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A DELAYED YULE PROCESS

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Abstract. In now classic work, David Kendall (1966) recognized that the Yule process and Poisson process could be related by a (random) time change. Furthermore, he showed that the Yule population size rescaled by its mean has an almost sure exponentially distributed limit as $t \to \infty$. In this note we introduce a class of coupled delayed continuous time Yule processes parameterized by $0 < \alpha \leq 1$ and find a representation of the Poisson process as a delayed Yule process at delay rate $\alpha = 1/2$. Moreover we extend Kendall's limit theorem to include a larger class of positive martingales derived from functionals that gauge the population genealogy. Specifically, the latter is exploited to uniquely characterize the moment generating functions of distributions of the limit martingales, generalizing Kendall's mean one exponential limit. A connection with fixed points of the Holley-Liggett smoothing transformation also emerges in this context, about which much is known from general theory in terms of moments, tail decay, and so on.

1. Introduction

The *basic Yule process* $Y = \{Y_t : t \geq 0\}$ is a continuous time branching process starting from a single progenitor in which a particle survives for a mean one, exponentially distributed time before being replaced by two offspring independently evolving in the same manner. Y_t represents the size of the population of particles at time $t \geq 0$, starting from $Y_0 = 1$. The *basic Poisson process* $N = \{N_t : t \geq 0\}$ is another continuous time Markov process in which a particle survives for a mean one, exponentially distributed time before being replaced by a single particle that evolves in the same manner. The shift $N_t + 1$ represents the number of replacements that have occurred by time $t \geq 0$, $N_0 = 0$. The multiplicative (geometric) growth of the process Y is in stark contrast to the additive growth of N.

Considerations of evolutionary processes, to be referred to as *delayed Yule processes,* arise somewhat naturally in the probabilistic analysis of quasi-linear evolution equations such as incompressible Navier-Stokes equations, and complex Burgers equation by probabilistic methods originating with Le Jan and Sznitman [10]. In particular, considerations of non-uniqueness and/or explosion problems in [4] for this framework prompted the present considerations. However this paper has a purely probabilistic focus and does not depend on such motivations. In fact, the probabilistic framework may also be of interest in the context of evolutionary biological processes.

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The principal results are extensions of the aforementioned theorems of Kendall (see [9]). In particular, a key result is that the representation of the Poisson process as a delayed Yule process at delay rate $\alpha = 1/2$ provides an exact coupling of the two processes through a binary tree-indexed family of i.i.d. exponential random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Secondly, complete criteria for the uniform integrability of positive martingales derived from a family of gauges of the genealogy of the Yule process, including cardinality, is also given. Once this is established the exact limit distribution is identified for these uniformly integrable martingales as unique (mean one) fixed points of the Holley-Liggett smoothing operator [7]. This characterization generalizes Kendall's mean one exponential limit in the case the gauge is the cardinality of the population; the latter limit distribution is the Gamma distributed fixed point solution corresponding to the uniform (Beta) smoothing factor in [7]. The characterization of the uniformly integrable martingale limits as fixed points to a smoothing transformation has numerous implications on the more detailed structure of the limit; e.g., see [6], [11] for further general theory and results on the nature of fixed points of smoothing recursions. As an illustration, simple conditions are noted for the existence of finite moments of the limit martingale. From the perspective of delayed Yule processes as continuous time Markov processes it is shown that $\alpha = 1/2$ is a critical transition value between bounded and unbounded infinitesimal generators defining the α-delayed Yule processes for $0 < \alpha \le 1$.

2. Delayed Yule process

To begin, consider the modification of the Yule process given by successively halving the previous generation branching frequencies, i.e., doubling the previous generation mean holding time of particles of each generation. That is, let $\{T_v : v \in \mathbf{T} = \bigcup_{k=0}^{\infty}\{1,2\}^k\}$, with $\{1,2\}^0 = \{\theta\}$, be a binary, tree-indexed family of i.i.d. mean one exponentially distributed random variables rooted at a single progenitor θ, and define

$$V^{(\frac{1}{2})}(t) = \Big\{v \in \mathbf{T} : \sum_{j=0}^{|v|-1} (1/2)^{-j} T_{v|j} \le t < \sum_{j=0}^{|v|} (1/2)^{-j} T_{v|j}\Big\}, \quad t \ge 0,$$

where $|\theta| = 0$, and $|v| = | < v_1, \ldots, v_k > | = k$ denotes the *height* of vertex $v \in \mathbf{T}$. Also $v|j = < v_1, \ldots, v_j >$ is the restriction of v to generation $j \le k$. Also, by convention, $\sum_{j=0}^{-1} = 0$.

Observe that

$$Y_t = \#V^{(1)}(t) = \Big\{v \in \mathbf{T} : \sum_{j=0}^{|v|-1} T_{v|j} \le t < \sum_{j=0}^{|v|} T_{v|j}\Big\}, \quad t \ge 0,$$

defines the basic Yule process; throughout $\#V$ will denote the cardinality of a set V.

