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ASYMPTOTIC LIPSCHITZ REGULARITY OF VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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Abstract. For each continuous initial data $\varphi(x) \in C(M,\mathbb{R})$, we obtain the asymptotic Lipschitz regularity of the viscosity solution of the following evolutionary Hamilton-Jacobi equation with convex and coercive Hamiltonians:

$$\begin{cases} \partial_t u(x,t) + H(x,\partial_x u(x,t)) = 0, \\ u(x,0) = \varphi(x). \end{cases}$$

1. Introduction and main result

Let M be an n-dimensional connected and closed smooth manifold. We are concerned with a Hamiltonian $H : T^*M \to \mathbb{R}$ satisfying the following assumptions:

(H1) Smoothness: $H(x,p)$ is a C^2 function;
(H2) Convexity: $H(x,p)$ is strictly convex with respect to p;
(H3) Coercivity: for each $x \in M$, $H(x,p) \to \infty$ uniformly as $|p| \to \infty$.

(H3) is equivalent to the topological statement that for each $c \in \mathbb{R}$, the set $\{(x,p) \in T^*M | x \in K, H(x,p) \leq c\}$ is compact.

We consider the following Hamilton-Jacobi equation under the assumptions (H1)-(H3):

$$\begin{cases} \partial_t u(x,t) + H(x,\partial_x u(x,t)) = 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

where $(x,t) \in M \times [0,\infty)$ and $\varphi(x) \in C(M,\mathbb{R})$.

We recall the Mañé critical value of $H(x,p)$ denoted by $c[0]$. By [3], one has

$$c[0] = \inf_{u \in C^1(M,\mathbb{R})} \max_{x \in M} H(x,\partial_x u). \tag{1.2}$$

Let $u(x,t)$ be the viscosity solution of (1.1). It was shown by [4] that the limit $v(x) := \lim_{t\to\infty}(u(x,t) + c[0]t)$ is a Lipschitz weak KAM solution of

$$H(x,\partial_x u) = c[0]. \tag{1.3}$$

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Recently, a convergence result for more general contact Hamilton-Jacobi equations was established in [7]. Note that the limit $v(x)$ is a Lipschitz function, while the initial data $\varphi(x)$ is only continuous. A question is:

When does the Lipschitz regularity of the viscosity solution of (1.1) *emerge?*

If $H(x,p)$ is superlinear with respect to p, then the Lipschitz regularity emerges after an arbitrarily small time, which is basically from the celebrated Fleming's lemma [5, Theorem 4.4.3]. Unfortunately, if $H(x,p)$ is coercive, the Fleming's lemma does not hold anymore. Then it is natural to ask

Will the Lipschitz regularity of the viscosity solution of (1.1) *emerge after a finite time (asymptotic Lipschitz regularity) or an infinite time (limit Lipschitz regularity)?*

In this note, we clarify that the asymptotic Lipschitz regularity of the viscosity solution of (1.1) is true. More precisely, we have:

Theorem 1.1. *Let $u(x,t)$ be a viscosity solution of* (1.1) *with continuous initial data $\varphi \in C(M,\mathbb{R})$. Then there exists $t_0 > 0$ such that for $t > t_0$, $u(x,t)$ is ι-Lipschitz continuous, where $t_0, \iota = \iota(t_0)$ are independent of φ.*

This note is outlined as follows. In Section 2, some properties of viscosity solutions are introduced as preliminaries. In Section 3, by introducing a modified Hamiltonian, the Mañé critical value and action minimizing orbits are located. The proof of Theorem 1.1 is completed in Section 4.

2. Preliminaries

In this section, we introduce some properties of the viscosity solutions in our settings. First of all, we introduce the notion of semiconcave functions.

Definition 2.1 (Semiconcavity on $\mathbb{R}^n$)**.** Let U be an open convex subset of $\mathbb{R}^n$ and let $u : U \to \mathbb{R}$ be a function. u is called a semiconcave function with linear modulus if there exists a finite constant K and for each $x \in U$ there exists a linear form $\theta_x : \mathbb{R}^n \to \mathbb{R}$ such that for any $y \in U$,

$$u(y) - u(x) \leq \theta_x(y-x) + K|y-x|^2. \tag{2.1}$$

For the sake of simplicity, we only consider the semiconcave functions with linear modulus defined as above. See [2] for a more general definition. In this context, the notion "semiconcave" means "semiconcave with a linear modulus".

Definition 2.2 (Semiconcavity on a manifold)**.** A function $u : M \to \mathbb{R}$ defined on the C^r ($r \geq 2$) differential k-dimensional manifold M is locally semiconcave if for each $x \in M$ there exists a C^r ($r \geq 2$) coordinate chart $\psi : U \to \mathbb{R}^n$ with $x \in U$ such that $u \circ \psi^{-1} : U \to \mathbb{R}$ is semiconcave.

Consider the stationary equation

$$H(x, \partial_x u) = 0 \tag{2.2}$$

and the evolutionary equation

$$\partial_t u + H(x, \partial_x u) = 0. \tag{2.3}$$

Based on [2, Theorem 5.3.1. and Theorem 5.3.6], we have the following results.

Proposition 2.3. *Letting $H \in C^2(T^*M, \mathbb{R})$, we have the following properties.*

(a) *Let u be a semiconcave function satisfying the equations* (2.2) *(resp.* (2.3)*) almost everywhere. If $H(x,p)$ is convex with respect to p, then u is a viscosity solution of the equations* (2.2) *(resp.* (2.3)*).*

(b) *Let u be a Lipschitz viscosity solution of the equations* (2.2) *(resp.* (2.3)*). If $H(x,p)$ is strictly convex with respect to p, then u is locally semiconcave on M (resp. $M \times (0, +\infty)$).*

Let us recall the notion of upper differentials (see [2, 5] for instance).

