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TRAVELING WAVE FRONT FOR PARTIAL NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. By using Schauder's point fixed theorem we study the existence of a traveling wave front for reaction-diffusion differential equations of the neutral type. Some examples arising in populations dynamics are presented.

1. Introduction

Using Schauder's point fixed theorem and monotonicity, we study the existence of a traveling wave front for neutral differential equations of the form

$$\frac{d}{dt}[u(t,x) - G(u_t)(x)] = \mathcal{D}\Delta u(t,x) + F(u_t)(x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.1}$$

where $\mathcal{D} = diag(d_i)$ is a matrix of order $N \times N$, $d_i > 0$ for every $i = 1, \ldots, N$, and $F \in C(C([-\tau, 0]; \mathbb{R}^N); \mathbb{R}^N)$, $G \in C^1(C([-\tau, 0]; \mathbb{R}^N); \mathbb{R}^N)$ $(\tau > 0)$ are functions to be specified later.

The literature on the existence and qualitative properties of traveling waves for reaction-diffusion equations is extensive. We cite the early papers by Fisher [4], Kolmogorov, Petrovskii and Piskunov [10], Britton [1], Fife [3], Murray [17] and Volpert et al. [20] regarding related differential equations. For the case of delayed differential equations, we refer the reader to Schaaf [19], Ma [16], Zou and Wu [21, 25] and the references therein.

To the best of our knowledge, the paper [14] is the unique work treating traveling waves for partial neutral differential equations. Using a variable transform which allows one to study neutral equations with discrete delay via a differential equation with an infinite number of constant delays, results on existence and invariance in reaction diffusion equations and techniques on the construction of upper and lower solutions, in [14] are proved some results on the existence and qualitative properties of traveling waves for neutral differential equations of the form

$$\begin{aligned}\frac{d}{dt}(u(t,x) - bu(t-r,x)) &= d\Delta[u(t,x) - bu(t-r,x)] \\ &\quad + f(u(t,x) - bu(t-r,x), u(t,x), u(t-r,x)).\end{aligned}$$

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Concerning partial neutral differential equations, we cite the early paper by Hale [8], where are proved some results on the existence, uniqueness and qualitative properties of solutions of neutral equations of the form $\partial_t \mathcal{L}(u_t)(\xi) = \partial_{\xi\xi}\mathcal{L}u_t(\xi) + f(u_t)(\xi)$, where $\mathcal{L}(\cdot)$ is a bounded linear operator on $C([-r,0];C(S^1;\mathbb{R}))$. In [23], Wu and Xia derived a neutral difference-differential system with diffusion from a ring array of coupled lossless transmission lines and investigated the problem of self-sustained oscillations of the considered transmission lines and the existence of multiple large amplitude phase-locked periodic solutions in the corresponding neutral system. In [24], Wu and Xia continued their studies in [23], proved some general results on the existence and global continuation of rotating waves for neutral partial differential equations and applied their results to study a concrete neutral problem of the form

$$\frac{d}{dt}(u(t,x) - bu(t-r,x)) = d\Delta[u(t,x) - bu(t-r,x)] - au(t,x) \\ - abu(t-r,x) - g(u(t,x) - bu(t-r,x)).$$

In the theory developed in [7,18], the internal energy and the heat flux are described as functionals of the temperature $u(\cdot)$ and their derivative $u_x(\cdot)$. The system

$$\frac{d}{dt}(\,u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\,) = d\triangle u(t,x) + \int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds.$$

has been used to describe this phenomena; see [15]. In this problem, d is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are the internal energy and the heat flux relaxation respectively. If we assume that the solution $u(\cdot)$ is known on $(-\infty, 0]$, we obtain a neutral equation with unbounded delay. Partial neutral differential equations can also be derived from the theory of population dynamic (see [2,5,6, 11–13]) where diffusion arises from the tendency of biological species to migrate from high to low population density regions.

In this work, for Banach spaces X, Y we use the symbol $\mathcal{L}(X;Y)$ for the space of bounded linear operators from X into Y endowed with the usual norm denoted by $\|\cdot\|_{\mathcal{L}(X,Y)}$, and for $z \in Z$ and $l > 0$, $B_l(z,Z) = \{x \in Z : \| z - x \|_Z \le l\}$. A function $H : C([-\tau,0];\mathbb{R}^N) \to \mathbb{R}^M$ is described in the form $H(\psi) = (H_1(\psi),\dots,H_M(\psi))$. For $c > 0$ and $\psi \in C([-c\tau,0];\mathbb{R}^N)$, we denote by $H^c(\cdot)$ and $\psi^c(\cdot)$ the functions $H^c : C([-c\tau;0];\mathbb{R}^N) \to \mathbb{R}^M$ and $\psi^c \in C([-c\tau;0];\mathbb{R}^N)$ given by $H^c(\psi) = H(\psi^c)$ and $\psi^c(\theta) = \psi(c\theta)$. If $H(\cdot)$ is a C^1 function, $DH(\cdot)$ denotes the differential of $H(\cdot)$ and $(DH)^c : C([-c\tau;0];\mathbb{R}^N) \to \mathcal{L}(C([-c\tau;0];\mathbb{R}^N);\mathbb{R}^M)$ is given by $(DH)^c(\psi) = (DH)(\psi^c)$. In this case, we note that $((DH)^c(\psi))^c(\phi) = (DH)(\psi^c)(\phi^c)$ and $(((DH)^c(\psi))^c(\phi))_i = (DH_i)(\psi^c)(\phi^c)$ for all $\phi \in C([-c\tau,0];\mathbb{R}^N)$ and $i = 1,\dots,M$.

A traveling wave solution of (1.1) is a solution of the form $u(t,x) = \phi(x+ct)$, where $\phi \in C^2(\mathbb{R};\mathbb{R}^N)$ and $c \in (0,\infty)$. If $u(t,x) = \phi(x+ct)$ is a traveling wave of (1.1), $F \in C(C([-\tau,0];\mathbb{R}^N);\mathbb{R}^N)$ and $G \in C^1(C([-\tau,0];\mathbb{R}^N);\mathbb{R}^N)$, then $\phi(\cdot)$ is a solution of the ordinary problem

$$\mathcal{D}w''(\xi) - cw'(\xi) + c((DG)^c(w_\xi))^c(w'_\xi) + F^c(w_\xi) = 0, \qquad \xi \in \mathbb{R}. \tag{1.2}$$

For $u = (u_1,\dots,u_N)$, $v = (v_1,\dots,v_N) \in \mathbb{R}^N$, we write $u \le v$ if $u_i \le v_i$ for all $i = 1,\dots,N$, and $u < v$ if $u \le v$ and $u \ne v$. A function $H : C([-\tau,0];\mathbb{R}^N) \to \mathbb{R}^N$ is described in the form $H(\psi) = (H_1(\psi),\dots,H_N(\psi))$.

For $g \in C(\mathbb{R};\mathbb{R}^+)$ with $\lim_{s\to\pm\infty} g(s) = 0$, we use the notation $C^1_g(\mathbb{R};\mathbb{R}^p)$ for the space formed by all the continuously differentiable functions $\xi : \mathbb{R} \to \mathbb{R}^p$ such that $\| \xi \|_{C^1_g(\mathbb{R};\mathbb{R}^p)} = \sup_{s\in\mathbb{R}} g(s)(\| \xi(s) \| + \| \xi'(s) \|) < \infty$, endowed with the norm $\| \cdot \|_{C^1_g(\mathbb{R};\mathbb{R}^p)}$. The definition of $(C_g(\mathbb{R};\mathbb{R}^p), \| \cdot \|_{C_g(\mathbb{R};\mathbb{R}^p)})$ is similar.

This paper has three sections. In the next section we study the existence of a traveling wave front for (1.1). In the last section some examples are presented.

2. Existence of a traveling wave front

Let $\eta_1 < 0 < \eta_2$. In the next lemmas, for $\xi \in C(\mathbb{R};\mathbb{R})$ we use the notation $Y(\xi)$ and $Z(\xi)$ for the functions $Y(\xi), Z(\xi) : \mathbb{R} \to \mathbb{R}$ given by $Y(\xi)(t) = \int_{-\infty}^t e^{\eta_1(t-s)}\xi(s)ds$ and $Z(\xi)(t) = \int_t^\infty e^{\eta_2(t-s)}\xi(s)ds$. The proof of our first lemma is easy and we omit it.

