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ALGEBRAIC EQUATIONS IN STATE CONDITION

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ABSTRACT. In this paper, we will prove that a problem deciding whether there is an upper-triangular coordinate in which a character is not in the state of a Hilbert point is NP-hard. This problem is related to the GIT-semistability of a Hilbert point.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero and ${}^rS = k[x_1, \ldots, x_r]$ be a polynomial ring of r variables graded by degree. We will omit the superscript ^{r} if there is no confusion. When non-negative integers d and b are fixed, there is a projective space

$$E^r_{d,b} = \mathbb{P}\left(\bigwedge^b {}^rS_d\right),$$

which is a $\mathrm{GL}_r(k)$-representation. Let T_r be the maximal torus of $\mathrm{GL}_r(k)$ which consists of diagonal matrices and U_r be the set of all upper-triangular matrices with 1's in the diagonal. There is a G-equivariant closed immersion

$$i_{r,P,d} : \mathrm{Hilb}^P(\mathbb{P}^{r-1}_k) \to E^r_{d,Q(d)}$$

for $d \geq g_P$ where g_P is the Gotzmann number associated to a Hilbert polynomial P, which is defined in [2]. Also $Q(d) = \binom{r+d-1}{d} - P(d)$.

For any point $v \in E^r_{d,b}$, the collection of states $\Xi_{G.v} = \{\Xi_{g.v}(T) | g \in \mathrm{GL}_r(k)\}$ (defined in [5]) of v determines whether v is semistable or not, as stated in [7]. If v is unstable, $\Xi_{G.v}$ determines the Hesselink strata of $\mathbb{P}\left(\bigwedge^b S_d\right)^{\mathrm{us}}$ that contains v, which is stated in [3]. For an arbitrary character χ of T, $Z_{v,\chi} = \{g \in \mathrm{GL}_r(k) | \chi \notin \Xi_{g.v}\}$ is a Zariski-closed subset of $\mathrm{GL}_r(k)$. In this paper, we will construct a solvability check problem (**SC**) which is equivalent to deciding if an arbitrary system of algebraic equations is solvable (**SysAl**) by *specializing* the defining equation of some $Z_{v,\chi}$ to the defining equation of $U_r \cap Z_{v,\chi}$ in U_r.

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It's a well-known fact that to decide whether an arbitrary system of algebraic equation is solvable is an NP-hard problem ([6]). We will show that this problem can be reduced to the problem asking whether there is a $g \in U_r$ such that $\chi \notin \Xi_{g.w}$, in polynomial time. This means that such a problem is NP-hard. This problem is related to the GIT-semistability of a Hilbert point. By solving finitely many such problems, we can decide whether a Hilbert point is semistable or not.

2. Definitions and notation

First of all, we need to define the notion of generalization of a system of algebraic equation.

Definition. Suppose I is an ideal of $S = k[x_1, \ldots, x_r]$. An ideal J of a finitely generated k algebra R is a *generalization* of I under π if there is a surjective ring homomorphism $\pi : R \to S$ and a minimal generator $\{z_1, \ldots, z_{r'}\}$ of R satisfying the following:

- For any $1 \leq i \leq r'$, $\pi(z_i) \in k \cup \{x_1, \ldots, x_r\}$.
- $\pi(J) = I$.

I is a *specialization* of J if J is a generalization of I.

For example, $I = \langle x^2 + y^2 \rangle \subset k[x, y]$ is a specialization of $J = \langle z(x^2 + y^2), zw \rangle \subset k[x, y, z, w]$ under the map $\pi : k[x, y, z, w] \to k[x, y]$ which satisfies $\pi(x) = x$, $\pi(y) = y$, $\pi(z) = 1$ and $\pi(w) = 0$.