Definition 2.4 (Upper differential on $\mathbb{R}^n$)**.** Let $u : U \to \mathbb{R}$ be a function defined on the open subset U of $\mathbb{R}^n$. The set

$$D^+u(x_0) := \left\{ \theta \in \mathbb{R}^n \mid \limsup_{x \to x_0} \frac{u(x) - u(x_0) - \theta(x - x_0)}{|x - x_0|} \leq 0 \right\}$$

is called an upper differential of u at x_0.

Definition 2.5 (Upper differential on a manifold)**.** Let $u : M \to \mathbb{R}$ be a function defined on the manifold M. The linear form $\theta \in T^*_{x_0}M$ is an upper differential of u at $x_0 \in M$ if there exist a neighborhood V of x_0 and a function $\varphi : V \to \mathbb{R}$, diffferentiable at x_0, with $\varphi(x_0) = u(x_0)$ and $d_{x_0}\varphi = \theta$ and such that $\varphi(x) \geq u(x)$ for each $x \in V$.

It is easy to verify the equivalence between the definition of upper differentials on a Euclidean space and the one on a manifold.

We use $\partial u(x_0, \theta)$ to denote a one-sided directional derivative along $\theta \in \mathbb{R}^n$ at x_0, namely

$$\partial u(x_0, \theta) := \lim_{h \to 0^+} \frac{u(x_0 + h\theta) - u(x_0)}{h}.$$

The upper differential and one-sided directional derivative of the semiconcave function enjoy the following properties ([2, Proposition 3.3.4 and Theorem 3.3.6]).

Proposition 2.6. *Let $u : M \to \mathbb{R}$ be a semiconcave function. Then the following properties hold true.*

(a) *$D^+u(x) \neq \emptyset$ for any $x \in M$.*

(b) *If $\{x_n\}$ is a sequence in M converging to x and if $p_n \in D^+u(x_n)$ converges to a vector p, then $p \in D^+u(x)$.*

(c) *$\partial u(x, \theta) = \min_{p \in D^+u(x)} \langle p, \theta \rangle$ for any $x \in M$ and $\theta \in \mathbb{R}^n$.*

Throughout this paper, we shall use $|\cdot|$ to denote the Euclidean norm, that is, $|\alpha| = \sqrt{\alpha_1^2 + \ldots + \alpha_i^2}$ for given $\alpha = (\alpha_1, \ldots, \alpha_i) \in \mathbb{R}^i$, $i = 1$ or $i = n$.

3. Mañé critical value and action minimizing orbits

3.1. **Modification of the Hamiltonian.** Let $H(x,p)$ be a Hamiltonian satisfying (H1)-(H3). We construct a new Hamiltonian denoted by $H_R(x,p)$ with $R > 1$ as follows. Without loss of generality, we assume $M = \mathbb{T}^n$, from which $T^*M = \mathbb{T}^n \times \mathbb{R}^n$,

$$H_R(x,p) = \alpha_R(p)H(x,p) + \mu_R\beta(|p|^2 - R^2), \tag{3.1}$$

where μ_R is a constant determined by (3.4) below and $\alpha_R(p)$ is a C^2 function satisfying

$$\alpha_R(p) = \begin{cases} 1, & |p| \le R+1, \\ 0, & |p| > R+2. \end{cases} \tag{3.2}$$

Without loss of generality, one can require $|\alpha'_R(p)| < 2$ and $\|\alpha''_R(p)\|_1 < 2$, where $\|\cdot\|_1$ denotes 1-norm, namely the maximum of the summation of the absolute values of elements in each column. $\beta(z)$ is defined as

$$\beta(z) = \begin{cases} 0, & |z| \le 0, \\ z^4, & |z| > 0. \end{cases} \tag{3.3}$$

It is easy to see that $H_R(x,p) = H(x,p)$ for $|p| \le R$. In the following, we show that $H_R(x,p)$ satisfies (H1), (H2) and superlinearity.

Claim 1. $H_R(x,p)$ satisfies (H1).

Proof of Claim 1. Note that $\alpha_R(p)$ and $H(x,p)$ are C^2 functions. By the construction, $\beta(z)$ is of class C^3. It follows that $H_R(x,p)$ is a C^2 function. □

Claim 2. $H_R(x,p)$ satisfies (H2).

Proof of Claim 2. It suffices to show that for given $x \in M$, $\partial^2 H_R/\partial p^2(x,p) > 0$.

(i) For $|p| \le R$,

$$H_R(x,p) = H(x,p).$$

Hence, we have

$$\frac{\partial^2 H_R}{\partial p^2}(x,p) = \frac{\partial^2 H}{\partial p^2}(x,p) > 0.$$

(ii) For $R < |p| \le R+1$,

$$H_R(x,p) = H(x,p) + \mu_R \beta(|p|^2 - R^2).$$

It follows that

$$\frac{\partial^2 H_R}{\partial p^2}(x,p) = \frac{\partial^2 H}{\partial p^2}(x,p) + 2\mu_R \left(2\beta''(|p|^2 - R^2)Z(p) + \beta'(|p|^2 - R^2)\cdot E\right) > 0,$$

where $Z(p) := (p_1, \dots, p_n)^T \cdot (p_1, \dots, p_n)$, and E denotes the $n \times n$ identity matrix.