Lemma 2.1. *Let $\xi \in C(\mathbb{R};\mathbb{R})$ and assume that $\lim_{t\to\pm\infty} \xi(t) = \beta_{\pm\infty}$. Then $\lim_{t\to\pm\infty} Y(\xi)(t) = -\frac{\beta_{\pm\infty}}{\eta_1}$, $\lim_{t\to\pm\infty} Z(\xi)(t) = \frac{\beta_{\pm\infty}}{\eta_2}$, the functions $Y(\xi)$, $Z(\xi)$ are differentiable, $Y(\xi)' = \eta_1 Y(\xi) + \xi$, $Z(\xi)' = \eta_2 Z(\xi) - \xi$ and $\lim_{t\to\pm\infty} Y(\xi)'(t) = \lim_{t\to\pm\infty} Z(\xi)'(t) = 0$. If, in addition, $\xi \in C^1(\mathbb{R},\mathbb{R})$ and ξ' is bounded, then $Y(\xi), Z(\xi) \in C^1(\mathbb{R},\mathbb{R})$, $Y(\xi)' = Y(\xi')$ and $Z(\xi)' = Z(\xi')$.*

Lemma 2.2. *If $0 < \theta < \min\{-\eta_1, \eta_2\}$, $r > 0$ and $g(\cdot) = e^{-\theta|\cdot|}$, then the map $W : B_r(0, C(\mathbb{R};\mathbb{R})) \subset C_g(\mathbb{R};\mathbb{R}) \to C^1_g(\mathbb{R};\mathbb{R})$ given by $W(\xi) = Y(\xi) + Z(\xi)$ is completely continuous.*

Proof. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $B_r(0, C(\mathbb{R};\mathbb{R}))$ and $\xi \in B_r(0, C(\mathbb{R};\mathbb{R}))$ such that $(\xi_n)_{n\in\mathbb{N}} \to \xi$ in $C_g(\mathbb{R};\mathbb{R})$. For $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $e^{-\theta|s|} \| \xi_n(s) - \xi(s) \| \leq \varepsilon$ for all $s \in \mathbb{R}$ and $n \geq N_\varepsilon$. For $n \geq N_\varepsilon$ and $t \in \mathbb{R}$, we get

$$\begin{aligned} e^{-\theta|t|} \mid Y(\xi_n)(t) - Y(\xi)(t) \mid &\leq \int_{-\infty}^t e^{\eta_1(t-s)} e^{-\theta|t|+\theta|s|} e^{-\theta|s|} \mid \xi_n(s) - \xi(s) \mid ds \\ &\leq \varepsilon \int_{-\infty}^t e^{\eta_1(t-s)} e^{-\theta|t|+\theta|s|} ds \\ &\leq \varepsilon[\frac{1}{-\eta_1 - \theta} + \frac{1}{-\eta_1 + \theta}], \end{aligned}$$

which proves that $Y : B_r(0, C(\mathbb{R};\mathbb{R})) \subset C_g(\mathbb{R};\mathbb{R}) \to C_g(\mathbb{R};\mathbb{R})$ is continuous.

We prove now that $Y(\cdot)$ is a compact map. From Lemma 2.1, it is easy to see that $\| Y'(\xi) \|_{C(\mathbb{R};\mathbb{R})} \leq 2r$ and $\| Y(\xi) \|_{C(\mathbb{R};\mathbb{R})} \leq \frac{r}{-\eta_1}$ for all $\xi \in B_r(0, C(\mathbb{R};\mathbb{R}))$, which implies that $Y(B_r(0, C(\mathbb{R};\mathbb{R})))_{|[-l,l]} = \{Y(\xi)_{|[-l,l]} : \xi \in B_r(0, C(\mathbb{R};\mathbb{R}))\}$ is relatively compact in $C([-l,l];\mathbb{R})$ for all $l > 0$.

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $B_r(0, C(\mathbb{R};\mathbb{R}))$. From the above remarks, there exists $\xi \in B_r(0, C(\mathbb{R};\mathbb{R}))$ and a subsequence of $(Y(\xi_n))_{n\in\mathbb{N}}$ (which we denote in the same form) such that $Y(\xi_n) \to \xi$ uniformly on compact set. Let $\varepsilon > 0$ be given. Let $K > 0$ and $N_\varepsilon \in \mathbb{N}$ such that $e^{-\theta K} 2\frac{r}{-\eta_1} \leq \frac{\varepsilon}{2}$ and $\| Y(\xi_n) - \xi \|_{C([-K,K];\mathbb{R})} \leq \frac{\varepsilon}{2}$ for all $n \geq N_\varepsilon$. Under these conditions, for $n \geq N_\varepsilon$ we see that

$$\| Y(\xi_n) - \xi \|_{C_g(\mathbb{R};\mathbb{R})} \leq \sup_{|s|\leq K} \mid Y(\xi_n)(s) - \xi(s) \mid + e^{-\theta K} \frac{2r}{-\eta_1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

which proves that $(Y(\xi_n))_{n\in\mathbb{N}} \to \xi$ in $C_g(\mathbb{R};\mathbb{R})$. Since $(\xi_n)_{n\in\mathbb{N}}$ is arbitrary, we infer that $Y(B_r(0, C(\mathbb{R};\mathbb{R})))$ is relatively compact in $C_g(\mathbb{R};\mathbb{R})$ and $Y : B_r(0, C(\mathbb{R};\mathbb{R})) \subset C_g(\mathbb{R};\mathbb{R}) \to C_g(\mathbb{R};\mathbb{R})$ is a compact map.

From the above, $Y : B_r(0, C(\mathbb{R};\mathbb{R}))\subset C_g(\mathbb{R};\mathbb{R}) \to C_g(\mathbb{R};\mathbb{R})$ is completely continuous and a similar procedure proves that $Z : B_r(0, C(\mathbb{R};\mathbb{R}))\subset C_g(\mathbb{R};\mathbb{R}) \to C_g(\mathbb{R};\mathbb{R})$ is completely continuous. Finally, since $W(\xi)' = \eta_1 Y(\xi) + \eta_2 Z(\xi)$ we can conclude that $W : B_r(0, C(\mathbb{R};\mathbb{R})) \subset C_g(\mathbb{R};\mathbb{R}) \to C^1_g(\mathbb{R};\mathbb{R})$ is completely continuous. □

From Hirsch et al. [9] and Gopalsamy [6] we note the followings results.

Lemma 2.3 ([9]). *If $v \in C^1(\mathbb{R}^+, \mathbb{R})$ and $v^+ = \limsup_{t\to\infty} v(t) > \liminf_{t\to\infty} v(t) = v^-$, then there exist sequences of real numbers $(t_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ such that $(t_n)_{n\in\mathbb{N}} \to \infty$, $(s_n)_{n\in\mathbb{N}} \to \infty$, $v'(t_n) = v'(s_n) = 0$ for all $n \in \mathbb{N}$, $v^+ = \limsup_{n\to\infty} v(t_n)$ and $v^- = \liminf_{n\to\infty} v(s_n)$.*

Lemma 2.4 ([6]). *If $v \in C^1(\mathbb{R}^+, \mathbb{R})$, $\lim_{t\to\infty} v(t)$ exists and $v'(\cdot)$ is uniformly continuous, then $\lim_{t\to\infty} v'(t) = 0$.*

Next, for $x \in \mathbb{R}^N$ we use the symbol $\widehat{x}$ for the function $\widehat{x} \in C([-c\tau, 0];\mathbb{R}^N)$ given by $\widehat{x}(\theta) = x$ for all $\theta \in [-c\tau, 0]$. For a function $\psi : \mathbb{R} \to \mathbb{R}^N$, we denote by ψ_+ and ψ_- the limits $\lim_{t\to\infty} \psi(t)$ and $\lim_{t\to-\infty} \psi(t)$, when the limit exists.

We include now the following lemma.

Lemma 2.5. *Assume that $\psi : \mathbb{R} \to \mathbb{R}^N$ is a twice continuously differentiable solution of (1.2), $\psi(\cdot)$ is bounded, monotone nondecreasing, the functions $F, (DG)$ takes bounded sets into bounded sets and $\{((DG)^c(\psi_t))^c(\psi'_t) : t \in \mathbb{R}\}$ is bounded. Then $\lim_{t\to\pm\infty} \psi'(t) = 0$, $F\left(\widehat{\psi_\pm}\right) = 0$ and $\lim_{t\to\pm\infty}((DG)^c(\psi_t))^c(\psi'_t) = 0$.*

Proof. To begin, we prove that ψ' is bounded. Assume that ψ' is unbounded on $[0, \infty)$ and let $i \in \{1, \ldots , N\}$ such that $\limsup_{t\to\infty} \psi'_i(t) = \infty$. If $\liminf_{t\to\infty} \psi'_i(t) = \infty$, then ψ_i is unbounded, which is absurd. If $\liminf_{t\to\infty} \psi'_i(t) < \infty$, from Lemma 2.3 there exists a sequence of real numbers $(t_n)_{n\in\mathbb{N}}$ such that $\psi''_i(t_n) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \psi'_i(t_n) = \infty$. Using this fact, we infer that

$$\lim_{t\to\infty} c[-\psi'_i(t_n) + (((DG)^c(\psi_{t_n}))^c(\psi'_{t_n}))_i = -F^c_i(\widehat{\psi_+}), \tag{2.1}$$

and $\lim_{n\to\infty}(((DG)^c(\psi_{t_n}))^c(\psi'_{t_n}))_i = \infty$, which is contrary to the assumptions. From the above, we have that ψ' is bounded on $[0, \infty)$. A similar argument proves that ψ' is bounded on $(-\infty, 0]$, which completes the proof that ψ' is bounded.

From the above and (1.2) we infer that ψ'' is bounded which implies that ψ' is uniformly continuous. Since $\lim_{t\to\pm\infty} \psi(s)$ exists, from Lemma 2.4 it follows that $\lim_{t\to\pm\infty} \psi'(t) = 0$, $\lim_{t\to\pm\infty} \psi'_t = \widehat{0}$ and $\lim_{t\to\pm\infty} ((DG)^c(\psi_t))^c(\psi'_t) = 0$. Moreover, from (1.2) we obtain that $\lim_{t\to\pm\infty} \mathcal{D}\psi''(t) = -F^c(\widehat{\psi_\pm})$, which allows us to conclude that ψ'' is uniformly continuous. Finally, from Lemma 2.4 we have that $F^c_i(\widehat{\psi_\pm}) = \lim_{t\to\pm\infty} \psi''(t) = 0$. □

To begin our studies on the existence of a traveling wave front for (1.1), we consider the quasi-monotone case.