Let $\tau_k, k = 1, 2, \ldots$ be the increasing sequence of jump times of the $\frac{1}{2}$-*delayed Yule process* defined by

$$N_t = \#V^{(\frac{1}{2})}(t) - 1, t \ge 0.$$

The following calculations provide a warm-up to our basic coupling of the Poisson and Yule processes. First observe that $\tau_0 = 0, \tau_1 = T_\theta$, so that $P(\tau_1 - \tau_0 > t) = e^{-t}, t \geq 0$. Next, for $k = 2$, one has by definition that $\tau_2 - \tau_1 = 2T_1 \wedge 2T_2$, and $\{T_1, T_2\}$ is independent of T_θ $(= \tau_1)$. In particular, therefore,

$$P(\tau_2 - \tau_1 > t | \sigma\{\tau_1 - \tau_0\}) = P(2T_1 \wedge 2T_2 > t) = e^{-\frac{t}{2}} e^{-\frac{t}{2}} = e^{-t}.$$

The independence of inter-arrival times is less obvious in the case $k = 3$ where one has

$$\tau_3 - \tau_2 = \begin{cases} (2T_1 - 2T_2) \wedge 4T_{21} \wedge 4T_{22}, & \text{if } 2T_2 < 2T_1, \\ (2T_2 - 2T_1) \wedge 4T_{11} \wedge 4T_{12}, & \text{if } 2T_1 < 2T_2. \end{cases} \tag{2.1}$$

However, using symmetry and the two logical operations: (i) $a > b$ & $c > b$ iff $a \wedge c > b$, and (ii) $a > b$ & $a > c$ iff $a > b \vee c$, one may easily express the event $[\tau_3 - \tau_2 > t_3, \tau_2 - \tau_1 > t_2, \tau_1 - \tau_0 > t_1]$ in terms of the underlying i.i.d. Yule times $T_v, v \in \mathbb{T}$. From here a direct computation yields

$$P(\tau_3 - \tau_2 > t_3, \tau_2 - \tau_1 > t_2, \tau_1 - \tau_0 > t_1) = e^{-t_3} e^{-t_2} e^{-t_1}, \quad t_1, t_2, t_3 > 0. \tag{2.2}$$

An inductive extension of this calculation is cumbersome, however the following essential property that couples the Yule and Poisson processes has a very simple inductive proof based on the evolutionary structure of the state space $\mathcal{E}$. Namely, $V \in \mathcal{E}$ if and only if V is a *finite* subset of $\mathbf{T} = \bigcup_{n=0}^{\infty} \{1,2\}^n$, such that

$$V = \begin{cases} \{\theta\} & \text{if } \#V = 1, \\ W \backslash \{w\} \cup \{< w1 >, < w2 >\} & \text{for some } W \in \mathcal{E},\ \#W = \#V - 1, \\ w \in W, \text{ else.} \end{cases} \tag{2.3}$$

Lemma 2.1 (Key Coupling Lemma)**.** *For any $V \in \mathcal{E}$ one has*

$$\sum_{v \in V} (1/2)^{|v|} = 1.$$

Proof. The assertion is clear for $V = \{\theta\}$ since $|\theta| = 0$. The induction follows directly from (2.3). □

Theorem 2.1. *The stochastic process $N_t = \#V^{(\frac{1}{2})}(t) - 1, t \geq 0$, is a Poisson process with unit intensity. In particular, $\tau_k - \tau_{k-1}, k = 1, 2, \dots$ is an i.i.d. sequence of mean one exponentially distributed random variables.*

Proof. By Watanabe's martingale characterization of the Poisson process, in view of the unit jump sample path structure of $\#V^{(\frac{1}{2})}$ it is sufficient to check that $\#V^{(\frac{1}{2})}(t) - t, \quad t \geq 0$, is a martingale with respect to the filtration

$$\mathcal{F}_t = \sigma\{T_v : v \in V^{(\frac{1}{2})}(s) : s \leq t\}, t \geq 0.$$

On $[T_\theta < \infty]$, an event with probability one, express the process $V^{(\frac{1}{2})}(t), t \geq 0$, as $\{\theta\}$ for $t < T_\theta$, and for $t \geq T_\theta$ as the disjoint union of two independent sets of vertices $V_{(j)}^{(\frac{1}{2})}(\frac{t - T_\theta}{2})$ rooted at $\theta = (1), (2)$, respectively for $j = 1, 2$. Then, taking expected values over this representation, one has for $\mu(t) = \mathbb{E}\#V^{(\frac{1}{2})}(t), t \geq 0$,

$$\mu(t) = e^{-t} + 2\int_0^t e^{-s} \mu(\frac{t - s}{2}) ds, t > 0, \quad \mu(0) = 1. \tag{2.4}$$

To see that this equation uniquely determines the non-negative continuous solution μ consider the difference ν of two solutions, $\nu(0) = 0$. Observe that for any $T < \log 2$, the map $\nu \to L\nu(t) = 2\int_0^t e^{-s}\nu(\frac{t-s}{2})ds, 0 \le t \le T$ defines a linear contraction map on $C[0,T]$. In particular $\nu(0) = 0$ implies that $\nu(t) = 0, 0 \le t \le T$. In view of the delay, it now follows from the equation for the difference ν that $\nu = 0$ on $[T, 2T]$ and, inductively, $\nu = 0$ on $[0,\infty)$ for $\nu(0) = 0$. In particular, it follows from (2.4) that

$$\mathbb{E}\#V^{(\frac{1}{2})}(t) = \mu(t) = t + 1, \quad t \ge 0. \tag{2.5}$$

The calculation of the conditional expectation proceeds similarly while taking advantage of (2.5) and the Key Coupling Lemma 2.1: For $0 \le s \le t$,

$$\begin{aligned} \mathbb{E}\{\#V^{(\frac{1}{2})}(t)|\mathcal{F}_s\} &= \mathbb{E}\{\sum_{v\in V^{(\frac{1}{2})}(s)} \#V^{(\frac{1}{2})}(2^{-|v|}(t-s))|\mathcal{F}_s\} \\ &= \sum_{v\in V^{(\frac{1}{2})}(s)} \{2^{-|v|}(t-s)+1\} \\ &= t - s + \#V^{(\frac{1}{2})}(s). \end{aligned} \tag{2.6}$$

