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ON THE EQUIVALENCE BETWEEN Θ_n-SPACES AND ITERATED SEGAL SPACES

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ABSTRACT. We give a new proof of the equivalence between two of the main models for (∞, n)-categories, namely the n-fold Segal spaces of Barwick and the $\mathbf{\Theta}_n$-spaces of Rezk, by proving that these are algebras for the same monad on the ∞-category of n-globular spaces. The proof works for a broad class of ∞-categories that includes all ∞-topoi.

1. Introduction

(∞, n)-categories are a homotopical version of n-categories. This means that they have i-morphisms between $(i-1)$-morphisms for $i = 1, \ldots, n$ and also homotopies between n-morphisms, homotopies of homotopies, etc. (in other words, invertible i-morphisms for $i > n$), with composition of i-morphisms only associative up to a coherent choice of higher homotopies. There are now a number of good models for (∞, n)-categories; the two that have seen the most use so far are *n-fold Segal spaces* and *$\mathbf{\Theta}_n$-spaces*. Iterated Segal spaces were first defined in Barwick's thesis [Bar05], building on Rezk's work on *Segal spaces* [Rez01], and were later generalized by Lurie [Lur09b, §1] to the setting of ∞-topoi; they are presheaves of spaces on the category $\mathbf{\Delta}^n$ satisfying iteratively defined "Segal conditions" and constancy conditions. $\mathbf{\Theta}_n$-spaces, which were introduced by Rezk [Rez10] (no doubt influenced by Joyal's unpublished work on $\mathbf{\Theta}_n$-sets and Berger's description of n-fold loop spaces [Ber07]), are similarly presheaves of spaces on categories $\mathbf{\Theta}_n$ that satisfy certain "Segal conditions"; in this paper we consider their natural generalization to ∞-topoi, which we will refer to as *Segal $\mathbf{\Theta}_n$-objects* for clarity.

In [BSP11], Barwick and Schommer-Pries give axioms that characterize the homotopy theory of (∞, n)-categories. They also prove that these axioms are satisfied in the case of n-fold Segal spaces and $\mathbf{\Theta}_n$-spaces, which implies that these two models are equivalent. Another comparison, which relates the two models directly in the setting of model categories, has been given more recently by Bergner and Rezk [BR14].

The goal of this short paper is to give a new, conceptual proof of this equivalence: we will show that both models are the ∞-categories of algebras for a monad on the ∞-category of n-globular spaces (i.e. presheaves of spaces on the n-globular category; cf. Definition 2.3), and that these two monads are equivalent. This also brings out the relation between (∞, n)-categories and n-categories: strict n-categories are the algebras for the analogous monad on the category of n-globular *sets*.

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Our proof only makes use of formal properties of the ∞-category of spaces that hold for all ∞-topoi, so we obtain a comparison between iterated Segal objects and Segal $\mathbf{\Theta}_n$-objects in any ∞-topos $\mathcal{X}$. In fact, our comparison works in sufficient generality to allow us to conclude iteratively that Segal $\mathbf{\Theta}_{n_1+\cdots+n_k}$-objects in $\mathcal{X}$ are equivalent to Segal $\mathbf{\Theta}_{n_1} \times \cdots \times \mathbf{\Theta}_{n_k}$-objects that are *reduced* (i.e. satisfy certain constancy conditions).

In this paper we focus on the "algebraic" theory of (∞, n)-categories, i.e. we do not invert the appropriate class of "fully faithful and essentially surjective morphisms". However, this localization is given for both n-fold Segal spaces and $\mathbf{\Theta}_n$-spaces by restricting to subcategories of *complete* objects, and our equivalence is easily seen to restrict to an equivalence between these subcategories.

1.1. **Notation.** This paper is written in the language of ∞-categories, and we reuse some of the terminology and notation of [Lur09a, Lur14] without comment. Moreover, we use the ∞-categorical terminology uniformly, even when the ∞-categories in question are ordinary categories. In particular, for us a *cofinal* functor $F: \mathcal{C} \to \mathcal{D}$ between ordinary categories is a functor that is cofinal in the ∞-categorical sense (also called *homotopy cofinal*). Similarly, we will speak of (co)Cartesian fibrations between categories rather than Grothendieck (op)fibrations.

If $\mathcal{C}$ and $\mathcal{X}$ are ∞-categories we will write $\mathcal{P}(\mathcal{C}; \mathcal{X})$ for the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ of presheaves on $\mathcal{C}$ valued in $\mathcal{X}$.

2. $\mathbf{\Theta}_n$-objects and Segal conditions

In this section we will define our main objects of study in this paper: *reduced Segal $\mathbf{\Theta}_n$-objects*, which are certain presheaves on categories $\mathbf{\Theta}_n$, whose definition we will now recall; these categories were originally introduced by Joyal, but here we make use of the inductive reformulation of the definition due to Berger [Ber07, Definition 3.1].

Definition 2.1. The category $\mathbf{\Theta}_n$ is defined inductively as follows: First set $\mathbf{\Theta}_0 := *$. Then define $\mathbf{\Theta}_n$ to be the category with objects $[i](I_1, \ldots, I_i)$ with $[i] \in \mathbf{\Delta}$ and $I_p \in \mathbf{\Theta}_{n-1}$; a morphism $[i](I_1, \ldots, I_i) \to [j](J_1, \ldots, J_j)$ is given by a morphism $\phi: [i] \to [j]$ in $\mathbf{\Delta}$ and morphisms $\psi_{pq}: I_p \to J_q$ in $\mathbf{\Theta}_{n-1}$ where $0 < p \leq i$ and $\phi(p-1) < q \leq \phi(p)$. The composite of $(\phi, \psi_{pq}): [i](I_1, \ldots, I_i) \to [j](J_1, \ldots, J_j)$ and $(\phi', \psi'_{qr}): [j](J_1, \ldots, J_j) \to [k](K_1, \ldots, K_k)$ is $(\phi' \circ \phi, \psi''_{pr})$ where $\psi''_{pr} := \psi'_{qr} \circ \psi_{pq}$ where q is the unique index with $\phi(p-1) < q \leq \phi(p)$ such that $\phi'(q-1) < r \leq \phi'(q)$. If $\mathcal{X}$ is an ∞-category, we will refer to presheaves $\mathbf{\Theta}_n^{\mathrm{op}} \to \mathcal{X}$ as *$\mathbf{\Theta}_n$-objects* in $\mathcal{X}$.

There is a useful factorization system on $\mathbf{\Theta}_n$ given by the inert and active morphisms, in the following sense:

Definition 2.2. A morphism $\phi: [n] \to [m]$ in $\mathbf{\Delta}$ is *inert* if it is the inclusion of a subinterval in $[m]$, i.e. $\phi(i) = \phi(0) + i$ for all i, and *active* if it preserves the endpoints, i.e. $\phi(0) = 0$ and $\phi(n) = m$. We then inductively say a morphism (ϕ, ψ_{ij}) in $\mathbf{\Theta}_n$ is *inert* if ϕ is inert in $\mathbf{\Delta}$ and each ψ_{ij} is inert in $\mathbf{\Theta}_{n-1}$, and *active* if ϕ is active in $\mathbf{\Delta}$ and each ψ_{ij} is active in $\mathbf{\Theta}_{n-1}$. We write $\mathbf{\Theta}_{n,\mathrm{i}}$ for the subcategory of $\mathbf{\Theta}_n$ containing only the inert maps and $i_n: \mathbf{\Theta}_{n,\mathrm{i}} \to \mathbf{\Theta}_n$ for the inclusion.

Every morphism in $\mathbf{\Theta}_n$ can be factored as an active morphism followed by an inert morphism — moreover, as objects of $\mathbf{\Theta}_n$ have no non-trivial automorphisms, this factorization is *strictly* unique. This factorization system seems to have been first constructed by Berger — it is a special case of [Ber02, Lemma 1.11] (where the inert maps are called *immersions* and the active ones *covers*). It is also a special case of [Web07, Proposition 4.20] and of [Bar13, Lemma 8.3]; moreover, using the inductive definition of $\mathbf{\Theta}_n$ it is easy to check directly by hand.