We define some notation. Let $<_{\text{lex}}$ be a lexicographic monomial order satisfying $x_{i+1} <_{\text{lex}} x_i$ and let $A^r_{d,b} = \bigwedge^b {}^rS_d$. Let ${}^rM_d$ be the set of all monomials in ${}^rS_d$ and

$$W^r_{d,b} = \left\{ \bigwedge_{i=1}^{b} m_i \middle| m_i \in {}^rM_d,\ m_i >_{\text{lex}} m_{i+1} \right\}.$$

$W^r_{d,b}$ is a basis of $A^r_{d,b}$. Suppose $v \in A^r_{d,b}$ and $w \in W^r_{d,b}$. We define v_w to be the w-component of the vector v. That is,

$$v = \sum_{w \in W^r_{d,b}} v_w w.$$

Let $[v] \in E^r_{d,b}$ be the line in $A^r_{d,b}$ through v and the origin of $A^r_{d,b}$. For any $g \in \mathrm{GL}_r(k)$, $g_{ij} \in k$ is the component of g in the i'th row and j'th column. That is,

$$g = \begin{bmatrix} g_{11} & \cdots & g_{1j} & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ g_{i1} & \cdots & g_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Also, $\mathrm{GL}_r(k)$ action on ${}^rS$ is given by $g.x_i = \sum_{1 \leq j \leq r} g_{ji} x_j$. Note that this action is a left action on ${}^rS$. For any $v \in A^r_{d,b}$ and $w \in k[W^r_{d,b}]$, $(g.v)_w$ means $((\mathrm{id}_{\Gamma(\mathrm{GL}_r(k), \mathcal{O}_{\mathrm{GL}_r(k)})} \otimes_k e_v) \circ \phi)(w)$ when g is an indeterminate. Here ϕ is the coaction map

$$\phi : k[W^r_{d,b}] \to \Gamma(\mathrm{GL}_r(k), \mathcal{O}_{\mathrm{GL}_r(k)}) \otimes_k k[W^r_{d,b}] \cong k[\{g_{ij}\}^r_{i,j=1}]_{\det g} \otimes_k k[W^r_{d,b}]$$

and e_v is the evaluation map $e_v : k[W^r_{d,b}] \to k$ at v. Let's define $\chi_i \in X(T_r)$ for all $1 \leq i \leq r$ as follows:

$$\chi_i(D) = D_{ii}$$

where $D \in T_r$. Let $\xi^r_{d,b} = \frac{db}{r}(1,\dots,1) \in X(T_r)_{\mathbb{R}} = X(T_r) \otimes_{\mathbb{Z}} \mathbb{R}$. Here $X(T_r)$ is the group of characters of the algebraic torus T_r.

Let $L_r = \{g \in \mathrm{GL}_r(k) | g \text{ is lower-triangular.}\}$. Let's define a specialization map $\theta_r : \Gamma(\mathrm{GL}_r(k), \mathcal{O}_{\mathrm{GL}_r(k)}) \to \Gamma(U_r, \mathcal{O}_{U_r})$ as follows:

$$\theta_r(z_{ij}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \\ z_{ij}, & \text{if } i < j, \end{cases}$$

where $\Gamma(\mathrm{GL}_r(k), \mathcal{O}_{\mathrm{GL}_r(k)}) = k[\{z_{ij}\}^r_{i,j=1}]_{\det z}$ and $\Gamma(U_r, \mathcal{O}_{U_r}) = k[\{z_{ij}\}_{1 \le i < j \le r}]$. For any $C \subset k[\{z_{ij}\}^r_{i,j=1}]_{\det z}$, let span$C$ be the k-subspace of $k[\{z_{ij}\}^r_{i,j=1}]_{\det z}$ spanned by C. Let Σ_r be the permutation group on the set $\{1, 2, \dots, r\}$, which is a subgroup of $\mathrm{GL}_r(k)$. Let Δ_v be the convex hull of Ξ_v in $X(T_r)_{\mathbb{R}}$ for all $v \in E^r_{d,b}$.

3. Polynomial coefficients in some special cases

Suppose $v \in A^r_{d,b}$. In this section, we will compute v_w for some special $w \in W^r_{d,b}$. Let's compute it when $b = 1$ first.