□

Replacing $\frac{1}{2}$ by a parameter $\alpha \in (0,1]$ in successive generations of the basic Yule process defines the *α-delayed Yule process*. Namely,

$$V^{(\alpha)}(t) = \Big\{v \in \mathbf{T} : \sum_{j=0}^{|v|-1} \alpha^{-j} T_{v|j} \le t < \sum_{j=0}^{|v|} \alpha^{-j} T_{v|j}\Big\}, \quad t \ge 0.$$

Accordingly, $V^{(\alpha)}$ is a continuous time jump Markov process taking value in the (countable) space $\mathcal{E}$ of *evolutionary sets* defined inductively by (2.3).

Although one may check that $V^{(\alpha)}$ is a Markov process on $\mathcal{E}$, the functional $\#V^{(\alpha)}$ is *not* generally Markov; exceptions being $\alpha = \frac{1}{2}, 1$. When $\alpha = 1$, $\#V^{(\alpha)}$ is the classical Yule process, and so it is obviously Markov. Similarly in the case $\alpha = \frac{1}{2}$, the Markov property is a direct consequence of Theorem 2.1.

In addition to cardinality, letting $\beta > 0$, the following functionals serve to gauge the genealogy of the evolution:

$$a_\beta(V) = \sum_{v\in V} \beta^{|v|}, \quad V \in \mathcal{E}. \tag{2.7}$$

By the Key Coupling Lemma 2.1, one has that $a_{1/2}(V) = 1$ for all $V \in \mathcal{E}$. The cardinality $\#V$ is obtained by taking $\beta = 1$, and the following provides a generalization of Kendall's classic limit theorem to other *gauges* of the genealogical structure of the Yule process.

Theorem 2.2. *For each $\beta \in (0,1]$, $A_\beta(t) = e^{-(2\beta-1)t} a_\beta(V^{(1)}(t)), t \ge 0$, is a positive martingale. Moreover, A_β is uniformly integrable if and only if $\beta \in (\beta_c, 1]$ where $\beta_c \approx 0.1866823$ is the unique solution in $(0,1]$ to*

$$\beta_c \ln \beta_c = \beta_c - \frac{1}{2}. \tag{2.8}$$

Proof. Let $m_\beta(t) = \mathbb{E}a_\beta(V^{(1)}(t)), t \ge 0$. First, let us check that

$$m_\beta(t) = e^{(2\beta-1)t}, \quad t \ge 0. \tag{2.9}$$

For this write

(2.10) $a_\beta(V^{(1)}(t)) = 1[T_\theta > t] + 1[T_\theta \le t]\beta\{a_\beta(V^{(1)+}(t-T_\theta)) + a_\beta(V^{(1)-}(t-T_\theta))\}$,

where $V^{(1)\pm}(\cdot)$ are independent copies of $V^{(1)}(\cdot)$. Taking expected values one has

$$m_\beta(t) = e^{-t} + 2\beta\int_0^t e^{-s}m_\beta(t-s)ds, \quad m_\beta(0) = 1.$$

The expression (2.9) now follows.

To establish the martingale property, let $0 \le s < t$ and write

$$a_\beta(V^{(1)}(t)) = \sum_{w\in V^{(1)}(s)}\sum_{v\in V^{(1),w}(t-s)}\beta^{|w|}\beta^{|v|},$$

where $V^{(1),w}$ are the delayed Yule processes rooted at $w \in V^{(1)}(s)$. Note that the respective processes $V^{(1),w}$, $w \in V^{(1)}(s)$, are conditionally independent given $V^{(1)}(s)$, and therefore

$$\begin{aligned}\mathbb{E}[e^{-(2\beta-1)t}a_\beta(V^{(1)}(t))|\mathcal{F}_s] &= e^{-(2\beta-1)t}m_\beta(t-s)a_\beta(V^{(1)}(s))\\ &= e^{-(2\beta-1)s}a_\beta(V^{(1)}(s)).\end{aligned}$$

Thus A_β is a positive martingale. So, by the martingale convergence theorem, it follows that

$$A_\beta(\infty) = \lim_{t\to\infty} e^{-(2\beta-a)t}a_\beta(V^{(1)}(t)),$$

exists almost surely. Moreover, from (2.10) one has the distributional recursion

$$A_\beta(\infty) = \beta e^{-(2\beta-1)T_\theta}(A^+_\beta(\infty) + A^-_\beta(\infty)), \tag{2.11}$$

where $A^+_\beta(\infty)$ and $A^-_\beta(\infty)$ are independent copies of $A_\beta(\infty)$.