2.1. **The quasi-monotone case.** By considering Lemma 2.5, in the remainder of this work we assume that there is $K \in \mathbb{R}^N$ such that $0 < K$, $F(\widehat{0}) = F(\widehat{K}) = G(\widehat{0}) = G(\widehat{K}) = 0$ and $F(\widehat{L}) \neq 0$ for all $0 < L < K$. Next, we always suppose that $F, G, (DG)$ are Lipschitz with Lipschitz constants L_F, L_G and L_{DG} respectively. We introduce now the next condition.

$\mathbf{H^1_{F,G}}$ There are diagonal matrices $\gamma = diag(\gamma_1, \dots, \gamma_n)$ and $\zeta = diag(\zeta_1, \dots, \zeta_n)$ such that $\gamma_i > 0, \zeta_i > 0$ for all $i = \dots, N$, and

$$[F_i^c(\psi) - F_i^c(\phi)] + \gamma_i(\psi_i(0) - \phi_i(0)) \geq 0, \tag{2.2}$$

$$\lambda_{1,i}c[G_i^c(\psi) - G_i^c(\phi)] + \zeta_i(\psi_i(0) - \phi_i(0)) \geq 0, \tag{2.3}$$

$$\lambda_{2,i}c[G_i^c(\psi) - G_i^c(\phi)] + \zeta_i(\psi_i(0) - \phi_i(0)) \geq 0, \tag{2.4}$$

for all $\psi, \phi \in C([-c\tau, 0]; \mathbb{R}^N)$ with $0 \leq \phi \leq \psi \leq K$, where $\lambda_{1,i} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}$, $\lambda_{2,i} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i}$ and $\beta_i = \gamma_i + \zeta_i$ for all $i \in \{1, \dots, N\}$.

From the general theory of traveling waves, we introduce the followings concepts.

Definition 2.1. A function $\overline{\rho} \in C^2(\mathbb{R}; \mathbb{R}^N)$ is called an upper solution of (1.2) if $\mathcal{D}\overline{\rho}''(t) - c\overline{\rho}'(t) + c\frac{d}{dt}G^c(\overline{\rho}_t) + F^c(\overline{\rho}_t) \leq 0$ for all $t \in \mathbb{R}$. The concept of lower solution of (1.2) is defined reversing the last inequality.

In the remainder of this section we always assume that the condition $\mathbf{H^1_{F,G}}$ is satisfied, θ is a real number such that $0 < \theta < \min\{-\lambda_{1,i}, \lambda_{2,i} : i = 1, \dots, N\}$, $g(\cdot) = e^{-\theta|\cdot|}$ and $\overline{\rho}, \underline{\rho}$ are an upper and a lower solution of (1.2) such that $0 \leq \underline{\rho} \leq \overline{\rho} \leq \widehat{K}$, $\underline{\rho} \neq 0$ and $\lim_{t \to -\infty} \overline{\rho}(t) = 0$. For $M > 0$, $U^M_{\underline{\rho},\overline{\rho}}$ is the set defined by

$$U^M_{\underline{\rho},\overline{\rho}} = \{\xi \in C^1(\mathbb{R}; \mathbb{R}^N) : \ 0 \leq \xi_i' \leq M, \ i = 1, \dots, N, \text{ and } \underline{\rho} \leq \xi \leq \overline{\rho}\}. \tag{2.5}$$

We introduce now the map $\Gamma : U^M_{\underline{\rho},\overline{\rho}} \subset C^1_g(\mathbb{R}; \mathbb{R}^N) \to C^1_g(\mathbb{R}; \mathbb{R}^N)$ given by

$$\begin{aligned}(\Gamma u)_i(t) = {} & \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)}(F_i^c(u_s) + (\gamma_i + \zeta_i)u_i(s) + c((DG_i)^c(u_s))^c(u_s')ds \\ & + \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)}(F_i^c(u_s) + (\gamma_i + \zeta_i)u_i(s) + c((DG_i)^c(u_s))^c(u_s'))ds,\end{aligned} \tag{2.6}$$

where $\theta_i = \frac{1}{d_i(\lambda_{2,i} - \lambda_{1,i})}$. The function $\Gamma u(\cdot)$ is a solution of

$$\begin{aligned}\mathcal{D}w''(\xi) - cw'(\xi) - (\gamma_i + \zeta_i)w(\xi) \\ = -F^c(u_\xi) - (\gamma_i + \zeta_i)u(\xi) - c((DG)^c(u_\xi))^c(u_\xi'), \ \ \xi \in \mathbb{R}.\end{aligned}$$

Using that $\frac{d}{ds}e^{\lambda_{j,i}(t-s)}G_i^c(u_s) = -\lambda_{j,i}e^{\lambda_{j,i}(t-s)}G_i^c(u_s) + e^{\lambda_{j,i}(t-s)}((DG_i)^c(u_s))^c(u_s')$ we obtain that $\Gamma u = \sum_{i=1}^4 \Gamma^i u$ where

$$(\Gamma^1 u)_i(t) = \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)}\widetilde{F}_i(u)(s)ds, \tag{2.7}$$

$$(\Gamma^2 u)_i(t) = \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)}\widetilde{F}_i(u)(s)ds, \tag{2.8}$$

$$(\Gamma^3 u)_i(t) = \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)}\widetilde{G}_i(u)(s)ds, \tag{2.9}$$

$$(\Gamma^4 u)_i(t) = \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)}\widehat{G}_i(u)(s)ds, \tag{2.10}$$

and $\widetilde{F}, \widetilde{G}, \widehat{G} : C(\mathbb{R}; \mathbb{R}^N) \to C(\mathbb{R}; \mathbb{R}^N)$ are given by $(\widetilde{F})_i(\psi)(s) = F_i^c(\psi_s) + \gamma_i\psi(s)$, $(\widetilde{G})_i(\psi)(s) = \lambda_{1,i}cG_i^c(\psi_s) + \zeta_i\psi(s)$ and $(\widehat{G})_i(\psi)(s) = \lambda_{2,i}cG_i^c(\psi_s) + \zeta_i\psi(s)$.

From [21, Lemma 3.1], we have the next result.

Lemma 2.6 ([21, Lemma 3.1])**.** *Assume that the condition $\mathbf{H^1_{F,G}}$ is satisfied, $\phi, \psi \in U^M_{\underline{\rho},\overline{\rho}}$ and $\phi \leq \psi$. If $H(\cdot)$ is some of the functions $\widetilde{F}, \widetilde{G}, \widehat{G}$, then $H(\phi)(s) \geq 0$ for all $s \in \mathbb{R}$, $H(\phi)(\cdot)$ is nondecreasing and $H(\phi)(t) \leq H(\psi)(t)$ for all $t \in \mathbb{R}$.*

Lemma 2.7. *If the condition $\mathbf{H^1_{F,G}}$ is satisfied and $u \in U^M_{\underline{\rho},\overline{\rho}}$, then Γu is nondecreasing and $\underline{\rho} \leq \Gamma\underline{\rho} \leq \Gamma u \leq \Gamma\overline{\rho} \leq \overline{\rho}$.*

Proof. Since $u_{s+h} \geq u_s$ for all $s \in \mathbb{R}, h > 0$, from Lemma 2.6 we have that $(\widetilde{G})_i(u)(s+h) - (\widetilde{G})_i(u)(s) \geq 0$ for all $s \in \mathbb{R}$. For $t \in \mathbb{R}$ and $h > 0$, we get

$$(\Gamma^3 u)_i(t+h) - (\Gamma^3 u)_i(t) = \theta_i \int_{-\infty}^{t} e^{\lambda_{1,i}(t-s)}[\widetilde{G}_i(u)(s+h) - \widetilde{G}_i(u)(s)]ds \geq 0,$$

which implies that $(\Gamma^3 u)_i(t+h) \geq (\Gamma^3 u)_i(t)$. The same argument allows us to show that $(\Gamma^j u)_i(t+h) \geq (\Gamma^j u)_i(t)$ for $j = 1, 2, 4$, $i = 1, \ldots, N$ and $t \in \mathbb{R}$, which proves that Γu is nondecreasing on $\mathbb{R}$.

We now prove the second assertion. Let $W = \overline{\rho} - \Gamma\overline{\rho}$. Since

$$\mathcal{D}(\Gamma\overline{\rho})''(t) - c(\Gamma\overline{\rho})'(t) - \beta(\Gamma\overline{\rho})(t) + F^c(\overline{\rho}_t) + \beta\overline{\rho}(t) + c((DG)^c(\overline{\rho}_t))^c(\overline{\rho}'_t) = 0,$$
$$\mathcal{D}\overline{\rho}''(t) - c\overline{\rho}'(t) - \beta\overline{\rho}(t) + F^c(\overline{\rho}_t) + \beta\overline{\rho}(t) + c((DG)^c(\overline{\rho}_t))^c(\overline{\rho}'_t) \leq 0,$$

for all $t \in \mathbb{R}$, we have that $\mathcal{D}W'' - cW' - \beta W(t) + \tau(t) = 0$ for some nonnegative bounded continuous function $\tau(\cdot)$. Since $W(\cdot)$ is a C^2 bounded function, we get

$$W_i(t) = \theta_i \int_{-\infty}^{t} e^{\lambda_{1,i}(t-s)}\tau_i(s)ds + \theta_i \int_{t}^{\infty} e^{\lambda_{2,i}(t-s)}\tau_i(s)ds, \qquad \forall\, t \in \mathbb{R},$$

which implies that $W_i(t) \geq 0$ and $\Gamma\overline{\rho} \leq \overline{\rho}$. A similar argument proves that $\Gamma(\underline{\rho}) \geq \underline{\rho}$.