Lemma 3.1. *Suppose $r \ge 2$. Let $p \in {}^rS_d = A^r_{d,1}$. For any $g \in \mathrm{GL}_r(k)$,*

$$(g.p)_{x_1^{d-j} x_2^j} = \sum_{i_1 + \dots + i_r = j} \frac{\prod_{1 \le a \le r} g_{2a}^{i_a}}{\prod_{1 \le a \le r} i_a!} \frac{\partial^j p}{\partial x_1^{i_1} \dots \partial x_r^{i_r}}\bigg|_{x_i = g_{1i}}.$$

Proof. Without loss of generality, we can assume that p is a monomial. When p is a monomial, expanding $g.p$ proves the equality. □

We can generalize Lemma 3.1 using the following lemma.

Lemma 3.2. *Suppose $r \ge 2$. Let $p_1, p_2 \in {}^rS_d = A^r_{d,1}$. For any $g \in \mathrm{GL}_r(k)$,*

$$(g.p_1 \wedge p_2)_{x_1^{d-j_1} x_2^{j_1} \wedge x_1^{d-j_2} x_2^{j_2}} = \begin{vmatrix} (g.p_1)_{x_1^{d-j_1} x_2^{j_1}} & (g.p_1)_{x_1^{d-j_2} x_2^{j_2}} \\ (g.p_2)_{x_1^{d-j_1} x_2^{j_1}} & (g.p_2)_{x_1^{d-j_2} x_2^{j_2}} \end{vmatrix}$$

for all $1 \le j_1 < j_2 \le d$.

Proof. It can be derived from the definition. □

In Lemma 3.1, we see that taking $(g.\star)_m$ of p *separates* each monomial with respect to the degrees of each variable of p and m. Our construction would make use of this *phenomenon.* That is, we will *control* the degree of one variable, say x_{r+1}.

Fix d. Let F be a sequence $\{F_i\}^{2l-1}_{i=0} \in ({}^rS_d)^{2l} \subset ({}^{r+1}S_d)^{2l}$. Let's define $v^r_d(F) \in A^{r+1}_{2l+d,2}$ as follows:

$$v^r_d(F) = \left\{ \sum_{i=0}^{2l-1} x_{r+1}^i x_1^{2l-i} F_i \right\} \wedge \left\{ \sum_{i=0}^{2l-1} x_{r+1}^{i+1} x_1^{2l-i-1} F_i \right\}.$$

Note that $[v^r_d(F)] \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ where

$$P(t) = \binom{r+t}{r} - \binom{r+t-2l-d+1}{r} + \binom{r+t-2l-d-1}{r-2}.$$

Indeed, the graded ideal

$$I_F = \left\langle \sum_{i=0}^{2l-1} x_{r+1}^i x_1^{2l-i} F_i, \sum_{i=0}^{2l-1} x_{r+1}^{i+1} x_1^{2l-i-1} F_i \right\rangle$$

of ${}^{r+1}S$ satisfies the following properties.

Lemma 3.3. *${}^{r+1}S/I_F$ has the Hilbert polynomial*

$$P(t) = \binom{r+t}{r} - \binom{r+t-2l-d+1}{r} + \binom{r+t-2l-d-1}{r-2}.$$

Also, $g_P = 2l+d$ so that $i_{r+1,P,d+2l}(I_F) = [v_d^r(F)]$. If $r \geq 2$, then I_F is saturated.

Proof. $I = I_F$ is isomorphic to $\langle x_1, x_{r+1}\rangle(-2l-d+1)$ as a graded ${}^{r+1}S$ module. Thus, $\dim_k(I_F)_{t+2l+d}$ is equal to the number of monomials in ${}^{r+1}S_{t+1}$ which is divisible by x_1 or x_{r+1}, for every $t \geq 0$. This implies that I_F has the Hilbert polynomial