The objects of $\mathbf{\Theta}_n$ can be thought of as n-dimensional pasting diagrams for compositions in n-categories. We now wish to define the appropriate *Segal conditions* for $\mathbf{\Theta}_n$-objects that make their values at such a pasting diagram decompose appropriately as a limit of the values at the basic i-morphisms ($i = 0, \dots, n$). These were originally specified by Rezk [Rez10], but we will use an alternative formulation influenced by the work of Barwick on operator categories [Bar13]; this is also a special case of the general version of Segal conditions considered in [Web07]. The definition requires introducing some notation:

Definition 2.3. We define objects $C_i \in \mathbf{\Theta}_n$ for $i = 0, \dots, n$ by $C_0 := [0]()$ and $C_i = [1](C_{i-1})$ for $i > 0$. (For $n = 0$, we let C_0 denote the unique object of $\mathbf{\Theta}_0 = *$.) Let $\mathbb{G}_n$, the *n-globular category*, be the full subcategory of $\mathbf{\Theta}_{n,\mathrm{i}}$ containing the objects $C_0, \dots, C_n$; we write γ_n for the inclusion $\mathbb{G}_n \hookrightarrow \mathbf{\Theta}_{n,\mathrm{i}}$. We can informally depict the category $\mathbb{G}_n$ as

$$C_0 \rightrightarrows C_1 \rightrightarrows \cdots \rightrightarrows C_n.$$

We refer to the object C_k as the *k-cell.* Given $I \in \mathbf{\Theta}_n$, we will write $\mathbb{G}_{n/I}$ for the category $\mathbb{G}_n \times_{\mathbf{\Theta}_{n,\mathrm{i}}} (\mathbf{\Theta}_{n,\mathrm{i}})_{/I}$, and refer to its objects as the *cells* of I.

Definition 2.4. Suppose $\mathcal{X}$ is a presentable ∞-category. A presheaf $F\colon \mathbf{\Theta}_n^{\mathrm{op}} \to \mathcal{X}$ is a *Segal $\mathbf{\Theta}_n$-object* if its restriction $F|_{\mathbf{\Theta}_{n,\mathrm{i}}^{\mathrm{op}}}$ is the right Kan extension along γ_n of its restriction to $\mathbb{G}_n^{\mathrm{op}}$ — in other words, for I in $\mathbf{\Theta}_n$ the natural map $F(I) \to \lim_{C \in \mathbb{G}_{n/I}^{op}} F(C)$ is an equivalence. We write $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{X})$ for the full subcategory of $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ spanned by the Segal $\mathbf{\Theta}_n$-objects, and $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$ for the analogous subcategory of $\mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$; these are accessible localizations of $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ and $\mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$, respectively.

This is equivalent to the inductive Segal condition given, for example, in [Rez10]:

Proposition 2.5. *$F \in \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ is a Segal presheaf if and only if the following conditions hold:*

(1) For every object $I = [i](I_1, \dots, I_i)$ ($i \neq 0$), the natural map

$$F(I) \to F([1](I_1)) \times_{F(C_0)} \cdots \times_{F(C_0)} F([1](I_i))$$

is an equivalence.

(2) The presheaf $F([1](-))\colon \mathbf{\Theta}_{n-1}^{\mathrm{op}} \to \mathcal{X}$ is a Segal $\mathbf{\Theta}_{n-1}$-object.

It is convenient for us to prove Proposition 2.5 using a result about general limit decompositions for Segal objects, which we turn to first.

Definition 2.6. Suppose $f\colon I \to J$ is an active morphism in $\mathbf{\Theta}_n$. For $\alpha\colon C \to I$ in $\mathbb{G}_{n/I}$, let $C \xrightarrow{f_\alpha} J_\alpha \xrightarrow{i_\alpha} J$ be the (unique) active-inert factorization of $f \circ \alpha\colon C \to J$. Given a factorization of α as $C \xrightarrow{\xi} C' \xrightarrow{\alpha'} I$ with ξ inert, the composite $C \to C' \to J_{\alpha'}$ has an active-inert factorization $C \to X \to J_{\alpha'}$. Since this also gives an active-inert factorization of $C \to J_{\alpha'} \to J$ we see that $X = J_\alpha$, and so ξ determines an inert morphism $J_\alpha \to J_{\alpha'}$. We thus get a functor $\mathbb{G}_{n/I} \to \mathrm{Cat}$ by sending α to $\mathbb{G}_{n/J_\alpha}$ and a morphism in $\mathbb{G}_{n/I}$ to the functor given by composition with the associated inert morphism $J_\alpha \to J_{\alpha'}$. Let $\mathbb{G}_{n/f} \to \mathbb{G}_{n/I}$ denote the corresponding coCartesian fibration. Composition with the inert morphisms $J_\alpha \to J$ gives a functor $\mathbb{G}_{n/f} \to \mathbb{G}_{n/J}$.

Proposition 2.7. *For any active morphism $f\colon I \to J$ in $\boldsymbol{\Theta}_n$, the functor $\mathbb{G}_{n/f} \to \mathbb{G}_{n/J}$ is cofinal.*

Before we prove this, we make the following simple observation:

Lemma 2.8. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a Cartesian fibration. Then p is cofinal if and only if the fibres $\mathcal{E}_b$ are weakly contractible for all $b \in \mathcal{B}$.*

Proof. By [Lur09a, Theorem 4.1.3.1], the functor p is cofinal if and only if the ∞-categories $\mathcal{E}_{b/} := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b/}$ are weakly contractible for all $b \in \mathcal{B}$. If p is Cartesian, for every $b \in \mathcal{B}$ the functor $\mathcal{E}_b \to \mathcal{E}_{b/}$ is coinitial (i.e. the op'ed functor is cofinal): for an object $\epsilon = (e \in \mathcal{E}, f\colon b \to p(e))$ in $\mathcal{E}_{b/}$, the ∞-category $(\mathcal{E}_b)_{/\epsilon}$ has a terminal object, given by a Cartesian morphism over f with target e. This functor is therefore in particular a weak homotopy equivalence by [Lur09a, Lemma 4.1.1.3(3)]. ☐

Proof of Proposition 2.7. The projection $\mathbb{G}_{n/f} \to \mathbb{G}_{n/J}$ is a Cartesian fibration, so by Lemma 2.8 it suffices to show that for $\gamma\colon C \to J$, the fibre $(\mathbb{G}_{n/f})_\gamma$ is weakly contractible; we will prove this by induction on n. The category $(\mathbb{G}_{n/f})_\gamma$ consists of diagrams

$$\begin{tikzcd} & C' \arrow[r, "\alpha"] \arrow[d, "f_\alpha"] & I \arrow[d, "f"] \\ C \arrow[r, "\gamma_\alpha"] \arrow[rr, bend right, "\gamma"'] & J_\alpha \arrow[r, "i_\alpha"] & J, \end{tikzcd}$$

where f_α is active and i_α is inert. Since inert maps are monomorphisms in $\boldsymbol{\Theta}_n$, we may identify this with the full subcategory of $\mathbb{G}_{n/I}$ spanned by those cells $\alpha\colon C' \to I$ such that γ factors through i_α.

First consider the case where $n = 1$. Then γ is a map $[a] \to J$ where a is either 0 or 1. If $a = 1$, then there is a *unique* cell $\alpha\colon [1] \to I$ such that γ factors through i_α, namely that where $f(\alpha(0)) \leq \gamma(0)$ and $f(\alpha(1)) \geq \gamma(1)$. Thus $(\mathbb{G}_{1/f})_\gamma = *$. On the other hand, if $a = 0$, then $(\mathbb{G}_{1/f})_\gamma$ is the full subcategory of $\mathbb{G}_{1/I}$ spanned by the inert maps $\alpha\colon [0] \to I$ such that $f(\alpha(0)) = \gamma(0)$ and $\alpha\colon [1] \to I$ such that $f(\alpha(0)) \leq \gamma(0)$ and $f(\alpha(1)) \geq \gamma(0)$. The nerve of this category is an iterated pushout of $\boldsymbol{\Delta}^1$'s along inclusions $\boldsymbol{\Delta}^0 \hookrightarrow \boldsymbol{\Delta}^1$, and so is weakly contractible, as required.