On the other hand, noting $\overline{\rho}_t \geq u_t$ for all $t \in \mathbb{R}$, from Lemma 2.6 we see that $\widetilde{G}(\overline{\rho})(t) - \widetilde{G}(u)(t) \geq 0$ for all $t \in \mathbb{R}$ and

$$(\Gamma^3\overline{\rho})_i(t) - (\Gamma^3 u)_i(t) = \theta_i \int_{-\infty}^{t} e^{\lambda_{1,i}(t-s)}(\widetilde{G}_i(\overline{\rho})(s) - \widetilde{G}_i(u)(s))ds \geq 0,$$

which shows that $(\Gamma^3\overline{\rho})_i(t) - (\Gamma^3 u)_i(t) \geq 0$ for all $i = 1, \ldots, N$. A similar procedure proves that $(\Gamma^j\overline{\rho})_i(t) - (\Gamma^j u)_i(t) \geq 0$ for $j = 1, 2, 4$ and $i = 1, \ldots, N$. From the above we have that $\Gamma u \leq \Gamma\overline{\rho} \leq \overline{\rho}$. The proof that $\underline{\rho} \leq \Gamma\underline{\rho} \leq \Gamma u$ is similar. ☐

We can prove now our first theorem on the existence of a traveling wave for (1.1). In this result, $\widetilde{L(G)} = (2L_G \parallel \overline{\rho} \parallel_{C(\mathbb{R};\mathbb{R}^N)} + \sup_{s\in\mathbb{R}} \parallel (DG)^c(\overline{\rho_s}) \parallel_{\mathcal{L}(C([-c\tau,0];\mathbb{R}^N);\mathbb{R}^N)}$.

Theorem 2.1. *Let condition $\mathbf{H^1_{F,G}}$ hold and assume $2 \max_{i=1,\ldots,N}\{\theta_i\}c\widetilde{L(G)}\sqrt{N} < 1$. Then there exists a nondecreasing traveling wave front solution $u(\cdot)$ of the problem (1.1) such that $\lim_{t\to-\infty} u(t) = 0$ and $\lim_{t\to\infty} u(t) = K$.*

Proof. To begin, we select $M > 0$ large enough such that

$$2(\max_{i=1,\ldots,N}\{\theta_i\beta_i K_i\} + \max_{i=1,\ldots,N}\{\theta_i K_i\}L_F + \max_{i=1,\ldots,N}\{\theta_i\}c\widetilde{L(G)}M)\sqrt{N} < M. \tag{2.11}$$

Since the sets of functions $\{\beta_i u_i : u \in U^M_{\underline{\rho},\overline{\rho}}, i = 1, \ldots, N\}$ and $\{s \to H^c_i(u_s) : u \in U^M_{\underline{\rho},\overline{\rho}}, i = 1, \ldots, N, H_i = F_i, \lambda_{1,i}, cG^c_i, \lambda_{2,i}cG^c_i\}$ are bounded in $C(\mathbb{R};\mathbb{R}^N)$, from Lemma 2.2 we have that the map $\Gamma : U^M_{\underline{\rho},\overline{\rho}} \subset C^1_g(\mathbb{R};\mathbb{R}^N) \to C^1_g(\mathbb{R};\mathbb{R}^N)$ defined by

(2.7)-(2.10) is completely continuous. To prove that $\Gamma(U^M_{\underline{\rho},\overline{\rho}}) \subset U^M_{\underline{\rho},\overline{\rho}}$, we use the decomposition $\Gamma u = \sum_{j=1}^3 (\Upsilon^j u)_i$, where

$$(\Upsilon^1 u)_i(t) = \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} F_i^c(u_s)ds + \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)} F_i^c(u_s)ds,$$
$$(\Upsilon^2 u)_i(t) = \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} \beta_i u_i(s)ds + \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)} \beta_i u_i(s)ds,$$
$$(\Upsilon^3 u)_i(t) = \theta_i \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} \lambda_{1,i} c G_i^c(u_s)ds + \theta_i \int_t^\infty e^{\lambda_{2,i}(t-s)} \lambda_{2,i} c G_i^c(u_s)ds.$$

Let $u \in U^M_{\underline{\rho},\overline{\rho}}$. Using that $u \leq \overline{\rho} \leq K$ and $F(0) = 0$, from Lemma 2.1 we get

$$\mid (\Upsilon^2 u)_i'(t) \mid \leq \theta_i \beta_i K_i (-\lambda_{1,i} \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} ds + \lambda_{2,i} \int_t^\infty e^{\lambda_{2,i}(t-s)} ds) \leq 2\theta_i \beta_i K_i,$$
$$\mid (\Upsilon^1 u)_i'(t) \mid \leq \theta_i L_F K_i (-\lambda_{1,i} \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} ds + \lambda_{2,i} \int_t^\infty e^{\lambda_{2,i}(t-s)} ds) \leq 2\theta_i L_F K_i.$$

To estimate $\mid (\Upsilon^j u)_i'(t) \mid$, for $j = 3$, $i = 1, \ldots, N$, we note that

$$\begin{aligned} &\| c(DG)^c(u_t) \|_{\mathcal{L}(C([-c\tau;0];\mathbb{R}^N);\mathbb{R}^N)} \\ &\quad \leq \| c(DG)^c(u_t) - c(DG)^c(\overline{\rho_t}) \|_{\mathcal{L}(C([-c\tau;0];\mathbb{R}^N);\mathbb{R}^N)} \\ &\quad + c \| (DG)^c(\overline{\rho_t}) \|_{\mathcal{L}(C([-c\tau;0];\mathbb{R}^N);\mathbb{R}^N)} \\ &\quad \leq cL_G 2 \| \overline{\rho} \|_{C(\mathbb{R};\mathbb{R}^N)} + c \sup_{s\in\mathbb{R}} \| (DG)^c(\overline{\rho_s}) \|_{\mathcal{L}(C([-c\tau;0];\mathbb{R}^N);\mathbb{R}^N)} = c\widetilde{L(G)}, \end{aligned}$$

and $c \mid ((DG)^c(u_t))^c(u_t') \mid_{\mathbb{R}^N} \leq c\widetilde{L(G)}M$. Using now Lemma 2.1, we get

$$\begin{aligned} \mid (\Upsilon^3 u)_i'(t) \mid &\leq -\theta_i \lambda_{1,i} \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} \mid c((DG_i)^c(u_s))^c(u_s') \mid ds \\ &\quad + \theta_i \lambda_{2,i} \int_t^\infty e^{\lambda_{2,i}(t-s)} \mid c((DG_i)^c(u_s))^c(u_s') \mid ds \\ &\leq -\theta_i \lambda_{1,i} c\widetilde{L(G)}M \int_{-\infty}^t e^{\lambda_{1,i}(t-s)} ds + \theta_i \lambda_{2,i} c\widetilde{L(G)}M \int_t^\infty e^{\lambda_{2,i}(t-s)} ds \\ &\leq 2c\theta_i \widetilde{L(G)}M. \end{aligned}$$

From the above estimates and (2.11),

$$\begin{aligned} \| (\Gamma u)'(t) \| &= (\sum_{i=1}^N (\sum_{j=1}^3 (\Upsilon^j u)_i'(t))^2)^{\frac{1}{2}} \leq (\sum_{i=1}^N (2\theta_i \beta_i K_i + 2\theta_i L_F K_i + 2\theta_i c\widetilde{L(G)}M)^2)^{\frac{1}{2}} \\ &\leq 2(\max_{i=1,\ldots,N} \{\theta_i \beta_i K_i\} + \max_{i=1,\ldots,N} \{\theta_i K_i\} L_F + \max_{i=1,\ldots,N} \{\theta_i\} c\widetilde{L(G)}M)\sqrt{N}, \end{aligned} \tag{2.12}$$

which shows $\| (\Gamma u)'(t) \| \leq M$ for all $t \in \mathbb{R}$. Moreover, from Lemma 2.7 we have that Γu is nondecreasing and $\underline{\rho} \leq \Gamma u \leq \overline{\rho}$, which complete the proof that $\Gamma(U^M_{\underline{\rho},\overline{\rho}}) \subset U^M_{\underline{\rho},\overline{\rho}}$.

From the above remarks, there exists $u \in U^M_{\underline{\rho},\overline{\rho}}$ such that $\Gamma u = u$. Since $u(\cdot)$ is nondecreasing and $\underline{\rho} \leq \Gamma \underline{\rho} \leq u = \Gamma u \leq \Gamma \overline{\rho} \leq \overline{\rho}$, we have that $u_- = \lim_{t\to-\infty} u(t) = 0$. Moreover, using that $u'(\cdot)$ is bounded and $\underline{\rho} \neq 0$, from Lemma 2.5 we infer that $F(\widehat{u}_+) = 0$ and $u_+ = K$. This completes the proof. □

Our result depends on the existence of upper and lower solutions, which is usually a nontrivial problem. Considering this fact and the developments in [16], we introduced the concepts of super and sub-solutions for the problem (1.2).