$$Q(t) = \binom{r+t-2l-d+1}{r} - \binom{r+t-2l-d-1}{r-2}.$$

Q admits the Macaulay representation

$$Q(t) = \binom{r+t-2l-d}{r} + \binom{r+t-2l-d-1}{r-1}.$$

By the definition of $n(Q)$ in [2, p. 65], $g_P = 2l+d$. The regularity of I_F is equal to the regularity of $\langle x_1, x_{r+1}\rangle(-2l-d+1)$, which is equal to $2l+d$. Let J be the saturation of I_F. The Hilbert polynomial of J is Q. This implies that the regularity of J is at most $g_P = 2l+d$. Therefore, $\dim_k J_t = Q(t) = \dim_k I_t$ for all $t \geq 2l+d$ by [2, (1.2) Satz, (2.9) Lemma]. Suppose $r \geq 2$. If there is a homogeneous $q \in J \setminus I$, then $q \in J_t$ for some $t < 2l+d$. We derive an inequality $2 = \dim_k I_{2l+d} = \dim_k J_{2l+d} \geq \dim_k \langle q\rangle_{2l+d} \geq r+1 \geq 3$, which is false. □

We can analyze the polynomial coefficient of $g.v_d^r(F)$ as follows:

Lemma 3.4. *$\{f_{a,r,l,F}\}_{a=0}^{l-1}$ is a basis for*

$$\operatorname{span}\{\theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d-a}x_{r+1}^a \wedge x_1^{d+a}x_{r+1}^{2l-a}}) | 0 \leq a \leq l-1\}$$

where

$$f_{a,r,l,F} = \sum_{0\leq i<j\leq 2l-1} \tilde{F}_i\tilde{F}_j g_{1r+1}^{i+j-2l+1}\left[\binom{i}{a}\binom{j}{2l-a-1} + \binom{i}{2l-a-1}\binom{j}{a}\right] + \sum_{i=0}^{2l-1} \tilde{F}_i^{\,2} g_{1r+1}^{2i-2l+1}\binom{i}{a}\binom{i}{2l-a-1}$$

and

$$\tilde{F}_i = F_i(1, g_{12}, \ldots, g_{1r}).$$

Proof. Using Lemma 3.1 and Lemma 3.2, we can compute that

$$f_{a,r,l,F} - f_{a-1,r,l,F} = \theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d-a}x_{r+1}^a \wedge x_1^{d+a}x_{r+1}^{2l-a}})$$

for all $1 \leq a \leq l-1$ and

$$f_{0,r,l,F} = \theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d} \wedge x_1^d x_{r+1}^{2l}}).$$

□

Let ψ be a sequence $\{\psi_i\}_{i=0}^{l-1} \in ({}^rS_d)^l \subset ({}^{r+1}S_d)^l$. Let's define a sequence $F_\psi \in ({}^rS_d)^{2l} \subset ({}^{r+1}S_d)^{2l}$ as follows:

- $(F_\psi)_i = 0$ for all $l \le i \le 2l-2$.
- $(F_\psi)_{2l-1} = x_1^d$.
- $(F_\psi)_i = \frac{1}{i!}\psi_i$ for all $0 \le i \le l-1$.

Lemma 3.5. *$\pi^\psi = \{\pi_j^\psi\}_{j=0}^{l-1}$ is a basis for*

$$\mathrm{span}\{\theta_{r+1}((g.v_d^r(F_\psi))_{x_1^{2l+d-a}x_{r+1}^a \wedge x_1^{d+a}x_{r+1}^{2l-a}})|0 \le a \le l-1\}$$

where

$$\pi_j^\psi = \frac{(2l-1)!}{(2l-1-j)!}\left[\sum_{a=0}^{l-1-j}\binom{2l-1-j}{a}(-1)^a\right] g_{1r+1}^{2l-1} + \tilde{\psi}_j g_{1r+1}^j$$

and

$$\tilde{\psi_j} = \psi_j(1, g_{12}, \ldots, g_{1r}).$$

Proof. By the definition of f_{a,r,l,F_ψ},

$$a!\binom{2l-1}{a}^{-1} f_{a,r,l,F_\psi} = \frac{(2l-1)!}{(2l-1-a)!} g_{1r+1}^{2l-1} + \sum_{i=a}^{l-1}\frac{1}{(i-a)!}\tilde{\psi}_i g_{1r+1}^i$$