Now suppose the result is known for $n-1$. The cell C is either $[0]()$ or $[1](\tilde{C})$ where $\tilde{C}$ is a cell in $\boldsymbol{\Theta}_{n-1}$. If $C = [1](\tilde{C})$, then the underlying diagram in $\boldsymbol{\Delta}$ is unique and of the form

$$\begin{tikzcd} & {[1]} \arrow[r, "\alpha_0"] \arrow[d] & {[i]} \arrow[d, "f_0"] \\ {[1]} \arrow[r] \arrow[rr, bend right, "\gamma_0"'] & {[l]} \arrow[r] & {[j].} \end{tikzcd}$$

Thus C' must also be of the form $[1](\tilde{C}')$. Let $\tilde{J} := J_{\gamma_0(1)}$ where $J = [j](J_1, \ldots, J_j)$, let $\tilde{\gamma}$ denote the map $\tilde{C} \to \tilde{J}$ induced by γ, set $\tilde{I} := I_{\alpha_0(1)}$ where $I = [i](I_1, \ldots, I_i)$, and take $\tilde{f}$ to be the induced map $\tilde{I} \to \tilde{J}$. Then the category $(\mathbb{G}_{n/f})_\gamma$ can be identified with $(\mathbb{G}_{n-1/\tilde{f}})_{\tilde{\gamma}}$, which is weakly contractible by assumption.

If $C = [0]()$, consider the projection $(\mathbb{G}_{n/f})_\gamma \to (\mathbb{G}_{1/f_0})_{\gamma_0}$ given by taking the underlying maps in $\mathbf{\Delta}$. This is a Cartesian fibration, so using Lemma 2.8 again, it suffices to show that the fibres $(\mathbb{G}_{n/f})_{\gamma,\Xi}$ at $\Xi \in (\mathbb{G}_{1/f_0})_{\gamma_0}$ are weakly contractible. The object Ξ is a diagram

$$\begin{tikzcd} & {[a]} \arrow[r, "\alpha_0"] \arrow[d] & {[i]} \arrow[d, "f_0"] \\ {[0]} \arrow[r] \arrow[rr, bend right, "\gamma_0"'] & {[l]} \arrow[r] & {[j]}, \end{tikzcd}$$

where $a = 0$ or 1. If $a = 0$, then the fibre is $*$, and if $a = 1$ it may be identified with $\mathbb{G}_{n-1/I_{\alpha_0(1)}}$, which is again weakly contractible by assumption. $\square$

Applying Proposition 2.7 to the unique (active) map $f\colon J \to [0]()$ we get as a special case:

Corollary 2.9. *For each I in $\mathbf{\Theta}_n$, the category $\mathbb{G}_{n/I}$ is weakly contractible.* $\square$

Corollary 2.10. *Suppose $F \in \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ is a Segal object. Then for any active morphism $f\colon I \to J$, the natural map*

$$F(J) \to \lim_{\alpha \in \mathbb{G}^{op}_{n/I}} F(J_\alpha)$$

is an equivalence.

Proof. Using the Segal conditions for J_α we have

$$\lim_{\alpha \in \mathbb{G}^{op}_{n/I}} F(J_\alpha) \simeq \lim_{\alpha \in \mathbb{G}^{op}_{n/I}} \lim_{C \to J_\alpha \in \mathbb{G}^{\mathrm{op}}_{n/J_\alpha}} F(C).$$

By [Hau16, Corollary 5.7] we can rewrite this limit as $\lim_{C \in \mathbb{G}^{\mathrm{op}}_{n/f}} F(C)$, and by Proposition 2.7 this limit is equivalent to $\lim_{C \to J \in \mathbb{G}^{\mathrm{op}}_{n/J}} F(C)$, which we know by the Segal condition for J is equivalent to $F(J)$. $\square$

Proof of Proposition 2.5. First suppose $F\colon \mathbf{\Theta}_n^{\mathrm{op}} \to \mathcal{X}$ is a Segal object. Given $I = [i](I_1, \ldots, I_i)$ $(i \neq 0)$, set $\tilde{I} := [i](C_{n-1}, \ldots, C_{n-1})$ and let $f\colon \tilde{I} \to I$ denote the (active) map given by the identity $[i] \to [i]$ and the unique active maps $C_{n-1} \to I_p$. If Λ denotes the full subcategory of $\mathbb{G}_{n/\tilde{I}}$ containing the i maps $C_n \to \tilde{I}$ and the $i+1$ maps $C_0 \to \tilde{I}$, then the inclusion $\Lambda \hookrightarrow \mathbb{G}_{n/\tilde{I}}$ is cofinal. Together with Corollary 2.10 this gives an equivalence

$$F(I) \xrightarrow{\sim} \lim_{\alpha \in \mathbb{G}^{\mathrm{op}}_{n/\tilde{I}}} F(I_\alpha) \xrightarrow{\sim} F([1](I_1)) \times_{F(C_0)} \cdots \times_{F(C_0)} F([1](I_i)),$$

which is condition (1). Condition (2) holds since the functor $\mathbb{G}_{n-1/J} \to \mathbb{G}_{n/[1](J)}$ induced by $[1](-)\colon \mathbf{\Theta}_{n-1} \to \mathbf{\Theta}_n$ is cofinal.

Now suppose conditions (1) and (2) hold for F; we then wish to show that $F(I) \to \lim_{C \to I \in \mathbb{G}^{\mathrm{op}}_{n/I}} F(C)$ is an equivalence for any $I = [i](I_1, \ldots, I_i) \in \mathbf{\Theta}_n$. With $f\colon \tilde{I} \to I$ as above, Proposition 2.7 implies that it suffices to prove that the natural map $F(I) \to \lim_{\mathbb{G}^{\mathrm{op}}_{n/f}} F$ is an equivalence. Using [Hau16, Corollary 5.7] we can rewrite this as an iterated limit $\lim_{\alpha \in \mathbb{G}_{n/\bar{I}}} \lim_{\mathbb{G}_{n/I_\alpha}} F$. But now using the same cofinal functors as above, we can rewrite this again as

$$\left(\lim_{C \to I_1 \in \mathbb{G}_{n-1/I_1}} F([1](C)) \right) \times_{F(C_0)} \cdots \times_{F(C_0)} \left(\lim_{C \to I_i \in \mathbb{G}_{n-1/I_i}} F([1](C)) \right),$$

which is equivalent to $F(I)$ by (1) and (2). □

For the ∞-category $\mathcal{S}$ of spaces, $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{S})$ is the ∞-category underlying Rezk's model category of $\mathbf{\Theta}_n$-spaces from [Rez10]. More generally, if $\mathcal{X}$ is, say, an ∞-topos, the ∞-category $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{X})$ gives the (algebraic) ∞-category of internal (∞, n)-categories in $\mathcal{X}$. We would like to be able to iterate this definition, so that we get a good definition of Segal $\mathbf{\Theta}_m$-objects in $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{X})$. Just as in Barwick's definition of n-fold Segal spaces, this requires forcing some of the images to be constant; to formalize this notion, it is convenient to introduce the following technical definition:

Definition 2.11. A *presentable ∞-category with good constants* is a pair $(\mathcal{X}, \mathcal{U})$ consisting of an ∞-category $\mathcal{X}$ together with a full subcategory $\mathcal{U}$ satisfying the following requirements:

(a) $\mathcal{X}$ and $\mathcal{U}$ are both presentable.
(b) The inclusion $\mathcal{U} \hookrightarrow \mathcal{X}$ preserves all limits and colimits (and hence, by the adjoint functor theorem, has both a left and a right adjoint).
(c) Coproducts in $\mathcal{U}$ are disjoint, i.e. for any two objects $U, U' \in \mathcal{U}$, the commutative square

$$\begin{array}{ccc} \emptyset & \longrightarrow & U \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \amalg U' \end{array}$$

is Cartesian.
(d) Coproducts over $\mathcal{U}$ are universal, i.e. for any morphism $f \colon X \to U$ in $\mathcal{X}$ with $U \in \mathcal{U}$, the functor $f^* \colon \mathcal{X}_{/U} \to \mathcal{X}_{/X}$, given by pullback along f, preserves the initial object and arbitrary coproducts.

