^{\nu m}\langle f, \varphi_{\nu,k}\rangle\}\|_{\dot{a}^{s,\tau}_{p,q}} \|g\|_{L^\infty} \\
&= \|\{\langle f, \varphi_{\nu,k}\rangle\}\|_{\dot{a}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} \simeq \|f\|_{\dot{A}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty}, \\
\|T_{\sigma^2}(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} &\lesssim \|\{2^{\nu m}\langle g, \varphi_{\nu,k}\rangle\}\|_{\dot{a}^{s,\tau}_{p,q}} \|f\|_{L^\infty} \\
&= \|\{\langle g, \varphi_{\nu,k}\rangle\}\|_{\dot{a}^{s+m,\tau}_{p,q}} \|f\|_{L^\infty} \simeq \|g\|_{\dot{A}^{s+m,\tau}_{p,q}} \|f\|_{L^\infty},
\end{aligned}
$$

from which the desired estimates follow. □

Remark 4.1. Let $m \in \mathbb{R}$ and $\sigma \in \dot{BS}^m_{1,1}$. The estimates in Theorem 1.2 hold true in $\dot{A}^{s,\tau}_{p,q}$ for $0 < p, q \leq \infty$, $s \leq J - n$ and $0 \leq \tau < \frac{1}{p} + \frac{1-(J-s)^*}{n}$ if the following cancellation conditions are satisfied:

$$T^{*1}_{\sigma^1}(x^\gamma, g) = T^{*1}_{\sigma^2}(f, x^\gamma) = 0 \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n),\ |\gamma| \leq [J - n - s].$$

We recall that if T is a bilinear operator continuous from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, T^{*1} and T^{*2} denote the adjoint operators of T defined from $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}'_0(\mathbb{R}^n)$ and from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'_0(\mathbb{R}^n)$, respectively, as $\langle h, T(f,g)\rangle = \langle T^{*1}(h,g), f\rangle = \langle T^{*2}(f,h), g\rangle$.

The proof of the estimates in this case is the same as above, with the only thing left to check being the zero moment conditions for $T_{\sigma^1}(\psi_{\nu,k}, g)$ and $T_{\sigma^2}(f, \psi_{\nu,k})$ (note that the range assumed for τ comes from the assumptions for the validity of (3.14)). We have, for $|\gamma| \leq [J - n - s]$,

$$\int_{\mathbb{R}^n} x^\gamma T_{\sigma^1}(\psi_{\nu,k}, g)\, dx = \langle x^\gamma, T_{\sigma^1}(\psi_{\nu,k}, g)\rangle = \langle T^{*1}_{\sigma^1}(x^\gamma, g), \psi_{\nu,k}\rangle = 0 \quad \forall g \in \mathcal{S}_0(\mathbb{R}^n),$$

and similarly for $T_{\sigma^2}(f, \psi_{\nu,k})$.

Remark 4.2. Let $1 \leq r \leq \infty$ and m, σ, p, q, s and τ be as in the hypothesis of Theorem 1.2 or Remark 4.1. By the same reasoning as in the proof of Theorem 1.2 and Remark 4.1, we also obtain

$$\|T_\sigma(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{A}^{s+m+\frac{n}{r},\tau}_{p,q}} \|g\|_{L^r} + \|g\|_{\dot{A}^{s+m+\frac{n}{r},\tau}_{p,q}} \|f\|_{L^r}.$$

Remark 4.3. The implicit constants in the inequalities of Theorem 1.1 and Theorem 1.2 depend linearly on $\|\sigma\|_{K,L}$ for some $K, L \in \mathbb{N}$, where

$$\|\sigma\|_{K,L} := \sup_{|\gamma| \leq K, |\alpha+\beta| \leq L} \|\sigma\|_{\gamma,\alpha,\beta}.$$

From the proofs, it follows that the implicit constants in the inequalities of Theorem 1.1 are multiples of $\|\sigma\|_{|\gamma|,2N}$, with $N \in \mathbb{N}$, $N > M + n$ and where γ and M are as in the statement of the theorem. In turn, this implies that the implicit constants in Theorem 1.2 can be taken to be multiples of $\|\sigma\|_{[s+n\tau]+1,2N}$ with $N > \max\{J + n, 2(s + n) - J + n\}$. The latter is also true for the inequalities from Remark 4.1 with $N > J + n + 2(1 - (J - s)^*)$.