Definition 2.2. A function $\rho \in C(\mathbb{R};\mathbb{R}^N)$ is called a super solution of (1.2) if there exist numbers $T_1,\dots,T_m$ such that ρ'' is continuous on $\mathbb{R}\setminus\{T_1,\dots,T_m\}$, ρ' and ρ'' are bounded, the function $t\to G^c(\rho_t)$ is differentiable a.e. on $\mathbb{R}$ and $\mathcal{D}\rho''(t)-c\rho'(t)+c\frac{d}{dt}G^c(\rho_t)+F^c(\rho_t)\le 0$ a.e. on $\mathbb{R}$. A sub-solution is defined in the same form by reversing the last inequality.

Remark 2.1. Arguing as in the proof of [16, Lemma 2.5], we can prove that if ρ is a super-solution of (1.2) such that $\rho'(t^+)\le\rho'(t^-)$ for all $t\in\mathbb{R}$ (resp. if ϱ is a sub-solution of (1.2) such that $\varrho'(t^+)\ge\varrho'(t^-)$ for all $t\in\mathbb{R}$), then $\Gamma\rho$ (resp. $\Gamma\varrho$) is an upper solution (resp. a lower) solution of (1.2). Moreover, from the proof of [16, Lemma 2.5] we also infer that $\Gamma(\rho)\le\rho$ (resp. $\Gamma(\varrho)\ge\varrho$).

2.2. **The nonquasi-monotone case.** To prove the results of this section and considering the results in [21], we introduce the following condition:

$\mathbf{H^2_{F,G}}$ There are positive matrix $\gamma = diag(\gamma_1,\dots,\gamma_n)$, $\zeta = diag(\zeta_1,\dots,\zeta_n)$ such that $G^c(\phi)-G^c(\psi)\le 0$,

$$[F^c_i(\phi)-F^c_i(\psi)]+\gamma_i(\phi(0)-\psi(0))\ge 0, \tag{2.13}$$

$$\lambda_{2,i}c[G^c_i(\phi)-G^c_i(\psi)]+\zeta_i(\phi_i(0)-\psi_i(0))\ge 0, \tag{2.14}$$

for all $\psi,\phi\in C([-c\tau,0];\mathbb{R}^N)$ such that $0\le\psi\le\phi\le K$ and the function $e^{(\gamma+\zeta)(\cdot)}[\phi(\cdot)-\psi(\cdot)]$ is nondecreasing on $[-c\tau,0]$.

Remark 2.2. In what follows, for $v,w\in C(\mathbb{R};\mathbb{R}^N)$ and $s>0$, we use the notation v^s and $\mathcal{L}_{v,w}$ for the functions $v^s:\mathbb{R}\to\mathbb{R}^N$ and $\mathcal{L}_{v,w}:\mathbb{R}\to\mathbb{R}^N$ given by $v^s(t)=v(t+s)$ and $(\mathcal{L}_{v,w})_i(t)=e^{\beta_i t}[v_i(t)-w_i(t)]$. We also introduce the set

$$S^M_{\underline{\rho},\overline{\rho}}=\{\phi\in U^M_{\underline{\rho},\overline{\rho}}:\mathcal{L}_{\overline{\rho},\phi},\mathcal{L}_{\phi,\underline{\rho}},\mathcal{L}_{\phi^s,\phi}\text{ are nondecreasing on }\mathbb{R}\text{ for all}\, s>0\}. \tag{2.15}$$

To prove our next theorem we need some additional lemmas.

Lemma 2.8. *Let $u\in S^M_{\underline{\rho},\overline{\rho}}$ and $s>0$. If $c>1-\min\{\beta_i d_i : i=1,\dots N\}$ and the condition $\mathbf{H^2_{F,G}}$ is verified, then $\mathcal{L}_{\overline{\rho},\Gamma u}$, $\mathcal{L}_{\Gamma u,\underline{\rho}}$ and $\mathcal{L}_{(\Gamma u)^s,\Gamma u}$ are nondecreasing.*

Proof. To begin we prove that $\mathcal{L}_{(\Gamma u)^s,\Gamma u}$ is nondecreasing. For $t\in\mathbb{R}$ we see that

$$\begin{aligned}
(\frac{d}{dt}(\mathcal{L}_{(\Gamma u)^s,\Gamma u})(t))_i &= e^{\beta_i t}\theta_i(\beta_i+\lambda_{1,i})\int_{-\infty}^t e^{\lambda_{1,i}(t-\tau)}(\widetilde{F}^c_i(u)(\tau+s)-\widetilde{F}^c_i(u)(\tau))d\tau\\
&+e^{\beta_i t}\theta_i(\beta_i+\lambda_{2,i})\int_t^{\infty} e^{\lambda_{2,i}(t-\tau)}(\widetilde{F}^c_i(u)(\tau+s)-\widetilde{F}^c_i(u)(\tau))d\tau\\
&+e^{\beta_i t}\theta_i(\beta_i+\lambda_{1,i})\int_{-\infty}^t e^{\lambda_{1,i}(t-\tau)}(\widetilde{G}^c_i(u)(\tau+s)-\widetilde{G}^c_i(u)(\tau))d\tau\\
&+e^{\beta_i t}\theta_i(\beta_i+\lambda_{2,i})\int_t^{\infty} e^{\lambda_{2,i}(t-\tau)}(\widehat{G}^c_i(u)(\tau+s)-\widehat{G}^c_i(u)(\tau))d\tau\\
&+e^{\beta_i t}\theta_i(\widetilde{G}^c_i(u)(t+s)-\widetilde{G}^c_i(u)(t))\\
&-e^{\beta_i t}\theta_i(\widehat{G}^c_i(u)(t+s)-\widehat{G}^c_i(u)(t)).
\end{aligned}$$

From condition $\mathbf{H^2_{F,G}}$ and the fact that $\lambda_{1,i}c<0$, we have that $(\widetilde{F}^c_i(u)(\tau+s)-\widetilde{F}^c_i(u)(\tau))\geq 0$, $(\widetilde{G}^c_i(u)(\tau+s)-\widetilde{G}^c_i(u)(\tau))\geq 0$ and $(\widehat{G}^c_i(u)(\tau+s)-\widehat{G}^c_i(u)(\tau))\geq 0$ for all $\tau\in\mathbb{R}$ and $i=1,\ldots,N$. Moreover, since $c>1-\min\{\ \beta_i d_i : i=1,\ldots,N\}$, we note that $(\beta_i+\lambda_{j,i})\geq 0$ for $j=1,2$ and $i=1,\ldots,N$, which allows us to conclude that the first four terms in the previous decomposition are nonnegative. In addition, from condition $\mathbf{H^2_{F,G}}$ we observe that

$$\begin{aligned}e^{\beta_i t}\theta_i(\widetilde{G}^c_i(u)(t+s)-\widetilde{G}^c_i(u)(t))-e^{\beta_i t}\theta_i(\widehat{G}^c_i(u)(t+s)-\widehat{G}^c_i(u)(t))\\ &= e^{\beta_i t}\theta_i(\lambda_{1,i}-\lambda_{2,i})(cG^c_i(u)(t+s)-cG^c_i(u)(t))\\ &= -\frac{ce^{\beta_i t}}{d_i}(G^c_i(u)(t+s)-G^c_i(u)(t))\geq 0.\end{aligned}$$

From the above remarks we obtain that $\frac{d}{dt}(\mathcal{L}_{(\Gamma u)^s,\Gamma u}(t))_i\geq 0$, which shows that $\mathcal{L}_{(\Gamma u)^s,\Gamma u}$ is nondecreasing.

To prove that $\mathcal{L}_{\overline{\rho},\Gamma u}$ is nondecreasing, we note that $\mathcal{L}_{\overline{\rho},\Gamma u}=\mathcal{L}_{\overline{\rho},\Gamma\overline{\rho}}+\mathcal{L}_{\Gamma\overline{\rho},\Gamma u}$ and we show that $\mathcal{L}_{\overline{\rho},\Gamma\overline{\rho}}$ and $\mathcal{L}_{\Gamma\overline{\rho},\Gamma u}$ are nondecreasing. Arguing as above, we have that

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}_{\Gamma\overline{\rho},\Gamma u})_i &= e^{\beta_i t}\theta_i(\beta_i+\lambda_{1,i})\int_{-\infty}^t e^{\lambda_{1,i}(t-\tau)}(\widetilde{F}^c_i(\overline{\rho})(\tau)-\widetilde{F}^c_i(u)(\tau))d\tau\\ &\quad+e^{\beta_i t}\theta_i(\beta_i+\lambda_{2,i})\int_t^{\infty} e^{\lambda_{2,i}(t-\tau)}(\widetilde{F}^c_i(\overline{\rho})(\tau)-\widetilde{F}^c_i(u)(\tau))d\tau\\ &\quad+e^{\beta_i t}\theta_i(\beta_i+\lambda_{1,i})\int_{-\infty}^t e^{\lambda_{1,i}(t-\tau)}(\widetilde{G}^c_i(\overline{\rho})(\tau)-\widetilde{G}^c_i(u)(\tau))d\tau\\ &\quad+e^{\beta_i t}\theta_i(\beta_i+\lambda_{2,i})\int_t^{\infty} e^{\lambda_{2,i}(t-\tau)}(\widehat{G}^c_i(\overline{\rho})(\tau)-\widehat{G}^c_i(u)(\tau))d\tau\\ &\quad-\frac{e^{\beta_i t}c}{d_i}(G^c_i(\overline{\rho})(t)-G^c_i(u)(t)),\end{aligned}$$

which allows us to conclude that $\mathcal{L}_{\Gamma\overline{\rho},\Gamma u}$ is nondecreasing. □

