PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 4, April 2018, Pages 1713–1729
http://dx.doi.org/10.1090/proc/13917
Article electronically published on December 4, 2017

LONG-TIME ASYMPTOTIC BEHAVIOR FOR THE GERDJIKOV-IVANOV TYPE OF DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH TIME-PERIODIC BOUNDARY CONDITION

SHOU-FU TIAN AND TIAN-TIAN ZHANG

(Communicated by Mourad Ismail)

Abstract. The Gerdjikov-Ivanov (GI) type of derivative nonlinear Schrödinger equation is considered on the quarter plane whose initial data vanish at infinity while boundary data are time-periodic, of the form $ae^{i\delta}e^{2i\omega t}$. The main purpose of this paper is to provide the long-time asymptotics of the solution to the initial-boundary value problems for the equation. For $\omega < a^2(\frac{1}{4}a^2 + 3b - 1)$ with $0 < b < \frac{a^2}{4}$, our results show that different regions are distinguished in the quarter plane $\Omega = \{(x,t) \in \mathbb{R}^2 | x > 0, t > 0\}$, on which the asymptotics admit qualitatively different forms. In the region $x > 4tb$, the solution is asymptotic to a slowly decaying self-similar wave of Zakharov-Manakov type. In the region $0 < x < 4t\left(b - \sqrt{2a^2\left(\frac{a^2}{4} - b\right)}\right)$, the solution takes the form of a plane wave. In the region $4t\left(b - \sqrt{2a^2\left(\frac{a^2}{4} - b\right)}\right) < x < 4tb$, the solution takes the form of a modulated elliptic wave.

1. Introduction

In 1993, Deift and Zhou [6] introduced the nonlinear steepest descent method to analyze the long time asymptotics of initial value problems of integrable nonlinear evolution equations. This approach was inspired by earlier works of Manakov [23] and Its [14]; for a detailed historical review see [7], further extended by Deift, Venakides, and Zhou [8]. In the context of initial value problems, the Riemann-Hilbert (RH) problem is formulated on the basis of certain spectral functions whose definitions involve the initial data of the solution [1, 6]. In the 1980-1990's, Bikbaev, Deift, Novokshenov, and Venakides had already developed the Riemann-Hilbert method for solving the shock problems. An approach was recently developed by Buckingham and Venakides [5] to study the problem of shock-type oscillating initial data for the focusing nonlinear Schrödinger equation. A central role in their implementation is played by the so-called "g-function mechanism" [9] allowing one

Received by the editors June 20, 2017.
2010 *Mathematics Subject Classification.* Primary 35Q55, 35Q51; Secondary 35P30, 81Q05.
Key words and phrases. Integrable system, Riemann-Hilbert problem, initial-boundary value problem, long-time asymptotic, nonlinear steepest-descent method.
Project supported by the Fundamental Research Fund for the Central Universities under Grant No. 2017XKQY101.
Shou-fu Tian and Tian-Tian Zhang serve as corresponding authors.

to deform the original Riemann-Hilbert problem to a form that can be asymptotically treated with the help of associated singular integral equations. Using the steepest descent method for oscillatory Riemann-Hilbert problems, the long-time asymptotics of several initial-boundary value (IBV) problems have already been studied in [10, 11, 22, 34].

It is well-known that the nonlinear Schrödinger equation (NLS)

$$iq_t + q_{xx} + \frac{1}{2}|q|^2 q = 0 \tag{1.1}$$

can be used to describe slowly varying wave envelopes in dispersive media from water waves, plasma physics, and nonlinear optics. In [24, 25], Ma and his collaborator have provided a direct and effective method for finding exact solutions to the NLS equation. Ma [26] has also proposed a generalized Wronskian method to find the exact solution of some nonlinear integrable equations. Ismail and his collaborator have provided the classical and quantum orthogonal polynomials in one variable [15] and have also studied the spectral analysis of certain Schrödinger operators [16, 17], etc. By considering a new construction of phase g-function, the long-time asymptotics for the focusing NLS equation is solved by Boutet de Monvel, Its, and Kotlyarov [2–4] on the quarter plane. They showed that the solution of the equation is asymptotically periodic for large t with the same period T. The given Dirichlet datum consisting of a single periodic exponential is investigated by Lenells and Fokas for the NLS equation in [20, 21]. Lenells [22] has also studied the long-time asymptotics of its solution in the quarter plane.