(iii) For $R+1 < |p| \le R+2$,

$$H_R(x,p) = \alpha_R(p) H(x,p) + \mu_R \beta(|p|^2 - R^2).$$

This yields that

$$\begin{aligned}\frac{\partial^2 H_R}{\partial p^2}(x,p) =& H(x,p)\alpha''_R(p) + W(x,p) + \alpha_R(p)\frac{\partial^2 H}{\partial p^2}(x,p) \\ &+ 2\mu_R \left(2\beta''(|p|^2 - R^2)Z(p) + \beta'(|p|^2 - R^2)\cdot E\right),\end{aligned}$$

where

$$W(x,p) := \alpha'_R(p)^T \cdot \frac{\partial H}{\partial p}(x,p) + \frac{\partial H}{\partial p}(x,p)^T \cdot \alpha'_R(p).$$

Since $W(x,p)$ is symmetric, $\partial^2 H_R/\partial p^2(x,p)$ is symmetric. We denote $\partial^2 H_R/\partial p^2(x,p) = (a_{ij})_{n\times n}$; then $\partial^2 H_R/\partial p^2(x,p)(x,p)$ is positive definite if $\sqrt{a_{ii}a_{jj}} > (n-1)|a_{ij}|$ for $i,j = 1, \dots, n$ and $i \ne j$.

Based on the construction of $\alpha_R(p)$ and the compactness of M, let

$$\gamma_R := 2 \sup_{(x,p)\in\mathbb{T}^n\times[R+1,R+2]^n} |H(x,p)| + (n-1) \sup_{(x,p)\in\mathbb{T}^n\times[R+1,R+2]^n} \|W(x,p)\|_1.$$

It is enough to take

$$\mu_R > \max\{\gamma_R, 1\}. \tag{3.4}$$

(iv) For $|p| > R+2$,

$$H_R(x,p) = \mu_R\beta(|p|^2 - R^2),$$

which implies that

$$\frac{\partial^2 H_R}{\partial p^2}(x,p) = 2\mu_R\left(2\beta''(|p|^2-R^2)Z(p) + \beta'(|p|^2-R^2)\cdot E\right) > 0.$$

Therefore, $H_R(x,p)$ satisfies (H2). □

Claim 3. $H_R(x,p)$ satisfies the superlinearity.

Proof of Claim 3. It suffices to verify the superlinearity of $H_R(x,p)$ for $|p| > R+2$. In this case, we have

$$H_R(x,p) \geq \mu_R\beta(|p|^2-R^2) \geq |p|^2.$$

Hence, for each $A > 0$, one can find $C_A > 0$ such that

$$H_R(x,p) \geq A|p| - C_A.$$

Therefore, $H_R(x,p)$ satisfies the superlinearity. □

It is easy to see that H_R converges uniformly on compact subsets to H in the C^2 topology as $R \to \infty$.

3.2. **Mañé critical value.** We use c_R to denote the Mañé critical value of $H_R(x,p)$. Then

$$c_R = \inf_{u\in C^1(M,\mathbb{R})} \max_{x\in M} H_R(x, \partial_x u). \tag{3.5}$$

The following lemma asserts that for R large enough, the Mañé critical value of H_R is independent of R. We denote

$$c[0] := \inf_{u\in C^1(M,\mathbb{R})} \max_{x\in M} H(x, \partial_x u), \tag{3.6}$$

which can be seen as the Mañé critical value of $H(x,p)$.

Lemma 3.1. *There exists $R_0 > 0$ such that for any $R > R_0$, we have*

$$c_R = c[0]. \tag{3.7}$$

Proof. From (3.5) and the construction of H_R, it follows that for any $R > 0$,

$$c_R \leq \max_{x\in M} H_R(x,0) = \max_{x\in M} H(x,0). \tag{3.8}$$

Let $A := \max_{x\in M} H(x,0) + 1$. We denote

$$\Lambda := \{(x,p)\in T^*M | x\in M, H(x,p)\leq A\}.$$

By (H3) and the compactness of M, Λ is compact. Hence, there exists $R_0 > 0$ such that

$$\Lambda \subset \{(x,p)\in T^*M | x\in M, |p|_x \leq R_0\},$$

where $|\cdot|_x$ denotes the Riemannian metric on $T^*_x M$. Based on the construction of H_R, it yields that for any $R > R_0$ and $(x,p) \in \Lambda$, we have

$$H_R(x,p) = H(x,p). \tag{3.9}$$

In terms of the definition of the Mañé critical value, one can find a sequence $u_n \in C^1(M,\mathbb{R})$ such that

$$\max_{x\in M} H_R(x, \partial_x u_n(x)) \to c_R. \tag{3.10}$$

Since $c_R < A$, we have $|\partial_x u_n(x)| \le R_0$ for n large enough. Moreover, we have $H_R(x,\partial_x u_n(x)) = H(x,\partial_x u_n(x))$ for any $R > R_0$, which yields for n large enough,

$$\begin{aligned} c[0] &= \inf_{u\in C^1(M,\mathbb{R})} \max_{x\in M} H(x,\partial_x u(x)) \\ &\le \max_{x\in M} H(x,\partial_x u_n(x)) \\ &= \max_{x\in M} H_R(x,\partial_x u_n(x)). \end{aligned}$$