Let us first investigate parameters $\beta \in (0,1]$ such that $A_\beta(\infty) = 0$ almost surely. For this let $h \in (0,1)$ and observe that, since $(x+y)^h \le x^h + y^h$ and $\mathbb{E}(e^{-\delta T_\theta}) = 1/(1+\delta)$, (2.11) yields

$$\mathbb{E}A^h_\beta(\infty) \le 2\beta^h\frac{1}{1+(2\beta-1)h}\mathbb{E}A^h_\beta(\infty), \quad 0 < h < 1.$$

Thus, if $A_\beta(\infty) > 0$ with positive probability, then

$$\frac{2\beta^h}{1+(2\beta-1)h} \ge 1, \quad 0 < h < 1. \tag{2.12}$$

By comparing the functions $\phi(h) = 2\beta^h$ and $\psi(h) = 1 + (2\beta-1)h$ on $h \in [0,1]$, it follows that (2.12) holds if and only if

$$\beta_c \le \beta \le 1,$$

where $\beta_c \approx 0.1866823$ is the unique solution in $(0,1]$ to the equation $2\beta_c \ln\beta_c = (2\beta_c - 1)$. To see this equivalence, note that for $\beta > 0, \phi(0) = 2 > \psi(0) = 1$, and $\phi(1) = \psi(1) = 2\beta$. Now, if (i) $\beta > 1$, then, since ψ is increasing, convex, ϕ must be a secant line for ψ on $[0,1]$ and, therefore, (2.12) fails. On the other hand, if (ii) $\frac{1}{2} \le \beta \le 1$, then ϕ is decreasing, ψ is increasing, and $\phi(1) = \psi(1)$, so that (2.12) holds. Finally, if (iii) $\beta < \frac{1}{2}$, then ϕ is decreasing, convex, and ψ decreases linearly to meet ϕ at $h = 1$. That is, ψ is a secant line to ϕ on $[0,1]$ and, therefore, (2.12) fails *unless* $\phi'(1) \ge \psi'(1)$, i.e., unless $2\beta\ln\beta \ge 2\beta - 1$. This makes $\beta > \beta_c$ necessary in order for (2.12) to hold. Conversely, if $\beta_c \le \beta < 1/2$, then $\phi'(1) \ge \psi'(1)$, so that (2.12) holds.

Now, $\beta < \beta_c$ implies $A_\beta(\infty) = 0$ almost surely. For the converse, i.e., uniform integrability of the positive martingale $\{A_\beta(t) : t \geq 0\}$, we will use an inequality from [12], attributed there to B. Chauvin and J. Neveu, especially suited for such problems. For present purposes, if $1 < p \leq 2$, and $X_1, X_2 \in L^p(\Omega, \mathcal{F}, P)$ are independent, positive random variables, then

$$v_p(X_1 + X_2) \leq v_p(X_1) + v_p(X_2), \tag{2.13}$$

where $v_p(X_j) = \mathbb{E}X_j^p - (\mathbb{E}X_j)^p, j = 1, 2$.

By the basic recursion (2.10), one has

$$\mathbb{E}A_\beta^p(t) = e^{-[(2\beta-1)p+1]t} + \beta^p \int_0^t e^{-[(2\beta-1)p+1]s} \mathbb{E}(A_\beta^+(t-s) + \mathbb{E}A_\beta^-(t-s))^p ds. \tag{2.14}$$

Applying (2.13) and using the submartingale property $\mathbb{E}A_\beta^p(t-s) \leq \mathbb{E}A_\beta^p(t), 0 \leq s \leq t$ together with the fact that $\mathbb{E}A_\beta(t-s) = 1$, we estimate

$$\begin{aligned}
&\mathbb{E}(A_\beta^+(t-s) + A_\beta^-(t-s))^p \\
&\quad = v_p(A_\beta^+(t-s) + A_\beta^-(t-s)) + (\mathbb{E}A_\beta^+(t-s) + \mathbb{E}A_\beta^-(t-s))^p \\
&\quad \leq v_p(A_\beta^+(t-s)) + v_p(A_\beta^-(t-s)) + 2^p(\mathbb{E}(A_\beta(t-s)))^p \\
&\quad \leq 2\mathbb{E}A_\beta^p(t-s) + 2^p \leq 2\mathbb{E}A_\beta^p(t) + 2^p.
\end{aligned}$$

Thus, (2.14) yields

$$\mathbb{E}A_\beta^p(t) \leq e^{-[(2\beta-1)p+1]t} + \frac{(2\mathbb{E}A_\beta^p(t) + 2^p)\beta^p}{(2\beta-1)p+1},$$

which implies

$$\frac{(2\beta-1)p+1-2\beta^p}{(2\beta-1)p+1}\mathbb{E}A_\beta^p(t) \leq e^{-[(2\beta-1)p+1]t} + \frac{(2\beta)^p}{(2\beta-1)p+1}, \quad t \geq 0.$$

In particular, uniform integrability follows under the condition that for some $p \in (1, 2]$,

$$(2\beta-1)p + 1 - 2\beta^p > 0.$$

Equivalently, $\beta > \beta_c$ where, as before, β_c denotes the solution of (2.8).