for all $0 \le a \le l-1$. Now

$$\sum_{a=j}^{l-1}\frac{(-1)^{a+j}}{(a-j)!}\left[a!\binom{2l-1}{a}^{-1} f_{a,r,l,F_\psi}\right] = (2l-1)!\sum_{a=j}^{l-1}\frac{(-1)^{a+j}}{(a-j)!(2l-1-a)!} g_{1r+1}^{2l-1}$$
$$+\sum_{a=j}^{l-1}\sum_{i=a}^{l-1}\frac{(-1)^{a+j}}{(a-j)!(i-a)!}\tilde{\psi}_i g_{1r+1}^i$$
$$= \frac{(2l-1)!}{(2l-1-j)!}\sum_{a=0}^{l-1-j}\binom{2l-1-j}{a}(-1)^a g_{1r+1}^{2l-1}$$
$$+\sum_{i=j}^{l-1}\sum_{a=0}^{i-j}\frac{1}{(i-j)!}\binom{i-j}{a}(-1)^a\tilde{\psi}_i g_{1r+1}^i = \pi_j^\psi.$$

Clearly $\{\pi_j|0 \le j \le l-1\}$ is a linearly independent set. This proves the lemma. $\square$

π^ψ has the following property. This property depends on the characteristic of k, which is zero in this paper.

Lemma 3.6. *The coefficient of g_{1r+1}^{2l-1} in π_j^ψ is non-zero. That is,*

$$\sum_{a=0}^{l-1-j}\binom{2l-1-j}{a}(-1)^a \ne 0$$

for any choice of integers l and j satisfying $l \ge 1$ and $0 \le j \le l-1$.

Proof. Note that

$$\binom{2l-1-j}{a} \le \binom{2l-1-j}{a+1}$$

for all a satisfying $0 \le a \le l-1-j$.

If $l-1-j$ is even,

$$\sum_{a=0}^{l-1-j}\binom{2l-1-j}{a}(-1)^a = 1+\sum_{a=1}^{\frac{l-1-j}{2}}\binom{2l-1-j}{2a}-\binom{2l-1-j}{2a-1} > 0.$$

Similarly, If $l-1-j$ is odd, we can show that

$$\sum_{a=0}^{l-1-j}\binom{2l-1-j}{a}(-1)^a < 0$$

because the first term is always strictly smaller than the absolute value of the second term. ☐

4. NP-hardness of a problem judging the existence of an upper-triangular coordinate

Suppose $l \geq 3$, $r \geq 2$ and $p = \{p_i\}_{i=0}^{l-3} \in k[x_2, \ldots, x_r]^{l-2}$. Assume that

$$d \geq \max\{\deg(p_i)|0 \leq i \leq l-3\}$$

where $\deg(p_i)$ means the non-homogeneous degree of p_i. Let's construct $\psi(p) = \{\psi_i(p)\}_{i=0}^{l-1}$.

- Define the first two terms as follows:

$$\psi_i(p) = -\frac{(2l-1)!}{(2l-1-i)!}\left[\sum_{a=0}^{l-1-i}\binom{2l-1-i}{a}(-1)^a\right]x_1^d \tag{1}$$

  for $i \in \{0,1\}$.
- For $2 \leq i \leq l-1$, let

$$\psi_i(p) = -\frac{(2l-1)!}{(2l-1-i)!}\left[\sum_{a=0}^{l-1-i}\binom{2l-1-i}{a}(-1)^a\right]x_1^d + x_1^d p_{i-2}\left(\frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1}\right). \tag{2}$$

Now we are ready to prove the following.