Taking the limit as $n \to \infty$, it follows from (3.10) that $c[0] \le c_R$. Similarly, we choose a sequence $v_n \in C^1(M,\mathbb{R})$ such that

$$\max_{x\in M} H(x,\partial_x v_n(x)) \to c[0]. \tag{3.11}$$

Since $c[0] \le \max_{x\in M} H(x,0) < A$, we have $|\partial_x v_n(x)| \le R_0$ for n large enough. Moreover, we have $H_R(x,\partial_x v_n(x)) = H(x,\partial_x v_n(x))$ for any $R > R_0$, which yields for n large enough,

$$\begin{aligned} c_R &= \inf_{u\in C^1(M,\mathbb{R})} \max_{x\in M} H_R(x,\partial_x u(x)) \\ &\le \max_{x\in M} H_R(x,\partial_x v_n(x)) \\ &= \max_{x\in M} H(x,\partial_x v_n(x)), \end{aligned}$$

which together with (3.11) implies that $c_R \le c[0]$ as $n \to \infty$. Therefore, one can find $R_0 > 0$ such that for any $R > R_0$, $c_R = c[0]$. This finishes the proof of Lemma 3.1. □

For the sake of simplicity, we assume $c[0] = 0$ in the following context.

3.3. **The viscosity solution of (1.3).**

Let $\bar{u}(x)$ be a viscosity solution of $H(x,\partial_x u) = 0$. Since $H(x,p)$ is coercive with respect to p, $\bar{u}(x)$ is a Lipschitz function on M, which together with Proposition 2.3 implies that $\bar{u}$ is semiconcave.

Let $\mathcal{D}$ be the set of all differentiable points of $\bar{u}$ on M. Due to the Lipschitz property of $\bar{u}$, it follows that $\mathcal{D}$ has full Lebesgue measure.

Lemma 3.2. *There exists $R_1 > 0$ such that for any $R > R_1$, $\bar{u}(x)$ is a viscosity solution of $H_R(x,\partial_x u) = 0$.*

Proof. Since $\bar{u}(x)$ is a Lipschitz function on M, for $x \in \mathcal{D}$, we have $H(x,\partial_x \bar{u}) = 0$. By (H3), there exists $R_1 > 0$ such that $|\partial_x \bar{u}| \le R_1$ for $x \in \mathcal{D}$. It follows from the construction of H_R that for $R > R_1$ and

$$(x,p) \in \{(x,p) \in T^*M \mid x \in \mathcal{D}, |p|_x \le R_1\},$$

we have $H_R(x,p) = H(x,p)$, which means that for $x \in \mathcal{D}$,

$$H_R(x,\partial_x \bar{u}) = 0.$$

Due to the semiconcavity of $\bar{u}(x)$, it follows from Proposition 2.3 that $\bar{u}(x)$ is a viscosity solution of $H_R(x, \partial_x u) = 0$ for any $R > R_1$. This completes the proof of Lemma 3.2. □

3.4. **Location of the action minimizing orbits.** Let Φ_H^t denote the flow generated by $H(x,p)$. Let $(x(t), p(t)) := \Phi_H^t(x_0, p_0)$. Let L_R be the Lagrangian associated to H_R. To fix the notion, for a given $R > 0$ and $(x_0, p_0) \in T^*M$, we call $(x_R(t), p_R(t)) := \Phi_{H_R}^t(x_0, p_0)$ the action minimizing orbit with $x_R(0) = x_0$ and $x_R(t) = y$ if

$$x_R(t) = \gamma_R(t), \quad p_R(t) = \frac{\partial L_R}{\partial \dot{x}}(\gamma_R(t), \dot{\gamma}_R(t)),$$

where $\gamma_R : [0,t] \to M$ is an action minimizing curve with $\gamma_R(0) = x_0$ and $\gamma_R(t) = y$. That is, γ_R achieves

$$\inf_{\substack{\gamma(0)=x_0 \\ \gamma(t)=y}} \int_0^t L_R(\gamma(s), \dot{\gamma}(s))ds.$$

Lemma 3.3 (*A priori* compactness)**.** *For $s \in [0,t]$, let $(x_R(s), p_R(s))$ be an action minimizing orbit with $x_R(0) = x_0$ and $x_R(t) = y$. There exists $\bar{R} > 1$ such that for any $R > \bar{R}$, one can find $t_0 := t_0(\bar{R}) > 0$ such that for any $s \in [0,t]$ with $t > t_0$, we have*

$$(x_R(s), p_R(s)) \in \Omega,$$

where $\Omega := \{(x,p) \mid H(x,p) \leq 1\}$.

In order to prove Lemma 3.3, we need to do some preparations. Based on Lemma 3.2, it yields that for $x \in \mathcal{D}$ and $R > R_1$,

$$H_R(x, \partial_x \bar{u}(x)) = 0. \tag{3.12}$$

We define

$$\widetilde{L}_R(x, \dot{x}) = L_R(x, \dot{x}) - \langle \partial_x \bar{u}(x), \dot{x} \rangle, \quad x \in \mathcal{D}. \tag{3.13}$$

Denote

$$\Gamma_R := \left\{ \left(x, \frac{\partial H_R}{\partial p}(x, \partial_x \bar{u}(x)) \right) \ : \ x \in \mathcal{D} \right\}, \tag{3.14}$$

where $\frac{\partial H_R}{\partial p}$ denotes the partial derivative of H_R with respect to the second argument. We have the following lemma.