We study now the function $\mathcal{L}_{\overline{\rho},\Gamma\overline{\rho}}$. Let $w=\overline{\rho}-\Gamma\overline{\rho}$. Using that $\overline{\rho}$ is an upper solution, we have that there exists a nonnegative bounded integrable function $h=(h_1,\ldots,h_N):\mathbb{R}\to\mathbb{R}^N$ such that $\mathcal{D}w''(\xi)-cw'(\xi)-\beta w(\xi)+h(\xi)=0$ for all $\xi\in\mathbb{R}$. From the above, there exist real numbers q_i,l_i, $i=1,\ldots,N$ such that

$$w_i(t)=p_ie^{\lambda_{1,i}t}+l_ie^{\lambda_{2,i}t}+\theta_i\int_{-\infty}^t e^{\lambda_{1,i}(t-s)}h_i(s)ds+\theta_i\int_t^{\infty} e^{\lambda_{2,i}(t-s)}h_i(s)ds. \tag{2.16}$$

Since the functions w_i are bounded, we have that $p_i=l_i=0$ for all i and

$$w_i(t)=\theta_i\int_{-\infty}^t e^{\lambda_{1,i}(t-s)}h_i(s)ds+\theta_i\int_t^{\infty} e^{\lambda_{2,i}(t-s)}h_i(s)ds,\quad \forall\, t\in\mathbb{R}. \tag{2.17}$$

Using this representation, we obtain that

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}_{\overline{\rho},\Gamma\overline{\rho}})_i(t) &= e^{\beta_i t}(\beta_i+\lambda_{1,i})\theta_i\int_{-\infty}^t e^{\lambda_{1,i}(t-s)}h_i(s)ds\\ &\quad+e^{\beta_i t}(\beta_i+\lambda_{2,i})\theta_i\int_t^{\infty} e^{\lambda_{2,i}(t-s)}h_i(s)ds,\quad\forall\, t\in\mathbb{R},\end{aligned}$$

which permit us to conclude that $\mathcal{L}_{\overline{\rho},\Gamma\overline{\rho}}$ is nondecreasing and completes the proof that $\mathcal{L}_{\overline{\rho},\Gamma u}$ in nondecreasing.

Arguing as above and using that $\underline{\rho}$ is a lower solution, we can prove that $\mathcal{L}_{\Gamma u,\Gamma\underline{\rho}}$ $\mathcal{L}_{\Gamma\underline{\rho},\underline{\rho}}$ and $\mathcal{L}_{\Gamma u,\underline{\rho}}$ are nondecreasing. This completes the proof. $\square$

The proof of the next lemma follows from the proof of [21, Lemma 4.1].

Lemma 2.9. *Assume $c > 1 - \min\{\beta_i d_i : i = 1, \dots N\}$ and the condition $\mathbf{H^2_{F,G}}$ is verified. If $u \in S^M_{\underline{\rho},\overline{\rho}}$, then $\underline{\rho} \leq \Gamma\underline{\rho} \leq \Gamma u \leq \Gamma\overline{\rho} \leq \overline{\rho}$ and Γu is nondecreasing on $\mathbb{R}$.*

Proof. Since $\mathcal{L}_{\overline{\rho},u}$ and $\mathcal{L}_{u^s,u}$ are nondecreasing, from the proof of [21, Lemma 4.1 (ii),(iii)] it follows that $H(\underline{\rho}) \leq H(u) \leq H(\overline{\rho})$ and $H(u)$ is nondecreasing for $H = \widetilde{F}, \widetilde{G}, \widehat{G}$. From the above and the definition of Γ it is easy to see that $\Gamma\underline{\rho} \leq \Gamma u \leq \Gamma\overline{\rho}$. Moreover, from the proof of Lemma 2.8 (see (2.17)) we have that

$$\overline{\rho}(t) - \Gamma\overline{\rho}(t) = \theta_i \int_{-\infty}^{t} e^{\lambda_{1,i}(t-s)} h_i(s) ds + \theta_i \int_{t}^{\infty} e^{\lambda_{2,i}(t-s)} h_i(s) ds, \tag{2.18}$$

where $h_i(\cdot)$ is a nonnegative bounded integrable function. This implies that $\Gamma\overline{\rho} \leq \overline{\rho}$. The proof that $\underline{\rho} \leq \Gamma\underline{\rho}$ is similar. This completes the proof. $\square$

In the next theorem, $\widetilde{L(G)}$ is the number introduced in Theorem 2.1.

Theorem 2.2. *If $c > 1 - \min\{\beta_i d_i : i = 1, \dots, N\}$, the condition $\mathbf{H^2_{F,G}}$ is satisfied and $2 \max_{i=1,\dots,N}\{\theta_i\} c\widetilde{L(G)}\sqrt{N} < 1$, then there exists a nondecreasing traveling wave solution $u(\cdot)$ of (1.1) such that $\lim_{t\to-\infty} u(t) = 0$ and $\lim_{t\to\infty} u(t) = K$.*

Proof. Let $M > 0$ and $\Gamma : S^M_{\underline{\rho},\overline{\rho}} \subset C^1_g(\mathbb{R};\mathbb{R}^N) \to C^1_g(\mathbb{R};\mathbb{R}^N)$ be defined as in the proof of Theorem 2.1. It is easy to see that $S^M_{\underline{\rho},\overline{\rho}}$ is a closed and convex subset of $U^M_{\underline{\rho},\overline{\rho}}$ and from the proof of Theorem 2.1 we infer that $\| (\Gamma\xi)' \| \leq M$ for all $\xi \in S^M_{\underline{\rho},\overline{\rho}}$ and that Γ is completely continuous. Moreover, from Lemma 2.8 and Lemma 2.9 it follows that $\Gamma(S^M_{\underline{\rho},\overline{\rho}}) \subset S^M_{\underline{\rho},\overline{\rho}}$, which implies that Γ has a fixed point $u \in S^M_{\underline{\rho},\overline{\rho}}$.

From the above, $u(\cdot)$ is nondecreasing and $\underline{\rho} \leq u \leq \overline{\rho}$, which implies that $u_+ = \lim_{t\to\infty} u(t)$ exists and $u_- = \lim_{t\to-\infty} u(t) = 0$. Finally, since $u'(\cdot)$ is bounded and $\underline{\rho} \neq 0$, from Lemma 2.5 we obtain that $F(\widehat{u}_+) = 0$ and $u_+ = K$. $\square$

3. Examples

In this section we present some examples motivated by ordinary neutral differential equations arising in population dynamic; see [2, 5, 6, 11–13]. For sake of simplicity, we assume $N = d = 1$ and η is a positive number. To begin, we study the neutral problem

$$\frac{d}{dt}[u(t,x) + \eta u(t-\tau,x)] = \Delta u(t,x) + u(t,x)(1 - u(t-\tau,x)), \;\; t \in \mathbb{R}, \; x \in \mathbb{R}. \tag{3.1}$$

To study this problem, we consider the equation

$$w''(t) - cw'(t) - \eta c w'(t - c\tau) + w(t)[1 - w(t - \tau c)] = 0, \qquad t \in \mathbb{R}, \tag{3.2}$$

submitted to the condition

$$\lim_{t\to-\infty} w(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} w(t) = 1. \tag{3.3}$$

Let $F^c(\cdot)$ and $G^c(\cdot)$ be given by $F^c(\phi) = \phi(0)[1 - \phi(-\tau c)]$ and $G^c(\phi) = -\eta\phi(-\tau c)$. Next, we study the condition $\mathbf{H^2_{F,G}}$ and we construct a super- and a sub-solution.

If ϕ, ψ are the function in condition $\mathbf{H^2_{F,G}}$, we note that

$$G^c(\phi) - G^c(\psi) = -\eta(\phi(-\tau c) - \psi(-\tau c)) \le 0, \tag{3.4}$$

$$\begin{aligned} F^c(\phi) - F^c(\psi) &\ge (\phi(0) - \psi(0))(1 - \phi(-\tau c) - \psi(0)e^{\beta\tau c}) \\ &\ge -(\phi(0) - \psi(0))e^{\beta\tau c}. \end{aligned} \tag{3.5}$$

From (3.5) we have that (2.13) is satisfied if $\gamma - e^{\beta\tau c} \ge 0$. For simplicity, we take $c > 2$, $\zeta = \gamma > 1$, $\beta = \gamma + \zeta$ and we assume τ small so that $\frac{\beta}{2} - e^{\beta\tau c} = \gamma - e^{\beta\tau c} \ge 0$. Moreover, for $\lambda_{1,1} = \frac{c-\sqrt{c^2+4\beta}}{2}$ and $\lambda_{2,1} = \frac{c+\sqrt{c^2+4\beta}}{2}$, we suppose $\eta > 0$ small such that $\frac{\beta}{2} - \lambda_{2,1}c\eta e^{\beta\tau c} = \zeta - \lambda_{2,1}c\eta e^{\beta\tau c} \ge 0$. Under these conditions,

$$\begin{aligned} &\lambda_{2,1}c[G^c(\phi) - G^c(\psi)] + \zeta(\phi(0) - \psi(0)) \\ &\qquad = -\lambda_{2,1}c\eta[\phi(-\tau c) - \psi(-\tau c)] + \zeta(\phi(0) - \psi(0)) \\ &\qquad \ge (-\lambda_{2,1}c\eta e^{\beta\tau c} + \frac{\beta}{2})(\phi(0) - \psi(0)) \ge 0. \end{aligned}$$

From the above remarks we have that the condition $\mathbf{H^2_{F,G}}$ is satisfied.