Theorem 4.1. *Let $l \geq 3$. There is $g \in U_{r+1}$ satisfying $\chi = \chi_1^{2d+2l}\chi_{r+1}^{2l} \notin \Xi_{[g.v_d^r(F_{\psi(p)})]}$ if and only if the ideal J of $k[x_2, \ldots, x_r]$ generated by $\{p_i|0 \leq i \leq l-3\}$ has a solution over k.*

Proof. By definition, $Z_{[v_d^r(F_{\psi(p)})],\chi} \cap U_{r+1}$ is the zero set of the ideal

$$I \subset \Gamma(U_{r+1}, \mathcal{O}_{U_{r+1}}) = k[\{g_{ij}\}_{1\leq i<j\leq r+1}]$$

generated by

$$\{\theta_{r+1}((g.v_d^r(F_{\psi(p)}))_{x_1^{2l+d-a}x_{r+1}^a \wedge x_1^{d+a}x_{r+1}^{2l-a}})|0 \leq a \leq l-1\}.$$

By Lemma 3.5, I is generated by

$$\{\pi_i^{\psi(p)}|0 \leq i \leq l-1\}.$$

It suffices to show that the zero set of I is non-empty if and only if the zero set of J is non-empty. If there is an element $\{x_{ij}\}_{1\leq i<j\leq r-1}$ in the zero set of I, then $g_{1r+1} = 1$ because $\pi_i^{\psi(p)} = 0$ for $i \in \{0,1\}$ if and only if $g_{1r+1}^{2l-1} = 1$ and $g_{1r+1}^{2l-1} - g_{1r+1} = 0$

by Lemma 3.6. Note that $(x_{12}, \dots, x_{1r})$ is a solution of the system of equations defined by

$$\{\pi_i^{\psi(p)}|_{g_{1r+1}=1} | 2 \le i \le l-1\} = \{p_i(g_{12}, \dots, g_{1r}) | 0 \le i \le l-3\}$$

so that J has non-empty zero set. If there is an element $\{x_i\}_{i=2}^r$ in the zero set of J, $\{z_{ij}\}_{1 \le i < j \le r+1}$ is in the zero set of I if $z_{ir} = x_i$ for all $2 \le i \le l-1$ and $z_{1r+1} = 1$. □

Theorem 4.1 implies the following.

Corollary 4.2. *For any ideal I of a polynomial ring, there is a Hilbert point $v \in \mathrm{Hilb}^P(\mathbb{P}_k^r)$, a choice of closed immersion $\mathrm{Hilb}^P(\mathbb{P}_k^r) \to \mathbb{P}(\bigwedge^{Q(d)} S_d)$ and a character $\chi \in X(T_{r+1})$ such that there is an ideal J of $\Gamma(\mathrm{GL}_{r+1}(k), \mathcal{O}_{\mathrm{GL}_{r+1}(k)})$ such that $Z_{v,\chi}$ is the zero locus of J and J is a generalization of I.*

Let's consider some decision problems. Let **SysAl** be a problem asking if a system of algebraic equations over $\mathbb{Q}$ has a solution over k and **HC** be a problem asking if a graph has a Hamiltonian cycle. Using the proof of Corollary 2.3.2 in [6, p. 21], we can prove that **HC** can be reduced to **SysAl** in polynomial time. By Theorem 10.23 of [4], **HC** is an NP-complete problem so that **SysAl** is an NP-hard problem. Let's describe a solvability check problem **SC** as follows:

- Given : A rational Hilbert point $v \in \mathrm{Hilb}^P(\mathbb{P}_k^{r-1})$, a choice of closed immersion $\mathrm{Hilb}^P(\mathbb{P}_k^{r-1}) \to \mathbb{P}(\bigwedge^{Q(d)} S_d)$ and a character $\chi \in X(T_r)$.
- Decide : Is there a coordinate $g \in U_r$ satisfying $\chi \notin \Xi_{g.v}$?