Lemma 3.4. *For any $x \in \mathcal{D}$, $\widetilde{L}_R(x, \dot{x}) \geq 0$. In particular, $\widetilde{L}_R(x, \dot{x}) = 0$ if and only if $(x, \dot{x}) \in \Gamma_R$.*

Proof. By (3.13) and (3.14), we have

$$\widetilde{L}_R \Big|_{\Gamma_R} = -H_R(x, \partial_x \bar{u}(x)) = 0. \tag{3.15}$$

In addition, we have

$$\frac{\partial \widetilde{L}_R}{\partial \dot{x}} \Big|_{\Gamma_R} = \frac{\partial L_R}{\partial \dot{x}}(x, \dot{x}) - \partial_x \bar{u}(x) = 0. \tag{3.16}$$

By the superlinearity of L_R, it follows from (3.15) that there exists $K_1 > 0$ large enough such that for $|\dot{x}| > K_1$,

$$\widetilde{L}_R(x, \dot{x}) \geq d > 0,$$

where d is a constant independent of $(x, \dot{x})$.

For $x \in \mathcal{D}$, $\bar{u}(x)$ satisfies the equation (3.12). Since $\bar{u}(x)$ is Lipschitz continuous, $\partial_x \bar{u}(x)$ is bounded. Let

$$\dot{x}_0 := \frac{\partial H_R}{\partial p}(x, \partial_x \bar{u}(x)).$$

Then there exists $K_2 > 0$ independent of x such that $|\dot{x}_0| \leq K_2$. Take $K_3 := \max\{K_1, K_2\}$. Note that $\frac{\partial^2 L_R}{\partial \dot{x}^2}(x, \dot{x})$ is positive definite, for $|\dot{x}| \leq K_3$, it follows from (3.15) and (3.16) that there exists $\Lambda > 0$ independent of $(x, \dot{x})$ such that

$$\widetilde{L}_R(x, \dot{x}) \geq \Lambda \left| \dot{x} - \frac{\partial H_R}{\partial p}(x, \partial_x \bar{u}(x)) \right|^2. \tag{3.17}$$

Consequently, it is easy to see that

$$\widetilde{L}_R(x, \dot{x}) \begin{cases} = 0, & (x, \dot{x}) \in \Gamma_R, \\ > 0, & (x, \dot{x}) \notin \Gamma_R. \end{cases} \tag{3.18}$$

This completes the proof of Lemma 3.4. □

Let Ω^* denote the Legendre transformation of Ω via $\mathcal{L} : T^*M \to TM$. By (H3), there exist $R_2, R_2^* > 0$ such that

$$\Omega \subset \{(x, p) \in T^*M \mid x \in M,\ |p|_x \leq R_2\},$$
$$\Omega^* \subset \{(x, v) \in TM \mid x \in M,\ |v|_x \leq R_2^*\}.$$

Based on the preparations above, we will prove Lemma 3.3. First of all, we take

$$\bar{R} = \max\{R_0, R_1, R_2, R_2^*\}, \tag{3.19}$$

where R_0, R_1 are determined by Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.3. By the energy conservation of H, it suffices to prove $(x_0, p_0) \in \Omega$, where $(x_0, p_0) = (x_R(0), p_R(0))$ is the initial point of the flow $\Phi^t_{H_R}$. Let

$$\Delta := T^*M \backslash \Omega = \{(x, p) \mid H(x, p) > 1\}. \tag{3.20}$$

By contradiction, we assume $(x_0, p_0) \in \Delta$.

Let $\Sigma := \{(x, \partial_x \bar{u}(x)) \mid x \in \mathcal{D}\}$. Since $H(x, \partial_x \bar{u}(x)) = 0$ for $x \in \mathcal{D}$, $\Sigma \cap \Delta = \emptyset$. Let Σ^* and Δ^* denote the Legendre transformation of Σ and Δ via $\mathcal{L} : T^*M \to TM$ respectively. Since $\mathcal{L}$ is a diffeomorphism onto the image, we have

$$\Sigma^* \cap \Delta^* = \emptyset. \tag{3.21}$$

By virtue of Lemma 3.2, it yields that for $R > \bar{R}$ and $x \in \mathcal{D}$,

$$\frac{\partial H_R}{\partial p}(x, \partial_x \bar{u}(x)) = \frac{\partial H}{\partial p}(x, \partial_x \bar{u}(x)).$$

It follows that

$$\Sigma^* = \left\{ \left(x, \frac{\partial H}{\partial p}(x, \partial_x \bar{u}(x)) \right) \ : \ x \in \mathcal{D} \right\}. \tag{3.22}$$

We use Σ^*_κ to denote a κ-neighborhood of Σ^* in the fibers, namely

$$\Sigma^*_\kappa := \left\{ (x, \dot{x}) \mid x \in \mathcal{D}, \operatorname{dist}\left(\dot{x}, \frac{\partial H}{\partial p}(x, \partial_x \bar{u}(x)) \right) \leq \kappa \right\}.$$

By the C^2 regularity of H and L_R, for any $\epsilon > 0$, there exists $\kappa > 0$ such that for $(x,\dot{x}) \in \Sigma_\kappa^*$, we have

$$H\left(x, \frac{\partial L_R}{\partial \dot{x}}(x,\dot{x})\right) \leq \epsilon;$$

hence, for κ small enough, we have $\epsilon < 1$. Moreover

$$\Sigma_\kappa^* \cap \Delta^* = \emptyset.$$

By Lemma 3.4, for any $x \in \mathcal{D}$, $\widetilde{L}_R(x,\dot{x}) \geq 0$ and $\widetilde{L}_R(x,\dot{x}) = 0$ if and only if $(x,\dot{x}) \in \Sigma^*$. Then for each $R > \bar{R}$, there exists a constant $\eta := \eta(\bar{R}) > 0$ such that for $x \in \mathcal{D}$ and $(x,\dot{x}) \in \Delta^*$,

$$\widetilde{L}_R(x,\dot{x}) \geq \eta, \tag{3.23}$$

where

$$\widetilde{L}_R(x,\dot{x}) = L_R(x,\dot{x}) - \langle \partial_x \bar{u}(x), \dot{x}\rangle.$$