References

[1] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011. MR2768550

[2] Árpád Bényi, *Bilinear pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces*, J. Math. Anal. Appl. **284** (2003), no. 1, 97–103, DOI 10.1016/S0022-247X(03)00245-2. MR1996120

[3] Árpád Bényi, Andrea R. Nahmod, and Rodolfo H. Torres, *Sobolev space estimates and symbolic calculus for bilinear pseudodifferential operators*, J. Geom. Anal. **16** (2006), no. 3, 431–453, DOI 10.1007/BF02922061. MR2250054

[4] Árpád Bényi and Rodolfo H. Torres, *Symbolic calculus and the transposes of bilinear pseudodifferential operators*, Comm. Partial Differential Equations **28** (2003), no. 5-6, 1161–1181, DOI 10.1081/PDE-120021190. MR1986065

[5] Dongho Chae, *On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces*, Comm. Pure Appl. Math. **55** (2002), no. 5, 654–678, DOI 10.1002/cpa.10029. MR1880646

[6] F. M. Christ and M. I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), no. 1, 87–109, DOI 10.1016/0022-1236(91)90103-C. MR1124294

[7] Ronald R. Coifman and Yves Meyer, *Au delà des opérateurs pseudo-différentiels* (French), Astérisque, vol. 57, Société Mathématique de France, Paris, 1978. With an English summary. MR518170

[8] David Cruz-Uribe and Virginia Naibo, *Kato-Ponce inequalities on weighted and variable Lebesgue spaces*, Differential Integral Equations **29** (2016), no. 9-10, 801–836. MR3513582

[9] Matts Essén, Svante Janson, Lizhong Peng, and Jie Xiao, *Q spaces of several real variables*, Indiana Univ. Math. J. **49** (2000), no. 2, 575–615, DOI 10.1512/iumj.2000.49.1732. MR1793683

[10] Michael Frazier and Björn Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), no. 4, 777–799, DOI 10.1512/iumj.1985.34.34041. MR808825

[11] Michael Frazier and Björn Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), no. 1, 34–170, DOI 10.1016/0022-1236(90)90137-A. MR1070037

[12] Michael Frazier, Björn Jawerth, and Guido Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991. MR1107300

[13] Loukas Grafakos and Seungly Oh, *The Kato-Ponce inequality*, Comm. Partial Differential Equations **39** (2014), no. 6, 1128–1157, DOI 10.1080/03605302.2013.822885. MR3200091

[14] Loukas Grafakos and Rodolfo H. Torres, *Pseudodifferential operators with homogeneous symbols*, Michigan Math. J. **46** (1999), no. 2, 261–269, DOI 10.1307/mmj/1030132409. MR1704146

[15] Loukas Grafakos and Rodolfo H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164, DOI 10.1006/aima.2001.2028. MR1880324

[16] Archil Gulisashvili and Mark A. Kon, *Exact smoothing properties of Schrödinger semigroups*, Amer. J. Math. **118** (1996), no. 6, 1215–1248. MR1420922

[17] J. Hart, R.H. Torres, and X. Wu. Smoothing properties of bilinear operators and Leibniz-type rules in Lebesgue and mixed Lebesgue spaces. Transactions of the American Mathematical Society DOI: http://doi.org/10.1090/tran/7312

[18] Tosio Kato and Gustavo Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907, DOI 10.1002/cpa.3160410704. MR951744

[19] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620, DOI 10.1002/cpa.3160460405. MR1211741

[20] K. Koezuka and N. Tomita. Bilinear pseudo-differential operators with symbols in $BS^m_{1,1}$ on Triebel-Lizorkin spaces. *J. Fourier Anal. Appl. DOI:10.1007/s00041-016-9518-2*, 2016.

[21] Andrei K. Lerner, Sheldy Ombrosi, Carlos Pérez, Rodolfo H. Torres, and Rodrigo Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), no. 4, 1222–1264, DOI 10.1016/j.aim.2008.10.014. MR2483720

[22] Camil Muscalu and Wilhelm Schlag, *Classical and multilinear harmonic analysis. Vol. II*, Cambridge Studies in Advanced Mathematics, vol. 138, Cambridge University Press, Cambridge, 2013. MR3052499

[23] Virginia Naibo, *On the bilinear Hörmander classes in the scales of Triebel-Lizorkin and Besov spaces*, J. Fourier Anal. Appl. **21** (2015), no. 5, 1077–1104, DOI 10.1007/s00041-015-9398-x. MR3393696

[24] Thomas Runst and Winfried Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996. MR1419319

[25] Yoshihiro Sawano, Dachun Yang, and Wen Yuan, *New applications of Besov-type and Triebel-Lizorkin-type spaces*, J. Math. Anal. Appl. **363** (2010), no. 1, 73–85, DOI 10.1016/j.jmaa.2009.08.002. MR2559042

[26] Robert S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060. MR0215084

[27] Rodolfo H. Torres, *Continuity properties of pseudodifferential operators of type* $1, 1$, Comm. Partial Differential Equations **15** (1990), no. 9, 1313–1328, DOI 10.1080/03605309908820726. MR1077277

[28] Dachun Yang and Wen Yuan, *A new class of function spaces connecting Triebel-Lizorkin spaces and Q spaces*, J. Funct. Anal. **255** (2008), no. 10, 2760–2809, DOI 10.1016/j.jfa.2008.09.005. MR2464191

[29] Dachun Yang and Wen Yuan, *New Besov-type spaces and Triebel-Lizorkin-type spaces including Q spaces*, Math. Z. **265** (2010), no. 2, 451–480, DOI 10.1007/s00209-009-0524-9. MR2609320

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