Here, $v \in \mathrm{Hilb}^P(\mathbb{P}_k^{r-1})$ is rational if it represents a saturated homogeneous ideal of ${}^rS$ generated by rational polynomials. Theorem 4.1 shows that there is a polynomial time reduction from **SysAl** to **SC**. That is,

Corollary 4.3. *The problem* **SC** *is NP-hard.*

There is an extended version of **SC**, which would be called **ESC**, described as follows:

- Given : A rational Hilbert point $v \in \mathrm{Hilb}^P(\mathbb{P}_k^{r-1})$, a choice of closed immersion $\mathrm{Hilb}^P(\mathbb{P}_k^{r-1}) \to \mathbb{P}(\bigwedge^{Q(d)} S_d)$ and a finite set of characters $C \subset X(T_r)$.
- Decide : Is there a coordinate $g \in U_r$ satisfying $C \cap \Xi_{g.v} = \emptyset$?

SC can be reduced to **ESC** in polynomial time so that we can prove the following:

Corollary 4.4. *The problem* **ESC** *is NP-hard.*

On the other hand, we can use Buchberger's algorithm in [1] to solve the problem **ESC** because the zero set of an ideal $I \subset {}^rS$ is non-empty if and only if $1 \notin I$ if and only if the Gröbner basis of I with respect to the lexicographic (or graded reverse-lexicographic) monomial order contains 1.

Let's construct an example. Fix natural numbers r and d. Suppose $l = 3$, $p_0 \in k[x_2, \dots, x_r]$ and $\deg(p_0) \le d$. In this case, p is a sequence of length 1 and the ideal generated by $\{p_i | 0 \le i \le l-3\}$ has empty zero locus if and only if p_0 is a non-zero constant polynomial. Let

$$F' = -6x_1^{d+5} + 15x_1^{d+4}x_{r+1} - 10x_1^{d+3}x_{r+1}^2 + x_1^d x_{r+1}^5 + \frac{x_1^{d+3}x_{r+1}^2}{2} p_0\left(\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}\right).$$

By the definition, $I_{F_{\psi(p)}} = \langle x_1F', x_{r+1}F'\rangle$. This means that there is a $g \in U_{r+1}$ such that $\chi_1^{2d+6}\chi_{r+1}^6 \notin \Xi_{[g.v_d^r(F_{\psi(p)})]}$ if and only if p_0 is the zero polynomial or $\deg(p_0) \geq 1$.

5. A relation between the problem ESC and GIT-semistability

In this section, every GIT problem is related to the action of $\mathrm{GL}_r(k)$ on $E^r_{d,b}$. It will be proved that we can decide whether a rational Hilbert point is GIT-semistable by solving finitely many **ESC**. As a consequence of [7, Criterion 3.3], we have the following lemma.

Lemma 5.1. *A rational point $v \in E^r_{d,b}$ is GIT-semistable if and only if $\xi^r_{d,b} \in \Delta_{g.v}$ for all $g \in \mathrm{GL}_r(k)$.*

Proof. v is semistable if and only if it is semistable under the action of every maximal torus of $\mathrm{GL}_r(k)$ by [8, Theorem 2.1]. Since every two maximal tori are conjugate, [7, Criterion 3.3] proves the lemma. □

A point in $X(T)_{\mathbb{R}}$ is not in a polytope Δ if and only if there is a separating hyperplane in $X(T)_{\mathbb{R}}$. That is,

Lemma 5.2. *For any $g \in \mathrm{GL}_r(k)$ and $v \in E^r_{d,b}$, $\xi^r_{d,b} \notin \Delta_{g.v}$ if and only if there is an $\omega \in X(T)^\vee_{\mathbb{R}}$ such that*

$$\omega(\xi^r_{d,b}) < \min \omega(\Xi_{g.v} \otimes_{\mathbb{R}} 1). \tag{3}$$

For some special choices of $\omega \in X(T)^\vee_{\mathbb{R}}$ and v, we can still guarantee (3) for every $g \in L_r$.