Let $\gamma_R : [0,t] \to M$ be an action minimizing curve with $\gamma_R(0) = x_0$, $\gamma_R(t) = y$. Then we have $\dot{\gamma}_R(s) = \frac{\partial H_R}{\partial p}(x_R(s), p_R(s))$ for $s \in [0,t]$. Since $(x_0, p_0) \in \Delta$, for $s \in [0,t]$, we have

$$(\gamma_R(s), \dot{\gamma}_R(s)) \in \Delta^*. \tag{3.24}$$

Let Θ be the set of $\gamma_R(s)$ along which the one-sided directional derivative denoted by $\partial\bar{u}(\gamma_R(s), \dot{\gamma}_R(s))$ exists. For $\gamma_R(s) \in \Theta$, we denote

$$\widehat{L}_R(\gamma_R(s), \dot{\gamma}_R(s)) := L_R(\gamma_R(s), \dot{\gamma}_R(s)) - \partial\bar{u}(\gamma_R(s), \dot{\gamma}_R(s)).$$

Note that $\bar{u}$ is locally semiconcave. By virtue of Proposition 2.6(b), one can find a sequence $x_n^s \in \mathcal{D}$ with $x_n^s \to \gamma_R(s)$ and $\partial_x \bar{u}(x_n^s) \to p_s \in D^+\bar{u}(\gamma_R(s))$ as $n \to \infty$ for a given $s \in [0,t]$. By virtue of Proposition 2.6(c), for n large enough, extracting a subsequence if necessary, we have

$$\begin{aligned}\partial\bar{u}(\gamma_R(s), \dot{\gamma}_R(s)) &= \min_{p \in D^+\bar{u}(\gamma_R(s))} \langle p, \dot{\gamma}_R(s)\rangle \\ &\leq \langle p_s, \dot{\gamma}_R(s)\rangle \\ &\leq \langle \partial_x \bar{u}(x_n^s), \dot{\gamma}_R(s)\rangle + \frac{1}{n}.\end{aligned}$$

Note that Δ^* is an open set; then $(x_n^s, \dot{\gamma}_R(s)) \in \Delta^*$ for n large enough. It follows from (3.23) that for every $s \in [0,t]$ and n large enough,

$$\widehat{L}_R(\gamma_R(s), \dot{\gamma}_R(s)) \geq L_R(x_n^s, \dot{\gamma}_R(s)) - \langle \partial_x \bar{u}(x_n^s), \dot{\gamma}_R(s)\rangle - \frac{2}{n} \geq \frac{\eta}{2}. \tag{3.25}$$

Moreover, we have

$$\int_0^t \widehat{L}_R(\gamma_R(s), \dot{\gamma}_R(s))ds \geq \frac{\eta}{2}t.$$

On the other hand, we have

$$\begin{aligned}\int_0^t \widehat{L}_R(\gamma_R(s), \dot{\gamma}_R(s))ds &= \int_0^t L_R(\gamma_R(s), \dot{\gamma}_R(s)) - \partial\bar{u}(\gamma_R(s), \dot{\gamma}_R(s))ds \\ &= \int_0^t L_R(\gamma_R(s), \dot{\gamma}_R(s))ds - (\bar{u}(\gamma_R(t)) - \bar{u}(\gamma_R(0))).\end{aligned}$$

It follows from the semiconcavity and the compactness of M that $\bar{u}$ has a uniform bound denoted by C_0. Hence, we have

$$\int_0^t L_R(\gamma_R(s), \dot{\gamma}_R(s))ds \geq \frac{\eta}{2}t - 2C_0. \tag{3.26}$$

On the other hand, γ_R is an action minimizing curve of L_R. Let γ_{R_2} be an action minimizing curve of L_{R_2}. It follows from Lemma 3.1 that for $R > \bar{R}$, there exists a constant $C_1 > 0$ independent of R such that

$$\begin{aligned} \int_0^t L_R(\gamma_R(s), \dot{\gamma}_R(s))ds &\leq \int_0^t L_R(\gamma_{R_2^*}(s), \dot{\gamma}_{R_2^*}(s))ds \\ &= \int_0^t L_{R_2^*}(\gamma_{R_2^*}(s), \dot{\gamma}_{R_2^*}(s))ds \\ &= h^t_{R_2^*}(x_0, y) \leq C_1, \end{aligned} \tag{3.27}$$

where $h^t_{R_2^*}(x_0, y)$ denotes the minimal action of $L_{R_2^*}$. It is clear to see that (3.27) contradicts (3.26) if we take $t > (4C_0 + 2C_1)/\eta$. Let $t_0 := (4C_0 + 2C_1)/\eta$; then we have $(x_0, p_0) \notin \Delta$ for $t > t_0$. Obviously, t_0 only depends on $\bar{R}$. This completes the proof of Lemma 3.3. □