To the best of the authors' knowledge, the long-time asymptotics for the derivative nonlinear Schrödinger (DNLS) equation with time-periodic boundary condition on the quarter plane has not been investigated before. The main purpose of this paper is to study the following IBV problems for the Gerdjikov-Ivanov (GI) type of derivative nonlinear Schrödinger equation [12, 13], whose form is

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0, \tag{1.2}$$

where $q(x,t)$ is a complex-valued function of x and t, the overbar denotes the complex conjugation (similarly hereinafter), and the subscripts denote differentiation with respect to the corresponding variables. The GI equation (1.2) has several applications in plasma physics. In plasma physics, it is a model for Alfvén waves propagating parallel to the ambient magnetic field, q being the transverse magnetic field perturbation and x and t being space and time coordinates, respectively. Kitaev and Vartanian obtained the leading order long-time asymptotic for the KN-type DNLS equation with the decaying initial value [18, 19], and the higher order long-time asymptotic in [31]. Xu, Fan, and Chen have studied the Cauchy problem for the GI equation with steplike initial data [33] and have also studied the DNLS equation with decaying initial value problem [34]. Recently, we have studied the IBV problems for the general coupled nonlinear Schrödinger equation on the interval and on the half-line [27, 28]. We have also studied a generalized derivative nonlinear Schrödinger equation with time-periodic boundary condition [29] and with step-like initial data [30], respectively. Very recently, we have provided the characteristics of the breather and rogue waves in a (2+1)-dimensional nonlinear Schrödinger equation [32]. In this paper, we will study the long-time asymptotics of GI equation (1.2) with time-periodic boundary condition on the quarter plane.

We consider the following IBV problems for the GI equation, whose form is

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0, \tag{1.3a}$$

$$q(x,0) = q_0(x), \tag{1.3b}$$

$$q(0,t) = g_0(t) = ae^{2i\omega t + i\delta}, \tag{1.3c}$$

$$q_0(0) = g_0(0) = ae^{i\delta}, \tag{1.3d}$$

on the quarter plane Ω:

$$\Omega = \{(x,t) \in \mathbb{R}^2 | x > 0, t > 0\}, \tag{1.4}$$

where $q_0(x)$ vanishes for $x \to \infty$, $a > 0$, δ and ω are real constants. Let $q(x,t)$ be the solution of the IBV problems for $(x,t) \in \Omega$. Let $q(x,t)$ be C^∞, continuous with all its derivatives up to the boundary $\{xt = 0\}$ of Ω, and $q(x,t), q_0(x) \in \mathcal{S}(\mathbb{R}_+)$ in x for any fixed $t \in \mathbb{R}_+$, where $\mathcal{S}(\mathbb{R}_+)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}_+$:

$$\mathcal{S}(\mathbb{R}_+) = \{u(x) \in \mathcal{C}^\infty(\mathbb{R}_+) | x^n u^{(m)}(x) \in L^\infty(\mathbb{R}_+) \text{ for any } n, m \geq 0\}. \tag{1.5}$$

Let the boundary condition (1.3c) be of its natural weaker version,

$$q(0,t) = g_0(t) = ae^{2i\omega t + i\delta} + v_0(t), \tag{1.6}$$

with $v_0(t) \in \mathcal{S}(\mathbb{R}_+)$. Then all the results provided here are actually valid.

In this paper, we mainly focus on the case $\omega < a^2(\frac{1}{4}a^2 + 3b - 1)$, where a and b are the parameters of the following Floquet solution:

$$q^p(x,t) = ae^{2ibx + 2i\omega t + i\delta}, \ \omega := -2b^2 - a^2 b + \frac{1}{4}a^4, \ a > 0. \tag{1.7}$$

Assumptions. Throughout this paper, let the function $q(x,t)$ be a global solution of the Dirichlet IBV problems (1.3a)-(1.3d), sufficiently smooth and with sufficient decay for $x \to +\infty$. We also take the Neumann boundary values in the following form:

$$q_x(0,t) := g_1(t) = 2iabe^{2i\omega t} + v_1(t), \tag{1.8a}$$

$$-a^2 b + \frac{1}{4}a^4 - \omega = 2b^2 > 0, \tag{1.8b}$$

with $v_1(t) \in \mathcal{S}(\mathbb{R}_+)$.

Organization of this paper. In Sections 2 and 3, we provide the Lax representation and a Floquet solution of (1.2), based on which we study its domains of boundedness. Then, we use these functions in Section 4 to construct the Riemann-Hilbert problem of the IBV problems (1.3a)-(1.3d). In Section 5, we study the asymptotic analysis of this Riemann-Hilbert problem leading to asymptotic formulas for the solution of the IBV problems.

2. Preliminaries

The GI equation (1.2) is lax integrable and admits a Lax spectral problem associated with 2×2 matrices for the linear x-equation,

$$\Phi_x + ik^2\sigma_3\Phi = \left[kQ(x,t) + \frac{i}{2}|q|^2\sigma_3\right]\Phi, \tag{2.1a}$$

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{pmatrix} \text{ with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1b}$$

and for the linear t-equation,

(2.2a)

$$\Phi_t + 2ik^4\sigma_3\Phi = \tilde{Q}(x,t)\Phi,$$

(2.2b)

$$\tilde{Q}(x,t) = 2k^3Q(x,t) + ik^2|q|^2\sigma_3 - ikQ_x(x,t)\sigma_3 + \frac{1}{2}(q\bar{q}_x - \bar{q}q_x)\sigma_3 + \frac{i}{4}|q|^4\sigma_3,$$

where $\Phi(x,t,k)$ is a 2×2 matrix-valued function and $k \in \mathbb{C}$ is a spectral parameter. The compatibility condition of Lax pair (2.1a) and (2.2a) gives the GI equation (1.2).