Example 2.12. If $\mathcal{X}$ is an ∞-topos, then $(\mathcal{X}, \mathcal{X})$ is a presentable ∞-category with good constants by [Lur09a, Theorem 6.1.0.6].

Remark 2.13. Since we are requiring pullbacks over $\mathcal{U}$ to preserve *all* coproducts in $\mathcal{X}$, not just coproducts in $\mathcal{U}$, a *distributor* in the sense of Lurie [Lur09b, Definition 1.2.1] is not necessarily a presentable ∞-category with good constants. However, the key examples — ∞-topoi and iterated $\mathbf{\Theta}_n$-objects in ∞-topoi — are both distributors *and* presentable ∞-categories with good constants.

Definition 2.14. Suppose $(\mathcal{X}, \mathcal{U})$ is a presentable ∞-category with good constants. We say a presheaf $X \in \mathcal{P}(\mathbb{G}_n; \mathcal{X})$ is *reduced* if $X(C_i)$ is in $\mathcal{U}$ for all $i < n$; we write $\mathcal{P}_{\mathrm{r}}(\mathbb{G}_n; \mathcal{X}, \mathcal{U})$ for the full subcategory of $\mathcal{P}(\mathbb{G}_n; \mathcal{X})$ spanned by the reduced objects. A Segal object X in $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$ or $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{X})$ is then called *reduced* if $X|_{\mathbb{G}_n^{\mathrm{op}}}$ is reduced; we write $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ and $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ for the full subcategories of $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ and $\mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$, respectively, spanned by the reduced Segal objects.

Proposition 2.15. *Suppose $(\mathcal{X}, \mathcal{U})$ is a presentable ∞-category with good constants.*

(i) The ∞-category $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ is presentable, and the inclusion

$$\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \hookrightarrow \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$$

admits a left adjoint L_n.

(ii) The functor $c^*\colon \mathcal{U} \to \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ that takes an object in $\mathcal{U}$ to the constant presheaf with that value is fully faithful and takes values in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$.

(iii) The pair $(\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}), \mathcal{U})$, with $\mathcal{U}$ viewed as the full subcategory of constant presheaves, is a presentable ∞-category with good constants.

Before we give the proof of this proposition, we need some technical lemmas:

Lemma 2.16. *Let $(\mathcal{X}, \mathcal{U})$ be a presentable ∞-category with good constants. Suppose given maps of sets $f\colon A \to B$ and $g\colon C \to B$, objects $X_a \in \mathcal{X}$ for $a \in A$, $Y_c \in \mathcal{X}$ for $c \in C$, $U_b \in \mathcal{U}$ for $b \in B$, and morphisms $\phi_a\colon X_a \to U_{f(a)}$ and $\psi_c\colon Y_c \to U_{g(c)}$ in $\mathcal{X}$ for all $a \in A$ and $c \in C$. Then the natural map*

$$\coprod_{(a,b,c)\in A\times_B C} X_a \times_{U_b} Y_c \to \left(\coprod_{a\in A} X_a\right) \times_{(\amalg_{b\in B} U_b)} \left(\coprod_{c\in C} Y_c\right)$$

is an equivalence in $\mathcal{X}$.

Proof. For $b \in B$, let A_b and C_b denote the fibres of f and g at b. Then condition (d) in Definition 2.11 gives equivalences

$$\coprod_{(a,b,c)\in A\times_B C} X_a \times_{U_b} Y_c \simeq \coprod_{b\in B} \coprod_{(a,c)\in A_b\times C_b} X_a \times_{U_b} Y_c \simeq \coprod_{b\in B} \left(\coprod_{a\in A_b} X_a\right) \times_{U_b} \left(\coprod_{c\in C_b} Y_c\right),$$

$$\left(\coprod_{a\in A} X_a\right) \times_{(\amalg_{b\in B} U_b)} \left(\coprod_{c\in C} Y_c\right) \simeq \coprod_{b',b''\in B} \left(\coprod_{a\in A_{b'}} X_a\right) \times_{(\amalg_{b\in B} U_b)} \left(\coprod_{c\in C_{b''}} Y_c\right).$$

Let $\tilde{X}_b := \amalg_{a\in A_b} X_a$ and $\tilde{Y}_b := \amalg_{c\in C_b} Y_c$; then it remains to show that

$$\tilde{X}_{b'} \times_{(\amalg_{b\in B} U_b)} \tilde{Y}_{b''} \simeq \begin{cases} \emptyset, & b' \neq b'', \\ \tilde{X}_{b'} \times_{U_{b'}} \tilde{Y}_{b'}, & b' = b''. \end{cases}$$

Since $\tilde{X}_{b'} \times_{(\amalg_{b\in B} U_b)} \tilde{Y}_{b''} \simeq \tilde{X}_{b'} \times_{U_{b'}} U_{b'} \times_{(\amalg_{b\in B} U_b)} U_{b''} \times_{U_{b''}} \tilde{Y}_{b''}$ and pullbacks over objects in $\mathcal{U}$ preserve the initial object, it is enough to show that

$$U_{b'} \times_{(\amalg_{b\in B} U_b)} U_{b''} \simeq \begin{cases} \emptyset, & b' \neq b'', \\ U_{b'}, & b' = b''. \end{cases}$$

To see this we observe that, setting $V := \coprod_{b\neq b'} U_b$, for $b' \neq b''$ we have

$$U_{b'} \times_{(\amalg_{b\in B} U_b)} U_{b''} \simeq U_{b'} \times_{U_{b'}\amalg V} V \times_V U_{b''} \simeq \emptyset \times_V U_{b''} \simeq \emptyset,$$

using that coproducts in $\mathcal{U}$ are disjoint and pullbacks in $\mathcal{U}$ preserve the initial object. For $i = j$ we have

$$U_{b'} \simeq U_{b'} \times_{U_{b'}\amalg V} (U_{b'} \amalg V) \simeq (U_{b'} \times_{U_{b'}\amalg V} U_{b'}) \amalg (U_{b'} \times_{U_{b'}\amalg V} V) \simeq U_{b'} \times_{U_{b'}\amalg V} U_{b'}. \qquad \square$$

Lemma 2.17. *Given a set S and objects $Y_i \in \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ for $i \in S$, the coproduct $Y := \coprod_{i\in S} Y_i$ in $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ is a reduced Segal $\mathbf{\Theta}_n$-object.*

Proof. Since $\mathcal{U}$ is closed under colimits in $\mathcal{X}$, the object Y is reduced. Applying Lemma 2.16 we see that for $I = [i](I_1, \ldots, I_i)$ the natural map

$$Y(I) \to Y([1](I_1)) \times_{Y(C_0)} \cdots \times_{Y(C_0)} Y([1](I_i))$$

is an equivalence. By Proposition 2.5 this implies by induction on n that Y is a Segal object. □

Proof of Proposition 2.15. The ∞-category $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ fits in a commutative diagram

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \arrow[r] \arrow[d] & \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n; \mathcal{X}) \arrow[d] \\
\mathcal{P}_{\mathrm{r}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X}) \arrow[d] \\
\mathcal{P}(\mathbb{G}_{n-1}; \mathcal{U}) \arrow[r] & \mathcal{P}(\mathbb{G}_{n-1}; \mathcal{X}),
\end{tikzcd}$$

where both squares are Cartesian. Moreover, the bottom horizontal and the two right vertical functors are right adjoints between presentable ∞-categories. By [Lur09a, Theorem 5.5.3.18] limits in the ∞-category Pr^{R} of presentable ∞-categories and right adjoints are computed in that of large ∞-categories, hence all ∞-categories in this diagram are presentable and all functors are right adjoints. This proves (i).