4. Asymptotic Lipschitz regularity

In this section, we are devoted to proving Theorem 1.1, which is concerned with the following Hamilton-Jacobi equation under the assumptions (H1)-(H3):

$$\begin{cases} \partial_t u(x,t) + H(x, \partial_x u(x,t)) = 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{4.1}$$

where $(x,t) \in M \times [0,\infty)$ and $\varphi(x) \in C(M,\mathbb{R})$. Let $u_R(x,t)$ be the viscosity solution of the following equation:

$$\begin{cases} \partial_t u(x,t) + H_R(x, \partial_x u(x,t)) = 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{4.2}$$

Let T_t^R be the Lax-Oleinik semigroup generated by L_R associated to H_R via the Legendre transformation. Namely,

$$T_t^R\varphi(x) = \inf_{\gamma(t)=x} \left\{ \varphi(\gamma(0)) + \int_0^t L_R(\gamma(s), \dot{\gamma}(s))ds \right\}. \tag{4.3}$$

Then we have

$$u_R(x,t) = T_t^R\varphi(x). \tag{4.4}$$

First of all, we consider the viscosity solutions of (4.1) with t suitably large.

Lemma 4.1. *For any $R \geq \bar{R}$ where $\bar{R}$ is determined by* (3.19), *there exists $t_0 > 0$ such that for $t > t_0$, $u_R(x,t)$ is a viscosity solution of the following equation:*

$$\partial_t u(x,t) + H(x, \partial_x u(x,t)) = 0. \tag{4.5}$$

Proof. By Proposition 2.3(b), $u_R(x,t)$ is locally semiconcave on $M \times (0,\infty)$. Let $\mathcal{E}_R$ be the set of all differentiable points of $u_R(x,t)$ on $M \times (0,\infty)$. Then $\mathcal{E}_R$ has full Lebesgue measure. For $(x,t) \in \mathcal{E}_R$, we have $u_R(x,t)$ satisfies (4.2). For a given

$(\bar{x}, \bar{t}) \in \mathcal{E}_R$, let $\gamma_R : [0, \bar{t}] \to M$ be a curve achieving the infimum of (4.3) with $\gamma_R(\bar{t}) = \bar{x}$. Then we have

$$\partial_x u_R(\bar{x}, \bar{t}) = \frac{\partial L_R}{\partial \dot{x}}(\gamma_R(\bar{t}), \dot{\gamma}_R(\bar{t})). \tag{4.6}$$

Since $R \geq \bar{R}$, it follows from Lemma 3.3 that there exists $t_0 > 0$ independent of R such that for $\bar{t} > t_0$ and any $s \in [0, \bar{t}]$,

$$H\left(\gamma_R(s), \frac{\partial L_R}{\partial \dot{x}}(\gamma_R(s), \dot{\gamma}_R(s))\right) \leq 1.$$

Then $(\bar{x}, \partial_x u_R(\bar{x}, \bar{t})) \in \Omega$. Moreover, for each $(x, t) \in \mathcal{E}_R$ and $t > t_0$, we have

$$|\partial_x u_R(x, t)| \leq \bar{R},$$

since $\bar{R}$ is independent of (x, t). It follows that for $R > \bar{R}$, $(x, t) \in \mathcal{E}_R$ and $t > t_0$, $u_R(x, t)$ satisfies

$$\partial_t u(x, t) + H_R(x, \partial_x u(x, t)) = 0.$$

Hence, for $(x, t) \in \mathcal{E}_R$ and $t > t_0$, $u_R(x, t)$ satisfies (4.5). By Proposition 2.3(a), $u_R(x, t)$ is a viscosity solution of (4.5). This completes the proof of Lemma 4.1. □

Lemma 4.2. *Given $t > t_0$, $T_t^R \varphi(x)$ is uniformly bounded for each $R > \bar{R}$.*

Proof. Let $\gamma_R : [0, t] \to M$ be a curve achieving the infimum of (4.3) with $\gamma_R(t) = x$. By Lemma 3.3, for $R > \bar{R}$, there holds

$$\begin{aligned} T_t^R \varphi(x) &= \varphi(\gamma_R(0)) + \int_0^t L_R(\gamma_R(s), \dot{\gamma}_R(s)) ds \\ &= \varphi(\gamma_R(0)) + \int_0^t L(\gamma_R(s), \dot{\gamma}_R(s)) ds, \end{aligned}$$

which implies for any $x \in M$,

$$|T_t^R \varphi(x)| \leq \max_{x \in M} |\varphi(x)| + t \max_{(x, \dot{x}) \in \Omega^*} L(x, \dot{x}).$$

This completes the proof of Lemma 4.2. □

By Lemma 4.2 and Lemma 3.3, a standard argument shows that given $t > t_0$, $T_t^R \varphi(x)$ is equi-Lipschitz for each $R > \bar{R}$ (see [7, Proposition 5.5]). It follows from Lemma 4.1 that for $t > t_0$, the viscosity solution $u(x, t)$ of (4.1) can be represented as $\liminf_{R \to \infty} T_t^R \varphi(x)$. In the following, we consider the case with $t \in [0, t_0]$.