To complete the proof requires consideration of the case $\beta = \beta_c$. If, for the sake of contradiction, one assumes uniform integrability, then, as is elaborated in the proof of Proposition 2.1 below, the distribution of $A_{\beta_c}(\infty)$ provides a mean one fixed point to the Holley-Liggett smoothing map, see [7], where it is shown that there is not a mean one fixed point at β_c. □

For $\beta \in [0, 1]$, define the moment generating function

$$\varphi_\beta(r) = \mathbb{E}e^{-rA_\beta(\infty)}, \quad r \geq 0,$$

where $A_\beta(\infty) = \lim_{t\to\infty} A_\beta(t)$. Note that by Theorem 2.2 and its proof,

$$\varphi'_\beta(0) = 0 \quad \text{if } \beta < \beta_c \quad \text{and} \quad \varphi'_\beta(0) = -1 \quad \text{if } \beta > \beta_c.$$

Also define a probability measure ν_β on S_β where $S_\beta = [0, \beta]$ for $\beta > 1/2$, and $S_\beta = [\beta, \infty)$ for $0 < \beta < 1/2$, and

$$\nu_{\frac{1}{2}}(ds) = \delta_{\frac{1}{2}}(ds), \quad \nu_\beta(ds) = \frac{(s/\beta)^{\frac{1}{2\beta-1}}}{|2\beta-1|}\frac{ds}{s}, \ \beta \neq \frac{1}{2}. \tag{2.15}$$

Proposition 2.1. *For $\beta > \beta_c$, φ_β is uniquely determined within the class of probability distributions on $[0, \infty)$ whose moment generating function satisfies*

$$\varphi_\beta(r) = \int_{S_\beta} \varphi_\beta^2(rs)\nu_\beta(ds), \quad r \geq 0, \tag{2.16}$$

such that $\varphi_\beta(0) = 1$, $\varphi'_\beta(0) = -\mathbb{E}A_\beta(\infty)$. Equivalently, φ_β is uniquely determined by the delayed differential equation

$$\varphi'_\beta(r) = \frac{1}{r}\frac{1}{2\beta - 1}\varphi_\beta^2(\beta r) - \frac{1}{r}\frac{1}{2\beta - 1}\varphi_\beta(r), \quad \beta \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{2.17}$$

and the given initial conditions.

Proof. First we will show that (2.16) holds for $\beta \in [0, 1]$. When $\beta = 1/2$, by (2.11),

$$\varphi_{\frac{1}{2}}(r) = \varphi_{\frac{1}{2}}^2(r/2), \tag{2.18}$$

and thus (2.16) holds with $\nu_{1/2}$ – the Dirac measure as in (2.15). For $\beta \neq 1/2$, using the stochastic recursion (2.11), we obtain:

$$\begin{aligned}
\varphi_\beta(r) = \mathbb{E}\big(e^{-rA_\beta(\infty)}\big) &= \mathbb{E}\left(\exp\left[-r\beta e^{-(2\beta-1)T_\theta}\left(A_\beta^+(\infty)) + A_\beta^-(\infty)\right)\right]\right) \\
&= \int_0^\infty e^{-t}\,\mathbb{E}\exp\left[-r\beta e^{-(2\beta-1)t}\left(A_\beta^+(\infty)) + A_\beta^-(\infty)\right)\right] dt \\
&= \int_0^\infty e^{-t}\varphi_\beta^2\left(r\beta e^{-(2\beta-1)t}\right) dt.
\end{aligned}$$

Now (2.16) follows by the change of variables $s = \beta e^{-(2\beta-1)t}$.

For $\beta > \beta_c$, in view of the uniform integrality (see Theorem 2.2) one has $\mathbb{E}A_\beta(\infty) = 1$, and we may use early results of [7] on smoothing transformations. Specifically, it is simple to check that for $\beta_c < \beta \leq 1$, the random variable $W_\beta = 2\beta e^{-(2\beta-1)T_\theta}$ has mean one (in fact, $\frac{1}{2}W_\beta$ is a re-scaling of the distribution ν_β), while the recursion (2.11) takes the form

$$A_\beta(\infty) = W_\beta\Big(\frac{1}{2}A_\beta^+(\infty) + \frac{1}{2}A_\beta^-(\infty)\Big),$$

of a Holley-Liggett smoothing transformation within the framework of Theorem 7.1 in [7]. Accordingly, the distribution of $A_\beta(\infty)$ is the unique positive mean one solution to the stochastic recursion provided

$$\mathbb{E}(W_\beta \ln W_\beta) < \ln 2.$$

A direct calculation shows that $\mathbb{E}(W_\beta \ln W_\beta) = \ln(2\beta) - \frac{2\beta-1}{2\beta}$, and thus the inequality above is satisfied if and only if $\beta > \beta_c$.