Since $\mathbf{\Theta}_n$ is weakly contractible (as it has a terminal object) the image of the constant presheaf functor $c^* \colon \mathcal{U} \to \mathcal{P}(\mathbf{\Theta}_n; \mathcal{U}) \to \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ is fully faithful. Constant presheaves on objects in $\mathcal{U}$ satisfy the Segal condition by Corollary 2.9, so this functor factors through $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$, which gives (ii).

For (iii), we already know conditions (a) and (c) in Definition 2.11. Limits in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ are computed in $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$, i.e. objectwise, and colimits are given by the localizations of the corresponding colimits in $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$; since constant presheaves on objects in $\mathcal{U}$ are already local, this implies condition (b). It remains to check condition (d), i.e. given maps $Y_i \to c^*U$ for $i \in S$ we need to show that the natural map

$$\coprod_i X \times_{c^*U} Y_i \to X \times_{c^*U} \coprod_i Y_i$$

is an equivalence. By Lemma 2.17 these coproducts can be computed in $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$, so it suffices to show that for $I \in \mathbf{\Theta}_n$ we have that

$$\coprod_i X(I) \times_U Y_i(I) \to X(I) \times_U \coprod_i Y_i(I)$$

is an equivalence, which is true since U is in $\mathcal{U}$. □

Definition 2.18. For $(\mathcal{X}, \mathcal{U})$ a presentable ∞-category with good constants, we write $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n \times \mathbf{\Theta}_m; \mathcal{X}, \mathcal{U})$ for the full subcategory of $\mathcal{P}(\mathbf{\Theta}_n \times \mathbf{\Theta}_m; \mathcal{X})$ corresponding to $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_m; \mathcal{X}, \mathcal{U}), \mathcal{U})$. Similarly, we define $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n_1} \times \cdots \times \mathbf{\Theta}_{n_k}; \mathcal{X}, \mathcal{U})$ and $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n_1,\mathrm{i}} \times \cdots \times \mathbf{\Theta}_{n_k,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ by induction.

Example 2.19. The ∞-category $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}^n; \mathcal{S})$ is the ∞-category of Barwick's *n-fold Segal spaces* [Bar05]. More generally, $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}^n; \mathcal{X}, \mathcal{U})$ gives Lurie's n-fold $\mathcal{U}$-Segal spaces from [Lur09b].

3. The free reduced Segal $\mathbf{\Theta}_n$-object monad

Our goal in this section is to show that the ∞-category $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ is the ∞-category of algebras for a monad on $\mathcal{P}_{\mathrm{r}}(\mathbb{G}_n; \mathcal{X}, \mathcal{U})$, and to understand this monad explicitly. This is closely related to the arguments used by Berger in the proof of [Ber02, Theorem 1.12]. Before we state our precise result, we must introduce some notation:

Definition 3.1. For $I \in \mathbf{\Theta}_n$, let $\mathrm{Act}(I)$ denote the set of active morphisms $I \to J$ in $\mathbf{\Theta}_n$. A morphism $f\colon I' \to I$ determines a map of sets $f^*\colon \mathrm{Act}(I) \to \mathrm{Act}(I')$ by taking $\phi\colon I \to J$ to the active morphism $\phi'\colon I' \to J'$ that gives the (unique) active-inert factorization of $I' \to I \to J$. Since this factorization is unique, it is easy to see that this determines a functor $\mathrm{Act}\colon \mathbf{\Theta}_n^{\mathrm{op}} \to \mathrm{Set}$.

Definition 3.2. Define $\iota_n\colon \mathbf{\Theta}_{n-1} \to \mathbf{\Theta}_n$ inductively by taking $\iota_1\colon * = \mathbf{\Theta}_0 \to \mathbf{\Theta}_1 = \mathbf{\Delta}$ to be the inclusion of $[0]$ and setting

$$\iota_n([m](I_1, \ldots, I_m)) = [m](\iota_{n-1}(I_1), \ldots, \iota_{n-1}(I_m)).$$

Notice that ι_n is fully faithful. We write $\iota_k^n := \iota_n \circ \cdots \circ \iota_{k+1}\colon \mathbf{\Theta}_k \to \mathbf{\Theta}_n$.

Proposition 3.3. *Let $(\mathcal{X}, \mathcal{U})$ be a presentable ∞-category with good constants.*

(i) The functor $i_n^*\colon \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \to \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ has a left adjoint F_n.

(ii) The adjunction $F_n \dashv i_n^*$ is monadic.

(iii) The monad $T_n := i_n^* F_n$ on $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ satisfies

$$T_n X(I) \simeq \coprod_{I \to J \in \mathrm{Act}(I)} X(J).$$

In particular,

$$T_n X(C_k) \simeq \coprod_{J \in \mathbf{\Theta}_k} X(\iota_k^n J).$$

The proof relies on a simple description of the left Kan extension functor $i_{n,!}$, which we prove first:

Lemma 3.4. *The functor $i_{n,!}\colon \mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}) \to \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ can be described explicitly as*

$$i_{n,!} F(I) \simeq \coprod_{I \to J \in \mathrm{Act}(I)} F(J).$$

In particular, $i_{n,!} F(C_k) \simeq \coprod_{I \in \mathrm{ob}\, \mathbf{\Theta}_k} F(\iota_k^n(I))$.

Proof. We first show that the inclusion $\mathrm{Act}(I) \to (\mathbf{\Theta}_{n,\mathrm{i}}^{\mathrm{op}})_{/I} := \mathbf{\Theta}_{n,\mathrm{i}}^{\mathrm{op}} \times_{\mathbf{\Theta}_n^{\mathrm{op}}} (\mathbf{\Theta}_n^{\mathrm{op}})_{/I}$ is cofinal. By [Lur09a, Theorem 4.1.3.1] this is equivalent to the category $(\mathrm{Act}(I))_{/X}$ being weakly contractible for each $X = (J, f\colon I \to J)$ in $(\mathbf{\Theta}_{n,\mathrm{i}})_{I/}$. But this category consists of active-inert factorizations of f, and so is contractible as this factorization is unique. Hence the left Kan extension $i_{n,!}F$ is indeed given by

$$i_{n,!} F(I) \simeq \operatorname*{colim}_{(I \to J) \in (\mathbf{\Theta}_{n,\mathrm{i}}^{\mathrm{op}})_{/I}} F(J) \simeq \coprod_{I \to J \in \mathrm{Act}(I)} F(J).$$

If $I = C_k$, then the only objects of $\mathbf{\Theta}_n$ that admit an active map from C_k are those in the image of the fully faithful functor $\iota_k^n\colon \mathbf{\Theta}_k \to \mathbf{\Theta}_n$ (and these active maps are unique), which gives the expression for $i_{n,!} F(C_i)$. $\square$

We need one more observation:

Lemma 3.5. *The functor* $\mathrm{Act}\colon \mathbf{\Theta}_n^{\mathrm{op}} \to \mathrm{Set}$ *is a Segal* $\mathbf{\Theta}_n$*-object.*

Proof. We prove this by induction on n, using the criterion of Proposition 2.5. For $I = [i](I_1, \ldots, I_i) \in \mathbf{\Theta}_n$, the definition of active morphisms in $\mathbf{\Theta}_n$ immediately implies that $\mathrm{Act}(I) \cong \mathrm{Act}([1](I_1)) \times \cdots \times \mathrm{Act}([1](I_i))$ and $\mathrm{Act}(C_0) \cong *$, which gives condition (1). To prove (2), suppose $I = [1](J)$ for some $J \in \mathbf{\Theta}_{n-1}$. Then it is immediate from the definition of active maps in $\mathbf{\Theta}_n$ that

$$\mathrm{Act}(I) \cong \coprod_{i=0}^{\infty} \mathrm{Act}'(J)^{\times i}$$

(where for clarity we write Act' for the $\mathbf{\Theta}_{n-1}$-version of Act). By assumption we have $\mathrm{Act}'(J) \cong \lim_{C \to J \in \mathbb{G}^{\mathrm{op}}_{n-1/J}} \mathrm{Act}'(C)$, hence as limits commute and coproducts in Set commute with connected limits, we have isomorphisms

$$\begin{aligned}\mathrm{Act}(I) &\cong \coprod_{i=0}^{\infty} \left(\lim_{C \to J \in \mathbb{G}^{\mathrm{op}}_{n-1/J}} \mathrm{Act}'(C) \right)^{\times i} \cong \lim_{C \to J \in \mathbb{G}^{\mathrm{op}}_{n-1/J}} \left(\coprod_{i=0}^{\infty} \mathrm{Act}'(C)^{\times i} \right) \\ &\cong \lim_{C \to J \in \mathbb{G}^{\mathrm{op}}_{n-1/J}} \mathrm{Act}([1](C)),\end{aligned}$$

which is condition (2). □

Proof of Proposition 3.3. Let L_n denote the localization functor from $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ to $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$; then $L_n i_{n,!}$ clearly restricts to a left adjoint to i_n^*, which gives (i).