To obtain an upper and a lower solution, we construct a super-solution ρ and sub-solution ϱ such that $\rho'(t^+) \le \rho'(t^-)$ and $\varrho'(t^+) \ge \varrho'(t^-)$ for all $t \in \mathbb{R}$; see Remark 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(\lambda) = \lambda^2 - (c + \eta c e^{-\tau c\lambda})\lambda + 1$ and $\lambda_1 = \frac{c-\sqrt{c^2-4}}{2}$. Since $f(\lambda_1) = -\eta c e^{-\tau c\lambda_1} < 0$ and $f(0) = 1$, there exists $\vartheta_1 \in (0, \lambda_1)$ such that $f(\vartheta_1) = 0$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be given by $\rho(t) = \min\{e^{\vartheta_1 t}, 1\}$. For $t \le 0$, we see that

$$\begin{aligned} &\rho''(t) - c\rho'(t) - \eta c\rho'(t - \tau c) + F(\rho_t) \\ &\qquad = e^{\vartheta_1 t}[\vartheta_1^2 - (c + \eta c e^{-\tau c\vartheta_1})\vartheta_1 + 1] - \rho(t)\rho(t - \tau c) \\ &\qquad = -\rho(t)\rho(t - \tau c) \le 0, \end{aligned}$$

which permit us to conclude that ρ is a super-solution.

We now construct a sub-solution. Noting that $2\vartheta_1 - c < 2\lambda_1 - c < 0$ and assuming η small enough, we have that $f'(\vartheta_1) = 2\vartheta_1 - c + \eta c(\vartheta_1\tau c - 1)e^{-\vartheta_1\tau c} < 0$. In this case, we select $\vartheta_1 > \varepsilon > 0$ small and $M > 1$ large such that $f(\vartheta_1 + \varepsilon) < 0$ and $-Mf(\vartheta_1 + \varepsilon) - 1 > 0$. Let $\varrho : \mathbb{R} \to \mathbb{R}$ be given by $\varrho(t) = \max\{e^{\vartheta_1 t}(1 - Me^{\varepsilon t}), 0\}$ and $t^* < 0$ such that $\varrho(t^*) = 0$. For $t \le t^*$, we get

$$\begin{aligned} &\varrho''(t) - c\varrho'(t) - \eta c\varrho'(t - \tau c) + F^c(\varrho_t) \\ &= e^{\vartheta_1 t}[\vartheta_1^2 - c\vartheta_1 + 1] - Me^{(\vartheta_1+\varepsilon)t}[(\vartheta_1 + \varepsilon)^2 - c(\vartheta_1 + \varepsilon) + 1] \\ &\quad + [-\eta c\vartheta_1 e^{\vartheta_1 t}e^{-\vartheta_1\tau c} + M\eta c(\vartheta_1 + \varepsilon)e^{(\vartheta_1+\varepsilon)t}e^{-(\vartheta_1+\varepsilon)\tau c}] - \varrho(t)\varrho(t - \tau c) \\ &\ge e^{\vartheta_1 t}[\vartheta_1^2 - (c + \eta c e^{-\vartheta_1\tau c})\vartheta_1 + 1] \\ &\quad - Me^{(\vartheta_1+\varepsilon)t}[(\vartheta_1 + \varepsilon)^2 - (c + \eta c e^{-(\vartheta_1+\varepsilon)\tau c})(\vartheta_1 + \varepsilon) + 1] \\ &\quad - e^{2\vartheta_1 t}e^{-\vartheta_1\tau c}(1 - Me^{\varepsilon t})(1 - Me^{\varepsilon(t-\tau c)}) \\ &\ge -Me^{(\vartheta_1+\varepsilon)t}[(\vartheta_1 + \varepsilon)^2 - (c + \eta c e^{-(\vartheta_1+\varepsilon)\tau c})(\vartheta_1 + \varepsilon) + 1] - e^{2\vartheta_1 t}e^{-\vartheta_1\tau c} \\ &\ge e^{(\vartheta_1+\varepsilon)t}[-Mf(\vartheta_1 + \varepsilon) - e^{(\vartheta_1-\varepsilon)t}] \ge e^{(\vartheta_1+\varepsilon)t}[-Mf(\vartheta_1 + \varepsilon) - 1] \ge 0, \end{aligned}$$

and hence, ϱ is a sub-solution. Moreover, it is easy to see that $0 \le \varrho \le \rho \le 1$, $\rho'(t^+) \le \rho'(t^-)$ and $\varrho'(t^+) \ge \varrho'(t^-)$ for all $t \in \mathbb{R}$, which implies that there exists an upper and a lower solution $\overline{\rho}, \underline{\rho}$ verifying the general assumptions in Section 2.

From the above and Theorem 2.2, we have the next result. In this result, the condition $\frac{c\eta}{\sqrt{c^2+4\beta}} < 1$ is concerning the inequality $2 \max_{i=1,\dots,N}\{\theta_i\}c\widetilde{L(G)}\sqrt{N} < 1$ inTheorem 2.2.

Proposition 3.1. *Let* $\zeta = \gamma > 1$, $\beta = \gamma + \zeta$, $c > 2$ *and assume* τ, η *are small enough such that* $\beta - 2e^{\beta\tau c} \geq 0$, $\beta - 2\lambda_{2,1}\eta c e^{\beta\tau c} \geq 0$ *and* $\frac{c\eta}{\sqrt{c^2+4\beta}} < 1$. *Then there exists a nondecreasing traveling wave front solution of* (3.1) *satisfying* (3.3).

In the next example we study the existence of a traveling wave for the problem

$$\text{(3.6)} \quad \frac{d}{dt}[u(t,x)+\eta u(t,x)u(t-\tau,x)] = \Delta u(t,x)+u(t)[1-u(t-\tau,x)], \ \ t \in \mathbb{R}, x \in \mathbb{R}.$$

To this end, we study the equation

$$\text{(3.7)} \qquad w''(t) - cw'(t) - \eta c(w(t)w(t-c\tau))' + w(t)[1 - w(t-\tau c)] = 0, \qquad t \in \mathbb{R},$$

submitted to the condition (3.3). Next, $F^c(\cdot)$ is the function introduced in the first example and $G^c(\cdot)$ is given by $G^c(\psi) = -\eta\psi(0)\psi(-\tau c)$.

Let $\gamma = \zeta > 1$, $\beta = \gamma + \zeta$ and $c > 2$, and assume τ, η small enough such that $\beta - 2e^{\beta\tau c} \geq 0$ and $\beta - 2\lambda_{2,1}c\eta(1 + e^{\beta\tau c}) \geq 0$. From the first example, we infer that the inequality (2.13) is satisfied. Moreover, if ϕ, ψ are the functions in condition $\mathbf{H^2_{F,G}}$, we get

$$\begin{aligned}
[G^c(\phi) - G^c(\psi)] &= -\eta[(\phi(0) - \psi(0))\phi(-\tau c) + \psi(0)(\phi(-\tau c) - \psi(-\tau c))] \leq 0. \\
c[G^c(\phi) - G^c(\psi)] &\geq -\eta c[(\phi(0) - \psi(0))\phi(-\tau c) + \psi(0)e^{\beta\tau c}(\phi(0) - \psi(0))] \\
&\geq -\eta c(\phi(0) - \psi(0))(\phi(-\tau c) + \psi(0)e^{\beta\tau c}) \\
&\geq -\eta c(\phi(0) - \psi(0))(1 + e^{\beta\tau c}),
\end{aligned}$$

which implies that (2.14) is verified since $\beta - 2c\lambda_{2,1}\eta(1+e^{\beta\tau c}) \geq 0$. From the above we have that the condition $\mathbf{H^2_{F,G}}$ is satisfied. Next, we construct a super- and a sub-solution.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be defined by $\rho(t) = \min\{e^{\lambda_1 t}, 1\}$. For $t \geq 0$, we note that

$$\begin{aligned}
&d\rho''(t) - c\rho'(t) - ac(\rho(t)\rho(t-\tau c))' + F(\rho_t) \\
&\qquad \leq d\rho''(t) - c\rho'(t) + F(\rho_t) = e^{\lambda_1 t}[d\lambda_1^2 - c\lambda_1 + 1] - \rho(t)\rho(t-\tau c) \\
&\qquad = -\rho(t)\rho(t-\tau c) \leq 0,
\end{aligned}$$

and hence, ρ is a super-solution of (3.7).

Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(\lambda) = \lambda^2 - c\lambda + 1$ and $0 < \varepsilon < \lambda_1$ such that $\lambda_1 + \varepsilon \leq \frac{c}{2}$ and $g(\lambda_1 + \varepsilon) < 0$. Let $M > 1$ such that $-Mg(\lambda_1 + \varepsilon) > 1$, $\varrho : \mathbb{R} \to \mathbb{R}$ be the function given by $\varrho(t) = \max\{e^{\lambda_1 t}(1 - Me^{\varepsilon t}), 0\}$ and $t^* < 0$ such that $\varrho(t^*) = 0$. For $t \leq t^*$, we get

$$\begin{aligned}
\frac{d}{dt}[-\eta c\varrho(t)\varrho(t-c\tau)] &\geq \eta c[-2\lambda_1 e^{2\lambda_1 t}e^{-\lambda_1\tau c} - 2M^2(\lambda_1+\varepsilon)e^{2(\lambda_1+\varepsilon)t}e^{-(\lambda_1+\varepsilon)\tau c}] \\
&\geq \eta c[-2M^2(\lambda_1+\varepsilon)e^{2\lambda_1 t} - 2M^2(\lambda_1+\varepsilon)e^{(2\lambda_1+\varepsilon)t}] \\
&= -2\eta c(\lambda_1+\varepsilon)M^2(e^{2\lambda_1 t} + e^{(2\lambda_1+\varepsilon)t}) \\
&\geq -4\eta cM^2(\lambda_1+\varepsilon)e^{2\lambda_1 t},
\end{aligned}$$

$$-\varrho(t)\varrho(t-\tau c) = -e^{\lambda_1 t}(1 - Me^{\varepsilon t})e^{\lambda_1(t-\tau c)}(1 - Me^{\varepsilon(t-\tau c)}) \geq -e^{(\lambda_1+\varepsilon)t}e^{(\lambda_1-\varepsilon)t}.$$

From the above, we have that

$$\begin{aligned}
&\varrho''(t) - c\varrho'(t) - [\eta c\varrho(t)\varrho(t - c\tau)]' + F^c(\varrho_t)\\
&\quad \geq e^{\lambda_1 t}[\lambda_1^2 - c\lambda_1 + 1] - e^{(\lambda_1+\varepsilon)t}M[(\lambda_1 + \varepsilon)^2 - c(\lambda_1 + \varepsilon) + 1]\\
&\quad\quad - 4\eta c M^2(\lambda_1 + \varepsilon)e^{2\lambda_1 t} - \varrho(t)\varrho(t - \tau c)\\
&\quad \geq -e^{(\lambda_1+\varepsilon)t}M[(\lambda_1 + \varepsilon)^2 - c(\lambda_1 + \varepsilon) + 1] - 4\eta c M^2(\lambda_1 + \varepsilon)e^{2\lambda_1 t}\\
&\quad\quad - e^{(\lambda_1+\varepsilon)t}e^{(\lambda_1-\varepsilon)t}\\
&\quad \geq e^{(\lambda_1+\varepsilon)t}[-Mg(\lambda_1 + \varepsilon) - 4\eta c M^2(\lambda_1 + \varepsilon)e^{(\lambda_1-\varepsilon)t} - 1].
\end{aligned}$$

Thus, if η is sufficiently small such that $-Mg(\lambda_1 + \varepsilon) - 1 - 4\eta c M^2(\varepsilon + \lambda_1) > 0$, we have that ϱ is a sub-solution of (3.7). Moreover, we note that $0 \leq \varrho \leq \rho \leq 1$, $\rho'(t^+) \leq \rho'(t^-)$ and $\varrho'(t^+) \geq \varrho'(t^-)$ for all $t \in \mathbb{R}$.

The next result follows from Theorem 2.2. In this result, the condition $\frac{12c\eta}{\sqrt{c^2+4\beta}} < 1$ is equivalent to the inequality $2 \max_{i=1,\dots,N}\{\theta_i\} c\widetilde{L(G)}\sqrt{N} < 1$ in Theorem 2.2.

Proposition 3.2. *Let $\zeta > 1$, $\gamma = \zeta$, $\beta = \gamma+\zeta$ and $c > 2$. Let M, λ_1 and ε be defined as above. Assume τ, η small enough such that $\beta - 2e^{\beta\tau c} \geq 0$, $\beta - 2\lambda_{2,1}c\eta(1+e^{\beta\tau c}) \geq 0$, $-Mg(\lambda_1+\varepsilon) - 4\eta c^2M^2(\varepsilon+\lambda_1) > 1$ and $\frac{12c\eta}{\sqrt{c^2+4\beta}} < 1$. Then there exists a traveling wave front of (3.6) satisfying (3.3).*

To finish this section, we study the problem

$$\frac{d}{dt}[u(t,x) - \int_{-\tau}^{0} \xi(s)u(t+s,x)ds] = \Delta u(t,x) + u(t)[1 - u(t-\tau,x)], \tag{3.8}$$

where $\xi \in L^1([-\tau, 0]; \mathbb{R}^-)$, $\xi \neq 0$ and $0 < \tau$. We study this problem via the equation

$$w''(t) - cw'(t) + c\frac{d}{dt}\int_{\tau}^{0} \xi(s)w(t+cs)ds + w(t)[1 - w(t-\tau c)] = 0, \quad t \in \mathbb{R}, \tag{3.9}$$

submitted to the condition (3.3).

Let $F^c(\cdot)$ be defined as above and $G^c(\cdot)$ be given by $G^c(\phi) = \int_{-\tau}^{0}\xi(s)\phi(cs)ds$. Let $\gamma = \zeta > 1$, $\beta = \zeta + \gamma, c > 0$ and assume τ and $\| \xi \|_{L^1([-\tau,0];\mathbb{R}^-)}$ are small enough such that $\beta - 2e^{\beta\tau c} \geq 0$, $\beta - 2\lambda_{2,1}c \| \xi \|_{L^1([-\tau,0];\mathbb{R}^-)} e^{\beta c\tau} \geq 0$ and $\lambda_1 rc < 1$. If ϕ, ψ are the functions in condition $\mathbf{H^2_{F,G}}$, then

$$[G^c(\phi) - G^c(\psi)] = \int_{-\tau}^{0} \xi(s)[\phi(sc) - \psi(sc)]ds \leq 0, \tag{3.10}$$

$$\lambda_{2,1}c[G^c(\phi) - G^c(\psi)] \geq -(\phi(0) - \psi(0))\lambda_{2,1}c \| \xi \|_{L^1([-\tau,0];\mathbb{R}^-)} e^{\beta c\tau}. \tag{3.11}$$

From the above we have that the condition $\mathbf{H^2_{F,G}}$ is satisfied.

Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(\lambda) = \lambda^2 - c\lambda + \lambda\int_{-\tau}^{0}\xi(s)e^{\lambda sc}ds + 1$. Since $h(\lambda_1) = \lambda_1\int_{-\tau}^{0}\xi(s)e^{\lambda_1 sc}ds < 0$ and $h(0)$=1, there exists $\vartheta_2 \in (0, \lambda_1)$ such that $h(\vartheta_2) = 0$. Noting that $h'(\vartheta_2) \leq (\lambda_1\tau c - 1) \| \xi \|_{L^1([-\tau,0];\mathbb{R}^-)} + 2\lambda_1 - c < 0$, we can select $0 < \varepsilon < \theta_2$ small and $M > 0$ large such that $h(\vartheta_2 + \varepsilon) < 0$ and $-Mh(\vartheta_2 + \varepsilon) > 1$.

Let $\varrho, \rho : \mathbb{R} \to \mathbb{R}$ be defined by $\varrho(t) = \max\{e^{\theta_2 t} - Me^{(\theta_2+\varepsilon)t}, 0\}$, $\rho(t) = \min\{e^{\vartheta_2 t}, 1\}$ and $t^* < 0$ such that $\varrho(t^*) = 0$. It is easy to show that ρ is a super-solution. In addition, for $t \leq t^*$ we get

$$\begin{aligned}
&\varrho''(t) - c\varrho'(t) + c\frac{d}{dt}G^c(\varrho)(t) + F^c(\varrho_t)\\
&\geq e^{\vartheta_2 t}[\vartheta_2^2 - c\vartheta_2 + c\vartheta_2\int_{-\tau}^{0}\xi(s)e^{\vartheta_2 sc}ds + 1] - e^{(\vartheta_2+\varepsilon)t}e^{(\vartheta_2-\varepsilon)t}\\
&\quad - Me^{(\vartheta_2+\varepsilon)t}[(\vartheta_2+\varepsilon)^2 - c(\vartheta_2+\varepsilon) + c(\vartheta_2+\varepsilon)\int_{-\tau}^{0}\xi(s)e^{(\vartheta_2+\varepsilon)sc}ds + 1]\\
&\geq e^{(\vartheta_2+\varepsilon)t}[-Mh(\vartheta_2+\varepsilon) - 1] > 0,
\end{aligned}$$

which shows that $\varrho(\cdot)$ is a sub-solution.

Proposition 3.3 below is a consequence of Theorem 2.2. We note that the inequality $\frac{6c\eta}{\sqrt{c^2+4\beta}} \parallel \xi \parallel_{L^1([-\tau,0];\mathbb{R}^-)} < 1$ is related to the inequality in the statement of Theorem 2.2.

Proposition 3.3. *Let $\gamma = \zeta > 1$, $\beta = \zeta + \gamma$ and $c > 2$. Suppose, $\beta - 2e^{\beta\tau c} \geq 0$, $\beta - 2\lambda_{2,1}c \parallel \xi \parallel_{L^1([-\tau,0];\mathbb{R}^-)} e^{\beta c\tau} \geq 0$, $\lambda_1\tau c < 1$ and $\frac{6c\eta}{\sqrt{c^2+4\beta}} \parallel \xi \parallel_{L^1([-\tau,0];\mathbb{R}^-)} < 1$. Then there exists a nondecreasing traveling wave front of* (3.8) *verifying* (3.3).

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