To establish (2.17) we may use (2.16), as follows (noting that the implied differentiability is a property of a moment generating function of a probability distribution on $[0, \infty)$):

$$\varphi'_\beta(r) = \int_{S_\beta} \frac{d}{dr}\varphi_\beta^2(rs)\nu_\beta(ds) = \frac{1}{r}\int_{S_\beta} \frac{d}{ds}\varphi_\beta^2(rs)\, s\,\nu_\beta(ds).$$

Now use (2.15) and integrate by parts. In the case $\beta < 1/2$ we get:

$$\begin{aligned}\varphi_\beta'(r) &= \frac{1}{r}\int_\beta^\infty \frac{d}{ds}\varphi_\beta^2(rs)\,\frac{(s/\beta)^{\frac{1}{2\beta-1}}}{1-2\beta}\,ds\\ &= \frac{1}{r}\,\varphi_\beta^2(rs)\,\frac{(s/\beta)^{\frac{1}{2\beta-1}}}{1-2\beta}\Bigg|_{s=\beta}^{\infty} + \frac{1}{r}\int_\beta^\infty \varphi_\beta^2(rs)\,\frac{(s/\beta)^{\frac{1}{2\beta-1}}}{(1-2\beta)^2}\,\frac{ds}{s}\\ &= -\frac{1}{r}\frac{1}{1-2\beta}\,\varphi_\beta^2(\beta r) + \frac{1}{r}\frac{1}{1-2\beta}\,\varphi_\beta(r),\end{aligned}$$

which implies (2.17) for $\beta \in [0, 1/2)$. The case $\beta \in (1/2, 1]$ is treated analogously. □

Remark 2.1. While the martingale limit is clearly a fixed point of the Holley-Liggett smoothing transformation for any $\beta \in (0, 1]$, the proof of uniform integrability is essential to the identification of the critical parameter β_c for a positive martingale limit since the fixed point uniqueness theorem is within the class of mean one probability distributions on $[0, \infty)$. Once this is achieved then the existing theory of fixed points of smoothing transformations as given in [7], [11], among others, can be applied to discern more about the non-exponential cases of the limit distributions. As noted in [7] for particular Beta distributions of W, the fixed point distribution is a Gamma distribution. This includes the case of Kendall's theorem, [9], for $\beta = 1$ in which W is uniform on $(0, 1)$ and the martingale limit has a mean one exponential distribution as given below.

Corollary 2.1 (Kendall's theorem). *$A_1(t) = e^{-t}Y_t, t \geq 0$, is a uniformly integrable martingale, and $A_1(\infty) = \lim_{t\to\infty} A_1(t)$ is exponentially distributed with mean one.*

Proof. It is easy to see that the mean one exponential moment generating function $1/(1+r)$ satisfies (2.16) in case $\beta = 1$. Now the fact that the exponential is indeed the distribution of $A_1(\infty)$ follows from the uniqueness statement of Proposition 2.1. □

Remark 2.2. One can also obtain Kendall's result directly from (2.17). Indeed, when $\beta = 1$ we have

$$(r\varphi_1(r))' = \varphi_1^2(r), \qquad \varphi_1(0) = 1,\ \varphi_1'(0) = -1.$$

The non-zero solutions of the equation above can be obtained explicitly as

$$\varphi_1(r) = \frac{1}{1 + c_0 r},$$

while by the initial data, $c_0 = 1$, proving that the mean one exponential moment generating function is the only solution, and thus implying Kendall's theorem stated in Corollary 2.1.

The following result shows that for $\beta_c < \beta < 1/2$, $A_\beta(\infty)$ has heavy tails. As remarked earlier, this and more on the nature of the martingale limit distribution are also available from general theory, e.g., see [11]. However one may also give the following self-contained argument based on (2.17).

Proposition 2.2. *For any $\beta \in (\beta_c, 1/2)$, there exists $p_\beta \geq 2$ such that $\mathbb{E}(A_\beta^p(\infty)) = \infty$ for all $p \geq p_\beta$.*

Proof. Note that the finite moments of order $k \in \mathbb{N}$ satisfy:

$$m_k = (-1)^k \varphi_\beta^{(k)}(0),$$

and consequently, using (2.17) and the fact that $m_0 = m_1 = 1$ we obtain

$$\left((2\beta - 1)k - 2\beta^k + 1\right) \frac{m_k}{k!} = \beta^k \sum_{j=1}^{k-1} \frac{m_j}{j!} \frac{m_{k-j}}{(k-j)!}, \quad k \geq 2.$$

Since $Y_\beta(\infty) \geq 0$, we have $m_k > 0$ for all k, and thus

$$(2\beta - 1)k - 2\beta^k + 1 > 0 \quad \text{for all } k \geq 2.$$

Note that the above condition fails for large enough k if $\beta < 1/2$, implying that the higher-order moments of Y_β must be infinite. □

3. Infinitesimal generator and another critical value for the delayed Yule process

Give $\mathcal{E}$ the discrete topology and let $C_0(\mathcal{E})$ denote the space of (continuous) real-valued functions $f : \mathcal{E} \to \mathbb{R}$ that vanish at infinity; i.e., given $\epsilon > 0$, one has $|f(V)| < \epsilon$ for all but finitely many $V \in \mathcal{E}$. The subspace $C_{00}(\mathcal{E}) \subset C_0(\mathcal{E}) \subset L^\infty(\mathcal{E})$ of functions with compact (finite) support is clearly dense in $C_0(\mathcal{E})$ for the uniform norm.