To see that the adjunction is monadic it suffices by [Lur14, Theorem 4.7.4.5] to prove that i_n^* detects equivalences and that colimits of i_n^*-split simplicial objects exist in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ and are preserved by i_n^*. Since $\mathbf{\Theta}_{n,\mathrm{i}}$ is a subcategory of $\mathbf{\Theta}_n$ containing all the objects it is clear that i_n^* detects equivalences. Suppose we have an i_n^*-split simplicial object $X_\bullet$ in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$, i.e. $i_n^* X_\bullet$ extends to a split simplicial object $X'_\bullet\colon \mathbf{\Delta}^{\mathrm{op}}_{-\infty} \to \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ (where $\mathbf{\Delta}_{-\infty}$ is as in [Lur14, Definition 4.7.3.1]). If we consider $X_\bullet$ as a diagram in $\mathcal{P}(\mathbf{\Theta}_n; \mathcal{X})$ with colimit X, then this colimit is preserved by $i_n^*\colon \mathcal{P}(\mathbf{\Theta}_n; \mathcal{X}) \to \mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$ (since this functor is a left adjoint). But by [Lur14, Remark 4.7.3.3], the diagram $X'_\bullet$ is a colimit diagram also when viewed as a diagram in $\mathcal{P}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X})$, so $i_n^* X \simeq X'_{-\infty}$. This means that X is a reduced Segal $\mathbf{\Theta}_n$-object, and so it is also the colimit of $X_\bullet$ in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$, and its image in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ is $X'_{-\infty}$, as required. This proves (ii).

To prove (iii), we will show that if $X \in \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$, then $i_{n,!}X$ is a reduced Segal $\mathbf{\Theta}_n$-space, hence $F_n X$ is just given by the left Kan extension $i_{n,!}X$:

To see that $i_{n,!}$ is reduced, we observe that for $i < n$ the expression for $i_{n,!}F(C_i)$ in Lemma 3.4 is a coproduct of limits of objects in $\mathcal{U}$, and hence is also in $\mathcal{U}$ since this is closed in $\mathcal{X}$ under all limits and colimits.

Now since X is a Segal $\mathbf{\Theta}_{n,\mathrm{i}}$-object we have, using Proposition 2.7,

$$i_{n,!}X(I) \simeq \coprod_{I \to J \in \mathrm{Act}(I)} X(J) \simeq \coprod_{I \to J \in \mathrm{Act}(I)} \lim_{\alpha \in \mathbb{G}_{n/I}} X(J_\alpha).$$

These limits over $\mathbb{G}_{n/I}$ can be rewritten as iterated pullbacks over objects in $\mathcal{U}$, and by Lemma 3.5 we have that $\mathrm{Act}(I)$ is equivalent to $\lim_{\alpha\colon C \to I \in \mathbb{G}_{n/I}} \mathrm{Act}(C)$.

Applying Lemma 2.16 iteratively we can then conclude that the natural map

$$\coprod_{I\to J\in \mathrm{Act}(I)} \lim_{\alpha\in \mathbb{G}_{n/I}} X(J_\alpha) \to \lim_{\alpha\colon C\to I\in \mathbb{G}_{n/I}} \coprod_{C\to J_\alpha} X(J_\alpha)$$

is an equivalence. Here the target is equivalent to $\lim_{\alpha\colon C\to I\in\mathbb{G}_{n/I}} i_{n,!}X(C)$, i.e. $i_{n,!}X$ satisfies the Segal condition. The expression for $F_nX(C_i)$ is then immediate from Lemma 3.4. □

4. Comparison

Our goal in this section is to prove our comparison result. More precisely, we will show:

Theorem 4.1. *Let $\tau_{1,n}\colon \mathbf{\Delta}\times\mathbf{\Theta}_n\to\mathbf{\Theta}_{n+1}$ be the functor determined by sending $([n],I)$ to $[n](I,\ldots,I)$. Then composition with $\tau_{1,n}$ induces, for $(\mathcal{X},\mathcal{U})$ a presentable ∞-category with good constants, an equivalence*

$$\tau_{1,n}^*\colon \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n+1};\mathcal{X},\mathcal{U}) \xrightarrow{\sim} \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}\times\mathbf{\Theta}_n;\mathcal{X},\mathcal{U}).$$

Iterating this result, we get:

Corollary 4.2. *Let $\tau_{k,n}\colon \mathbf{\Delta}^k\times\mathbf{\Theta}_n\to\mathbf{\Theta}_{n+k}$ be defined inductively as*

$$\mathbf{\Delta}^k\times\mathbf{\Theta}_n \xrightarrow{\mathrm{id}_{\mathbf{\Delta}}\times\tau_{k-1,n}} \mathbf{\Delta}\times\mathbf{\Theta}_{n+k-1} \xrightarrow{\tau_{1,n+k-1}} \mathbf{\Theta}_{n+k}.$$

Then for $(\mathcal{X},\mathcal{U})$ any presentable ∞-category with good constants the functor

$$\tau_{k,n}^*\colon \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}^k\times\mathbf{\Theta}_n;\mathcal{X},\mathcal{U}) \to \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n+k};\mathcal{X},\mathcal{U})$$

is an equivalence. □

In particular, taking $\mathcal{X}$ to be an ∞-topos and $n=0$ we get an equivalence between the ∞-category $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}^k;\mathcal{X})$ of k-fold Segal spaces in $\mathcal{X}$ and the ∞-category $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_k;\mathcal{X})$ of Segal $\mathbf{\Theta}_k$-objects in $\mathcal{X}$.

Remark 4.3. Similarly, applying Theorem 4.1 inductively we get for any sequence of positive integers $(n_1,\ldots,n_k)$ an equivalence between $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n_1}\times\cdots\times\mathbf{\Theta}_{n_k};\mathcal{X},\mathcal{U})$ and $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n_1+\cdots+n_k};\mathcal{X},\mathcal{U})$.

To prove Theorem 4.1, we will use the following analogue of Proposition 3.3:

Proposition 4.4.

(i) Let $i_{1,n} := i_1\times i_n\colon \mathbf{\Delta}_{\mathrm{i}}\times\mathbf{\Theta}_{n,\mathrm{i}}\to\mathbf{\Delta}\times\mathbf{\Theta}_n$. The functor

$$i_{1,n}^*\colon \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}\times\mathbf{\Theta}_n;\mathcal{X},\mathcal{U})\to\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}_{\mathrm{i}}\times\mathbf{\Theta}_{n,\mathrm{i}};\mathcal{X},\mathcal{U})$$

has a left adjoint $F_{1,n}$.

(ii) The adjunction $F_{1,n}\dashv i_{1,n}^*$ is monadic.

(iii) The monad $T_{1,n} := i_{1,n}^*F_{1,n}$ on $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}_{\mathrm{i}}^n\times\mathbf{\Theta}_{n,\mathrm{i}};\mathcal{X},\mathcal{U})$ satisfies

$$T_{1,n}X([0],C_0)\simeq X([0],C_0),$$

$$T_{1,n}X([1],C_k)\simeq \coprod_{j=0}^{\infty} F_n\tilde{X}(C_k)\times_{X([0],C_0)}\cdots\times_{X([0],C_0)} F_n\tilde{X}(C_k),$$

where $\tilde{X} := X([1],–)$ and the factor $F_n\tilde{X}(C_k)$ occurs j times.