Lemma 4.3. *Let $\psi(x)$ be a Lipschitz function. Then there exists $\tilde{R} > 0$ such that for $(x, t) \in M \times [0, t_0]$, $u_{\tilde{R}}(x, t)$ is the viscosity solution of* (4.5) *with $u_{\tilde{R}}(x, 0) = \psi(x)$.*

Proof. Based on uniqueness and the regularity theory of viscosity solutions ([1, Theorem 8.2], [4, Theorem 2.5]), under the assumptions (H1)-(H3), there exists a unique Lipschitz viscosity solution $u(x, t)$ of (4.5) with $u(x, 0) = \psi(x)$. At the differentiable points of $u(x, t)$ on $M \times [0, t_0]$, we have

$$|\partial_x u(x, t)| \leq K, \tag{4.7}$$

where K is a constant. Taking $\tilde{R} \geq K$, it follows from a similar argument as the one in the proof of Lemma 4.1 that for $(x,t) \in M \times [0,t_0]$, $u(x,t)$ is the viscosity solution of

$$\begin{cases} \partial_t u(x,t) + H_{\tilde{R}}(x, \partial_x u(x,t)) = 0, \\ u(x,0) = \psi(x). \end{cases} \tag{4.8}$$

On the other hand, $u_{\tilde{R}}(x,t)$ is also a viscosity solution of (4.8). By the uniqueness of the viscosity solution of (4.8), we have $u(x,t) \equiv u_{\tilde{R}}(x,t)$ for $(x,t) \in M \times [0,t_0]$. This completes the proof of Lemma 4.3. □

Proof of Theorem 1.1. First of all, we consider the case of $t \in [0,t_0]$, where t_0 is determined by Lemma 4.3. For given initial data $\varphi(x) \in C(M,\mathbb{R})$, we choose a sequence of Lipschitz functions $\varphi_n(x)$ such that $\varphi_n \to \varphi(x)$ in the C^0-norm. Let $u_R^n(x,t)$ be the viscosity solution of the following equation:

$$\begin{cases} \partial_t u(x,t) + H_R(x, \partial_x u(x,t)) = 0, \\ u(x,0) = \varphi_n(x). \end{cases} \tag{4.9}$$

By (4.4), we have $u_R^n(x,t) = T_t^R \varphi_n(x)$. Let

$$u_n(x,t) := \liminf_{R\to\infty} T_t^R \varphi_n(x).$$

It follows from Lemma 4.3 that $u_n(x,t)$ is the viscosity solution of

$$\begin{cases} \partial_t u(x,t) + H(x, \partial_x u(x,t)) = 0, \\ u(x,0) = \varphi_n(x). \end{cases} \tag{4.10}$$

Claim.

$$\lim_{n\to\infty} u_n(x,t) = \liminf_{R\to\infty} T_t^R \varphi(x). \tag{4.11}$$

Proof of the Claim. It is easy to see that for given $\tilde{R} > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} \inf_{R>\tilde{R}} \left(T_t^R \varphi_n(x) - T_t^R \varphi(x)\right) &\leq \inf_{R>\tilde{R}} T_t^R \varphi_n(x) - \inf_{R>\tilde{R}} T_t^R \varphi(x) \\ &\leq \sup_{R>\tilde{R}} \left(T_t^R \varphi_n(x) - T_t^R \varphi(x)\right). \end{aligned}$$

By virtue of the non-expansiveness of T_t^R, we have

$$\|T_t^R \varphi_n(x) - T_t^R \varphi(x)\|_\infty \leq \|\varphi_n(x) - \varphi(x)\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the C^0-norm. Hence,

$$\|\inf_{R>\tilde{R}} T_t^R \varphi_n(x) - \inf_{R>\tilde{R}} T_t^R \varphi(x)\|_\infty \leq \|\varphi_n(x) - \varphi(x)\|_\infty. \tag{4.12}$$

Since $\liminf_{R\to\infty} = \lim_{\tilde{R}\to\infty} \inf_{R>\tilde{R}}$, we have

$$\|\liminf_{R\to\infty} T_t^R \varphi_n(x) - \liminf_{R\to\infty} T_t^R \varphi(x)\|_\infty \leq \|\varphi_n(x) - \varphi(x)\|_\infty. \tag{4.13}$$

Moreover, $u_n(x,t)$ converges to $\liminf_{R\to\infty} T_t^R \varphi(x)$ in the C^0-norm on $M \times [0,t_0]$ as $n \to \infty$, which verifies the claim (4.11).

Let $\bar{u}(x,t) := \liminf_{R\to\infty} T_t^R \varphi(x)$. It follows from the stability of viscosity solutions ([5, Theorem 8.1]) that for $(x,t) \in M \times [0,t_0]$, $\bar{u}(x,t)$ is the viscosity solution of (4.1).

Second, it follows from Lemma 4.1 that for $t > t_0$, the viscosity solution $u(x,t)$ of (4.1) can be represented as $\liminf_{R\to\infty} T_t^R\varphi(x)$. By virtue of the uniqueness of the viscosity solution of (4.1) under the assumptions (H1)-(H3) [4, Theorem 2.5], it follows that for $(x,t) \in M \times [0,\infty)$,

$$u(x,t) = \liminf_{R\to\infty} T_t^R\varphi(x).$$

In particular, there exists $t_0 > 0$ such that for $t > t_0$, $u(x,t) = T_t^{\hat{R}}\varphi(x)$ where $\hat{R} = \max\{\bar{R}, \tilde{R}\}$. Note that $T_t^{\hat{R}}\varphi(x)$ is Lipschitz continuous and its Lipschitz constant is independent of φ ([5, Proposition 4.6.6]). By Lemma 4.1, t_0 is also independent of φ. Hence, for $t > t_0$, $u(x,t)$ is ι-Lipschitz continuous and t_0, ι are independent of φ.

Thus, we have completed the proof of Theorem 1.1. □

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