The construction at the outset of the coupled stochastic processes $V^{(\alpha)}, 0 < \alpha \leq 1$, provides corresponding semigroups of positive linear contractions $\{T_t^{(\alpha)} : t \geq 0\}$ defined by

$$T_t f(V) = \mathbb{E}_V f(V^{(\alpha)}(t)), \quad t \geq 0, f \in C_0(\mathcal{E}),$$

with the usual branching process convention that given $V^{(\alpha)}(0) = V \in \mathcal{E}$, $V^{(\alpha)}(t)$ is the total progeny independently produced by single progenitors at each $v \in V$. In fact, one may consider the semigroup as defined on $L^\infty(\mathcal{E}) \supset C_0(\mathcal{E})$.

The usual considerations imply that the infinitesimal generator $(L^{(\alpha)}, \mathcal{D}_\alpha)$ of $V^{(\alpha)}$ is given on $C_{00}(\mathcal{E})$ via

$$L^{(\alpha)} f(V) = \sum_{v \in V} \alpha^{|v|} \{f(V^v) - f(V)\}, \quad f \in C_{00}(\mathcal{E}),$$

where

$$V^v = V \backslash \{v\} \cup \{< v1, v2 >\}, \quad v \in V.$$

One may naturally pursue the computation of a core for $L^{(\alpha)}$, however for the present purposes the above is sufficient to establish the following distinct role of $\alpha = \frac{1}{2}$ as a critical parameter.

Proposition 3.1. $(L^{(\alpha)}, \mathcal{D}_\alpha)$, $\mathcal{D}_\alpha \subset L^\infty(\mathcal{E})$ *– the domain of* $L^{(\alpha)}$*, is a bounded linear operator if and only if* $\alpha \leq \frac{1}{2}$.

Proof. The sufficiency follows from the Key Coupling Lemma 2.1, since for $\alpha \leq \frac{1}{2}$ one has the bound $\sum_{v \in V} \alpha^{|v|} \leq \sum_{v \in V} 2^{-|v|} = 1, V \in \mathcal{E}$. In particular, for $f \in C_0(\mathcal{E})$,

$$|L^{(\alpha)} f(V)| \leq 2 \sup_{W \in \mathcal{E}} |f(W)|, \quad V \in \mathcal{E}.$$

On the other hand, for $\alpha > \frac{1}{2}$, define a sequence of functions $f_n \in C_{00}(\mathcal{E})$ by

$$f_n(V) = h(V) \, \mathbf{1}_{[h(V) \leq n]}, \quad n = 1, 2, \ldots,$$

where $h(V) = \max\{|v| : v \in V\}, V \in \mathcal{E}$. Then for full binary branching $h(V) = n, |V| = 2^n$. Thus $\|f_n\|_\infty = n$, and for such V,

$$|L^{(\alpha)} f_n(V)| = \sum_{v \in V} \alpha^n = (2\alpha)^n.$$

In particular

$$\frac{|L^{(\alpha)} f_n(V)|}{\|f_n\|_\infty} = \frac{(2\alpha)^n}{n} \to \infty \quad \text{as } n \to \infty \quad \text{for } \alpha > \frac{1}{2}.$$

□

Remark 3.1. Although $a_\beta \notin C_0(\mathcal{E})$ for any $\beta \in (0,1]$, the following formal calculation for $\alpha \in (0,1]$,

$$L^{(\alpha)} a_\beta(V) = (2\beta - 1) a_{\alpha\beta}(V), \quad V \in \mathcal{E},$$

is intriguing from the perspective of precise identification of the generator. In particular, a_β is formally a positive eigenfunction of $L^{(1)}$ with non-positive eigenvalue $2\beta - 1 < 0$ for $\beta < \frac{1}{2}$ as required for a contraction semigroup of positive linear operators. To make this formal calculation rigorous obviously requires a modification of the function space beyond the standard choice $C_0(\mathcal{E})$.

Finally let us conclude by noting a closely related evolution that takes place in sequence space that may be of interest in other contexts. For $V \in \mathcal{E}$, let

$$g_k(V) = \#\{v \in V : |v| = k\}, \quad k = 0, 1, 2, \ldots.$$

Also define an equivalence relation on $\mathcal{E}$ by $V \sim W$, $V, W \in \mathcal{E}$, if and only if $g_k(V) = g_k(W)$ for all k. Then the space of equivalence classes $\mathcal{E}/\sim$ is in one-to-one correspondence with a subset of the sequence space $c_{00}(\mathbb{Z}_+) \subset \ell_1(\mathbb{Z}_+)$ defined inductively as follows: $n = (n_0, n_1, \ldots) \in c_{00}(\mathbb{Z}_+)$ belongs to the space $\mathcal{E}_0$ of *evolutionary sequences* if either $n = (1, 0, \ldots)$ or, otherwise, there is an $m \in \mathcal{E}_0 \subset c_{00}(\mathbb{Z}_+)$ such that $m = n^{(k)} := (n_0, n_1, \ldots, n_k - 1, n_{k+1} + 2, n_{k+2}, \ldots)$ for some $k \geq 0$ such that $n_k \geq 1$. Note that $\sum_{j=0}^\infty n_j = \sum_{j=0}^\infty m_j - 1$. For $0 < \alpha \leq 1$, the equivalence relation induces $N^{(\alpha)} = \{N^{(\alpha)}(t) : t \geq 0\}$ as the continuous time jump Markov process on $\mathcal{E}_0$ with generator given for $f \in C_{00}(\mathcal{E}_0)$ by

$$\tilde{L}^{(\alpha)} f(n) = \sum_{k=0}^\infty n_k \alpha^k (f(n^{(k)}) - f(n)), \quad n \in \mathcal{E}_0.$$