For the proof we need the following observations:

Lemma 4.5. *Suppose $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is an adjunction. Then for any $d \in \mathcal{D}$ there is an adjunction*

$$L_d : \mathcal{C}_{/Rd} \rightleftarrows \mathcal{D}_{/d} : R_d,$$

where $L_d(x \to Rd)$ is the composite $Lx \to LRd \to d$ using the counit, and $R_d(y \to d)$ is $Ry \to Rd$.

Proof. Let $\eta\colon LR \to \mathrm{id}$ be the counit for the adjunction. This determines a natural transformation $\eta_d\colon L_dR_d \to \mathrm{id}$, and the map

$$\mathrm{Map}_{\mathcal{C}_{/Rd}}(x, R_dy) \to \mathrm{Map}_{\mathcal{D}_{/LRd}}(L_dx, L_dR_dy) \to \mathrm{Map}_{\mathcal{D}_{/d}}(L_dx, y)$$

is the map on fibres at $x \to Rd$ of the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(x, Ry) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(Lx, y) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(x, Rd) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(Lx, d)
\end{tikzcd}$$

induced by η. Here both horizontal maps are equivalences, since η is the counit of the adjunction $L \dashv R$, hence so is the map on fibres. The natural transformation η_d is therefore the counit of an adjunction $L_d \dashv R_d$ by [Lur09a, Proposition 5.2.2.8]. ☐

Lemma 4.6. *If $\mathcal{C}$ is an ∞-category with finite products, then for all $x \in \mathcal{C}$ there is a pullback square*

$$\begin{tikzcd}
\mathcal{C}_{/x\times x} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathbb{G}_1^{\mathrm{op}}, \mathcal{C}) \arrow[d] \\
\{x\} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

where the right vertical map is given by evaluation at C_0.

Proof. Let Λ denote the category $0 \to \infty \leftarrow 1$, i.e. $\{0,1\}^{\triangleright}$. Then for $x, y \in \mathcal{C}$ we have by [Hau14, Lemma 8.4] a pullback square

$$\begin{tikzcd}
\mathcal{C}_{/x\times y} \arrow[r] \arrow[d] & \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{C}) \arrow[d] \\
\{x, y\} \arrow[r] & \mathcal{C} \times \mathcal{C},
\end{tikzcd}$$

where the right vertical map is given by evaluation at 0 and 1. Now $\mathbb{G}_1$ can be identified with the pushout $\Lambda \amalg_{\{0,1\}} *$, so we also have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(\mathbb{G}_1^{\mathrm{op}}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{C}) \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{C} \times \mathcal{C},
\end{tikzcd}$$

where the bottom horizontal map is the diagonal map. Putting these two squares together gives the result. ☐

Proof of Proposition 4.4. The functor $i_{1,n}\colon \mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_{n,\mathrm{i}} \to \mathbf{\Delta} \times \mathbf{\Theta}_n$ factors as the composite of inclusions $i'_{1,n} := \mathrm{id} \times i_n\colon \mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_{n,\mathrm{i}} \to \mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_n$ and $i''_{1,n} := i_1 \times \mathrm{id}\colon \mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_n \to \mathbf{\Delta} \times \mathbf{\Theta}_n$. Here $(i''_{1,n})^*$ is just i_1^* applied to the presentable ∞-category with good constants $(\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}), \mathcal{U})$, so by Proposition 3.3 it has a left adjoint, given by $i_{1,!}$. To prove (i) it therefore suffices to show that $(i'_{1,n})^*$ has a left adjoint.

Let $\mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ denote the full subcategory of $\mathcal{P}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ spanned by those presheaves X such that $X(C_0)$ is a constant presheaf valued in $\mathcal{U}$ and $X(C_1)$ is in $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$. Then right Kan extension along $\gamma_1 \times \mathrm{id}$ gives an equivalence between $\mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ and $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$. Similarly, $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ is equivalent to the analogous full subcategory $\mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ of $\mathcal{P}(\mathbb{G}_1 \times \mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})$. Under these equivalences $(i'_{1,n})^*$ corresponds to the functor j^* where $j := \mathrm{id}_{\mathbb{G}_1} \times i_n\colon \mathbb{G}_1 \times \mathbf{\Theta}_{n,\mathrm{i}} \to \mathbb{G}_1 \times \mathbf{\Theta}_n$.

We then have a commutative triangle

$$\begin{tikzcd} \mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \arrow[rr, "j^*"] \arrow[dr] & & \mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U}) \arrow[dl] \\ & \mathcal{U}, & \end{tikzcd}$$

where the diagonal morphisms are given by evaluation at C_0, since this takes values in the constant presheaves on $\mathbf{\Theta}_n$ valued in $\mathcal{U}$.

By Lemma 4.6 we can identify the morphism on fibres at $U \in \mathcal{U}$ with the functor

$$\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})_{/U \times U} \to \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n,\mathrm{i}}; \mathcal{X}, \mathcal{U})_{/U \times U}$$

given by composing with i_n, where $U \times U$ denotes the constant presheaf with this value. Lemma 4.5 therefore implies that the functor j^* has a left adjoint on the fibre over each $U \in \mathcal{U}$, given by applying F_n and composing with the counit map to the constant presheaf. By [Lur14, Proposition 7.3.2.6] this implies that $(i'_{1,n})^*$ has a left adjoint, provided the two diagonal functors are Cartesian fibrations and j^* preserves Cartesian morphisms.

The functor $\mathrm{ev}_{C_0}\colon \mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U}) \to \mathcal{U}$ has a right adjoint R, taking $U \in \mathcal{U}$ to the right Kan extension along $\{C_0\} \hookrightarrow \mathbb{G}_1$ of the constant presheaf on U. (Thus $R(U)(C_1)$ is the constant presheaf on $U \times U$.) To prove that ev_{C_0} is a Cartesian fibration we can therefore apply the criterion of [Hau17, Corollary 4.52]: We must show that given $X \in \mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ and a morphism $f\colon U \to X(C_0)$ in $\mathcal{U}$, if we define X' as the pullback of $X \to R(X(C_0))$ along $R(U) \to R(X(C_0))$, then the morphism $X'(C_0) \to R(U)(C_0) \simeq U$ is an equivalence; this is clear since pullbacks in $\mathcal{P}_{\mathrm{rSeg}}(\mathbb{G}_1 \times \mathbf{\Theta}_n; \mathcal{X}, \mathcal{U})$ are computed objectwise. The map $X' \to X$ is a Cartesian morphism over f with target X; the same argument shows that the other functor is likewise a Cartesian fibration, and j^* preserves Cartesian morphisms as it preserves pullbacks. This completes the proof of (i).

(ii) now follows by the same argument as in the proof of Proposition 3.3(ii), and (iii) by Proposition 3.3 and our description of the left adjoints to $(i'_{1,n})^*$ and $(i''_{1,n})^*$. $\square$

Proof of Theorem 4.1. The functor $\tau_{1,n}^*$ takes reduced Segal $\mathbf{\Theta}_{n+1}$-objects to reduced Segal $\mathbf{\Delta} \times \mathbf{\Theta}_n$-objects: condition (1) in Proposition 2.5 implies the Segal condition in the $\mathbf{\Delta}$-coordinate, and condition (2) implies it in the $\mathbf{\Theta}_n$-coordinate; since $\tau_{1,n}([0], –)$ is constant at $[0]()$, we see that $\tau_{1,n}^*$ also preserves reducedness.