Lemma 5.3. *Suppose there are $v \in E^r_{d,b}$ and $\omega \in X(T)^\vee_{\mathbb{R}}$ satisfying*

$$\omega(\xi^r_{d,b}) < \min \omega(\Xi_v \otimes_{\mathbb{R}} 1)$$

and $\omega(\chi_i) \leq \omega(\chi_{i+1})$ for all $1 \leq i < r$. Then, for any $l \in L_r$,

$$\omega(\xi^r_{d,b}) < \min \omega(\Xi_{l.v} \otimes_{\mathbb{R}} 1).$$

Proof. Suppose $\eta \in \Xi_{l.v}\otimes_{\mathbb{R}}1\backslash\Xi_v\otimes_{\mathbb{R}}1$. It suffices to show that $\omega(\eta) \geq \min \omega(\Xi_v\otimes_{\mathbb{R}}1)$. By definition, there is an $m \in W^r_{d,b}$ satisfying $\eta \in \Xi_{l.m}$ and $\Xi_m \subset \Xi_v$. By expanding $l.m$, we can prove that

$$\omega(\eta) \geq \min \omega(\Xi_m \otimes_{\mathbb{R}} 1)$$

using the condition $\omega(\chi_i) \leq \omega(\chi_{i+1}), \forall 1 \leq i \leq r-1$. Since $\Xi_m \subset \Xi_v$, we can deduce that $\min \omega(\Xi_m \otimes_{\mathbb{R}} 1) \geq \min \omega(\Xi_v \otimes_{\mathbb{R}} 1)$. Thus the claimed statement is true. □

Now, we can restate the condition for v to be unstable.

Theorem 5.4. *Suppose $v \in E^r_{d,b}$. v is unstable if and only if there are $u \in U_r$ and $q \in \Sigma_r$ satisfying*

$$\xi^r_{d,b} \notin \Delta_{uq.v}.$$

Proof. If part is obvious by Lemma 5.1. Suppose there is $g \in \mathrm{GL}_r(k)$ satisfying

$$\xi^r_{d,b} \notin \Delta_{g.v}.$$

By Lemma 5.2, there is $\omega \in X(T)^\vee_{\mathbb{R}}$ satisfying

$$\omega(\xi^r_{d,b}) < \min \omega(\Xi_{g.v} \otimes_{\mathbb{R}} 1).$$

There is a $p \in \Sigma_r$ satisfying $\omega(\chi_{p(i)}) \leq \omega(\chi_{p(i+1)})$ for all i. Let's define $\omega_p(\chi_i) = \omega(\chi_{p(i)})$. Then,

$$\omega_p(\xi^r_{d,b}) = \omega(\xi^r_{d,b}) < \min \omega(\Xi_{g.v} \otimes_{\mathbb{R}} 1) = \min \omega_p(\Xi_{p^{-1}g.v} \otimes_{\mathbb{R}} 1).$$

Now there are $l \in L_r$,$u \in U_r$ and $q \in \Sigma_r$ satisfying $p^{-1}g = luq$ by the LU-decomposition of general non-singular matrix. $p^{-1}g.v$ and ω_p satisfies the condition of Lemma 5.3. Thus,

$$\omega_p(\xi^r_{d,b}) < \min \omega_p(\Xi_{l^{-1}luq.v} \otimes_{\mathbb{R}} 1) = \min \omega_p(\Xi_{uq.v} \otimes_{\mathbb{R}} 1).$$

By Lemma 5.2, $\xi^r_{d,b} \notin \Delta_{uq.v}$, as desired. □

Using Theorem 5.4 and Lemma 5.2, we can solve **ESC** for each choice of $\omega \in X(T)^{\vee}_{\mathbb{R}}$ and $q \in \Sigma_r$ to check if

$$\{\chi \in X(T) | \omega(\chi) \leq \omega(\xi^r_{d,Q(d)})\} \cap \Xi_{uq.v} = \emptyset$$

for a rational $v \in \mathrm{Hilb}^P(\mathbb{P}^{r-1}_k)$ and an integer $d \geq g_P$. Note that we have to consider finitely many ω's because $A^r_{d,b}$ has only finitely many weights with respect to the action of T_r. In this way, we can check if v is semistable or not. This fact implies that there is an algorithm deciding if a rational $v \in \mathrm{Hilb}^P(\mathbb{P}^{r-1}_k)$ is GIT-semistable or not.

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