4. Connections with other work

(a) After this article was recommended for publication the authors learned from David Aldous about [1], and related references, [2], [3], that analyze processes of the same type as here, but from different perspectives and objectives. That model is one of a large class of random tree-growth models studied within the probabilistic analysis of algorithms; see [8]. In [1] the primary focus is on limiting properties of a class of trees that grow randomly, one node at a time [in discrete time], in connection with Ziv's entropy estimation algorithm and various models from physics and computer science. As a "standard trick in probability", the authors also consider the continuous time model that would correspond to a $\frac{1}{c}$-delayed Yule process ($c > 1$) in sequence space. The main results are a strong law of large numbers and a central limit theorem. The authors note in passing (Section 7, page 539) that the case $c = 2$ corresponds to a Poisson process, and they remark that the relative slowing ($c > 2$), and speeding ($c < 2$) of the process are artifacts of the continuous-time formulation and have no direct interpretations for their problem. As shown in the present paper, $\alpha^{-1} = c = 2$ is a critical parameter with respect to the boundedness of the infinitesimal generator. In [5], where the continuous parameter models are the essential structures, this criticality together with certain monotonicities with respect to α play an essential role in the analysis of the complex Burgers equation.

(b) In [1] the authors single out a special value of the parameter of the form $\alpha^{-1} = c = 2^{\frac{1}{2}}$ for its (conjectured) role in a Gaussian central limit theorem. A non-Gaussian limit is conjectured for $c < 2^{\frac{1}{2}}$, and is partially confirmed to them through privately communicated results of H. Kesten. In [5], special roles for this and other parameters in the general form $c^{-1} = \alpha = 2^{-\frac{k}{2}}, k = 1, 2, \dots$ are shown to correspond to polynomial mean numbers of offspring at (continuous) time t.

(c) In [2] the focus is on a process that is termed a "discounted branching random walk"for which $\alpha > 1$. The question concerns the unique determination of the distribution of the position of the rightmost particle via an ordinary differential equation. Although mentioned as a possibility, the author remarks that this case $\alpha < 1$ is not dealt with in [2].

(d) In [3] the authors refer to the model in the case $\alpha = \frac{1}{2}$ as "directed diffusion-limited aggregation". The authors obtain some limit results on the tree height of the type in [1], but weaker.

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References

[1] David Aldous and Paul Shields, *A diffusion limit for a class of randomly-growing binary trees*, Probab. Theory Related Fields **79** (1988), no. 4, 509–542, DOI 10.1007/BF00318784. MR966174

[2] K. B. Athreya, *Discounted branching random walks*, Adv. in Appl. Probab. **17** (1985), no. 1, 53–66, DOI 10.2307/1427052. MR778593

[3] RM Bradley and PN Strenski, *Directed diffusion-limited aggregation on the bethe lattice: exact results*, Physical Review B **30** (1984), no. 11, 6788.

[4] Radu Dascaliuc, Nicholas Michalowski, Enrique Thomann, and Edward C. Waymire, *Symmetry breaking and uniqueness for the incompressible Navier-Stokes equations*, Chaos **25** (2015), no. 7, 075402, 16, DOI 10.1063/1.4913236. MR3405857

[5] Radu Dascaliuc, Nicholas Michalowski, Enrique Thomann, Edward C. Waymire. *Complex Burgers Equation: A probabilistic perspective.* Festscrift in Honor of Charles M. Newman, V. Sidoravicius, D. Stein eds., to appear, Springer, NY.

[6] Richard Durrett and Thomas M. Liggett, *Fixed points of the smoothing transformation*, Z. Wahrsch. Verw. Gebiete **64** (1983), no. 3, 275–301, DOI 10.1007/BF00532962. MR716487

[7] Richard Holley and Thomas M. Liggett, *Generalized potlatch and smoothing processes*, Z. Wahrsch. Verw. Gebiete **55** (1981), no. 2, 165–195, DOI 10.1007/BF00535158. MR608015

[8] Philippe Jacquet and Wojciech Szpankowski, *Analytic pattern matching*, Cambridge University Press, Cambridge, 2015. From DNA to Twitter; With a foreword by Robert Sedgewick. MR3524836

[9] David G. Kendall, *Branching processes since 1873*, J. London Math. Soc. **41** (1966), 385–406. (1 plate), DOI 10.1112/jlms/s1-41.1.385. MR0198551

[10] Y. Le Jan and A. S. Sznitman, *Stochastic cascades and 3-dimensional Navier-Stokes equations*, Probab. Theory Related Fields **109** (1997), no. 3, 343–366, DOI 10.1007/s004400050135. MR1481125

[11] Quansheng Liu, *On generalized multiplicative cascades*, Stochastic Process. Appl. **86** (2000), no. 2, 263–286, DOI 10.1016/S0304-4149(99)00097-6. MR1741808

[12] J. Neveu, *Multiplicative martingales for spatial branching processes*, Seminar on Stochastic Processes, 1987 (Princeton, NJ, 1987), Progr. Probab. Statist., vol. 15, Birkhäuser Boston, Boston, MA, 1988, pp. 223–242, DOI 10.1007/978-1-4684-0550-7_10. MR1046418

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