We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n+1};\mathcal{X},\mathcal{U}) \arrow[r, "\tau_{1,n}^*"] \arrow[d, "i_{n+1}^*"] & \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}\times\mathbf{\Theta}_n;\mathcal{X},\mathcal{U}) \arrow[d, "i_{1,n}^*"] \\
\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n+1,\mathrm{i}};\mathcal{X},\mathcal{U}) \arrow[r, "\tau_{1,n,\mathrm{i}}^*"] \arrow[d, "\gamma_{n+1}^*"] & \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}_{\mathrm{i}}\times\mathbf{\Theta}_{n,\mathrm{i}};\mathcal{X},\mathcal{U}) \arrow[d, "\gamma_{1,n}^*"] \\
\mathcal{P}_{\mathrm{r}}(\mathbb{G}_{n+1};\mathcal{X},\mathcal{U}) \arrow[r, "\beta_n^*"] & \mathcal{P}_{\mathrm{r}}(\mathbb{G}_1\times\mathbb{G}_{n+1};\mathcal{X},\mathcal{U}),
\end{tikzcd}$$

where $\gamma_{1,n} := \gamma_1 \times \gamma_n$ and β is the restriction of $\tau_{1,n}$ to a functor : $\mathbb{G}_1 \times \mathbb{G}_n \to \mathbb{G}_{n+1}$ (this sends (C_0, C_i) to C_0 and (C_1, C_i) to C_{i+1}).

We will now prove that the functor $\tau_{1,n,\mathrm{i}}^*$ is an equivalence. Observe that the vertical morphisms in the bottom square are equivalences, since the Segal presheaves on $\mathbf{\Theta}_{n+1,\mathrm{i}}$ and $\mathbf{\Delta}_{\mathrm{i}} \times \mathbf{\Theta}_{n,\mathrm{i}}$ are precisely those presheaves that are right Kan extensions along γ_{n+1} and $\gamma_1 \times \gamma_n$ of presheaves on $\mathbb{G}_{n+1}$ and $\mathbb{G}_1 \times \mathbb{G}_n$, respectively. It therefore suffices to show that the functor β_n^* is an equivalence. The reduced presheaves on $\mathbb{G}_1 \times \mathbb{G}_{n+1}$ are precisely those that are in the image under β_n^* of $\mathcal{P}_{\mathrm{r}}(\mathbb{G}_{n+1}; \mathcal{X}, \mathcal{U})$, so to see this it is enough to prove that β_n^* is fully faithful. Consider the left adjoint $\beta_{n,!} \colon \mathcal{P}(\mathbb{G}_1 \times \mathbb{G}_n; \mathcal{X}) \to \mathcal{P}(\mathbb{G}_{n+1}; \mathcal{X})$, given by left Kan extension along $\beta^{\mathrm{op}} \colon \mathbb{G}_1^{\mathrm{op}} \times \mathbb{G}_n^{\mathrm{op}} \to \mathbb{G}_{n+1}^{\mathrm{op}}$. The category $(\mathbb{G}_1^{\mathrm{op}} \times \mathbb{G}_n^{\mathrm{op}})_{/C_k}$ has a terminal object for every $k = 0, \ldots, n+1$, namely $((C_1, C_{i-1}), \beta(C_1, C_{i-1}) \xrightarrow{\mathrm{id}} C_i)$ for $i > 0$ and $(C_0, C_0), \beta(C_0, C_0) \xrightarrow{\mathrm{id}} C_0)$ for $i = 0$. Thus for $F \in \mathcal{P}(\mathbb{G}_1 \times \mathbb{G}_n; \mathcal{X})$ we have

$$\beta_! F(C_i) \simeq \begin{cases} F(C_1, C_{i-1}), & i > 0, \\ F(C_0, C_0), & i = 0. \end{cases}$$

The counit $\beta_! \beta^* \to \mathrm{id}$ is therefore an equivalence, which implies that β^* is fully faithful, as required.

The vertical maps in the top square above are monadic right adjoints by Proposition 3.3 and Proposition 4.4. To see that $\tau_{1,n}^*$ is an equivalence it then suffices, by [Lur14, Corollary 4.7.4.16], to show that for every $X \in \mathcal{P}_{\mathrm{r}}(\mathbb{G}_{n+1}; \mathcal{X}, \mathcal{U}) \simeq \mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Theta}_{n+1,\mathrm{i}}; \mathcal{X}, \mathcal{U})$ the unit map $X \to i_{n+1}^* F_{n+1} \simeq i_{1,n}^* \tau_{1,n}^* F_{n+1}$ induces an equivalence $F_{1,n} X \xrightarrow{\sim} \tau_{1,n}^* F_{n+1} X$, or (since $i_{1,n}^*$ detects equivalences) the induced map $(F_{1,n} X)(C_k) \to (F_{n+1} X)(C_k)$ is an equivalence for $k = 0, \ldots, n+1$.

To prove this we will rewrite our expression for $(F_{n+1} X)(C_k)$ from Proposition 3.3, which says

$$F_{n+1} X(C_k) \simeq \coprod_{I \in \mathrm{ob}\, \mathbf{\Theta}_k} \iota_k^{n+1,*} X(I).$$

Let $(\mathrm{ob}\, \mathbf{\Theta}_k)_i$ denote the subset of $\mathrm{ob}\, \mathbf{\Theta}_k$ consisting of objects of the form $[i](\cdots)$.

By Proposition 2.5 we get for every object $I = [i](I_1, \ldots, I_i)$ in $\mathbf{\Theta}_k$ an equivalence

$$\iota_k^{n+1,*} X(I) \simeq \iota_k^{n+1,*} X(\sigma_k I_1) \times_{\iota_k^{n+1,*} X(C_0)} \cdots \times_{\iota_k^{n+1,*} X(C_0)} \iota_k^{n+1,*} X(\sigma_k I_i),$$

where $\sigma_k \colon \mathbf{\Theta}_{k-1} \to \mathbf{\Theta}_k$ is the functor $[1](\text{–})$. There is a bijection $(\mathrm{ob}\, \mathbf{\Theta}_k)_i \cong (\mathrm{ob}\, \mathbf{\Theta}_{k-1})^{\times(i-1)}$, and since coproducts over $\mathcal{U}$ are universal we can rewrite our expression for $F_n X(C_k)$ as

$$\coprod_{i=0}^{\infty} \left(\coprod_{I_1 \in \mathbf{\Theta}_{k-1}} \iota_k^{n+1,*} X(\sigma_k I_1) \right) \times_{X(C_0)} \cdots \times_{X(C_0)} \left(\coprod_{I_i \in \mathbf{\Theta}_{k-1}} \iota_k^{n+1,*} X(\sigma_k I_i) \right).$$

Here, as $\iota_k^{n+1}\sigma_k = \sigma_{n+1}\iota_{k-1}^n$, we have equivalences

$$\coprod_{I' \in \mathbf{\Theta}_{k-1}} \iota_k^{n+1,*}X(\sigma_k I') \simeq \coprod_{I' \in \mathbf{\Theta}_{k-1}} \iota_{k-1}^{n,*}(\sigma_{n+1}^* X)(I') \simeq F_n(\sigma_{n+1}^* X)(C_{k-1}).$$

Comparing this to the expression for $F_{1,n}$ in Proposition 4.4 then completes the proof. $\square$

Remark 4.7. Let E^n denote the nerve of the (contractible) category with n objects and a unique morphism between any two objects, viewed as a Segal space. Then a Segal space is *complete* if it is local with respect to the map $E^1 \to E^0$. We can then inductively define a Segal $\mathbf{\Delta} \times \mathbf{\Theta}_n$-space X to be complete if $X(-, C_0)$ is a complete Segal space and $X([1], -)$ is a complete Segal $\mathbf{\Theta}_n$-space, where a Segal $\mathbf{\Theta}_n$-space Y is complete if $\tau_{1,n-1}^* Y$ is a complete $\mathbf{\Delta} \times \mathbf{\Theta}_{n-1}$-space. Expanding this out, it is easy to see that it recovers Rezk's definition of complete $\mathbf{\Theta}_n$-spaces, and that under our equivalence $\mathcal{P}_{\mathrm{rSeg}}(\mathbf{\Delta}^n) \simeq \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Theta}_n)$ the complete $\mathbf{\Theta}_n$-spaces correspond to the complete n-fold Segal spaces as defined by Barwick.

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