The GI equation (1.3a) admits the Floquet solution (1.7), which is consistent with (1.3b)-(1.3d) for $x > 0$.

Suppose $\tilde{Q}^p(t;k) = \tilde{Q}^p(0,t;k)$, where $\tilde{Q}^p(x,t;k)$ is introduced as $\tilde{Q}(t;k)$ by replacing $q(x,t)$ with $q^p(x,t)$, i.e.,

(2.3)

$$\tilde{Q}^p(x,t) = 2k^3Q^p(x,t) + ik^2|q^p|^2\sigma_3 - ikQ^p_x(x,t)\sigma_3 + \frac{1}{2}(q^p\bar{q}^p_x - \bar{q}^pq^p_x)\sigma_3 + \frac{i}{4}|q^p|^4\sigma_3,$$

where

$$Q^p(t) := Q^p(0,t) = \begin{pmatrix} 0 & ae^{2i\omega t+i\delta} \\ -ae^{-2i\omega t-i\delta} & 0 \end{pmatrix},$$
$$Q^p_x(t) := Q^p_x(0,t) = \begin{pmatrix} 0 & 2iabe^{2i\omega t+i\delta} \\ 2iabe^{-2i\omega t-i\delta} & 0 \end{pmatrix},$$
$$Q^p(x,t) := \begin{pmatrix} 0 & q^p(x,t) \\ -\bar{q}^p(x,t) & \end{pmatrix}.$$

We now study the t-part (2.2a) of the Lax pair associated with $Q^p(t)$, i.e.,

$$\Psi_t(t;k) + 2ik^4\sigma_3\Psi(t;k) = \tilde{Q}^p(t;k)\Psi(t;k),\ t > 0,\ k \in \mathbb{C}, \tag{2.4}$$

where $\Psi(t;k)$ is a 2×2 matrix-valued function. A particular (Floquet) solution of (2.4) is given by

$$\Psi(t;k)=\mathcal{E}(t;k)e^{i(\omega-\Omega(k))\sigma_3 t}, \tag{2.5a}$$

$$\mathcal{E}(t;k)=e^{i\omega\hat{\sigma}_3 t}E(k):=e^{i\omega\sigma_3 t}E(k)e^{-i\omega\sigma_3 t}, \tag{2.5b}$$

$$E(k)=\frac{1}{2}\begin{pmatrix} \varphi(k)+\frac{1}{\varphi(k)} & e^{i\delta}\left(\varphi(k)-\frac{1}{\varphi(k)}\right) \\ e^{-i\delta}\left(\varphi(k)-\frac{1}{\varphi(k)}\right) & \varphi(k)+\frac{1}{\varphi(k)} \end{pmatrix}, \tag{2.5c}$$

$$\Omega(k)=(k^2-b)X(k), \tag{2.5d}$$

$$X(k)=2\sqrt{(k^2+b)^2+\frac{a^2}{4}-a^2b}, \tag{2.5e}$$

$$\varphi(k)=\left(\frac{k^2+b-\frac{a^2}{2}-ika}{k^2+b-\frac{a^2}{2}+ika}\right)^{\frac{1}{4}}. \tag{2.5f}$$

The branches of the square roots are fixed by their asymptotics, for $k\to\infty$:

$$\begin{aligned} X(k)&=2\sqrt{(k^2+b)^2+\frac{a^2}{4}-a^2b}=2(k^2+b)+O(k),\\ \varphi(k)&=\left(\frac{k^2+b-\frac{a^2}{2}-ika}{k^2+b-\frac{a^2}{2}+ika}\right)^{\frac{1}{4}}=1-\frac{ia}{2k}+O\left(\frac{1}{k^2}\right), \end{aligned} \tag{2.6}$$

on the complex k-plane cut along any curve connecting the two branch points E and $\bar{E}$.

In [29], we have derived the formulation of a Riemann-Hilbert problem whose solution yields the solution of the IBV problems of the GI equation (1.3a)-(1.3d). In this paper, we will study the asymptotic analysis of the Riemann-Hilbert problem of the equation, which is formulated below.

3. Domains of boundedness

Let

$$\Sigma:=\{k\in\mathbb{C}|\mathrm{Im}\ \Omega(k)=0\}. \tag{3.1}$$

Introducing $\lambda=k^2$ and taking $\lambda_1=\mathrm{Re}\ \lambda$, $\lambda_2=\mathrm{Im}\ \lambda$, the equation $\mathrm{Im}\ \Omega(k)=0$ yields

$$\lambda_2=0 \tag{3.2}$$

or

(3.3)

$$\lambda_1\lambda_2^2=(\lambda_1-b)\left(\lambda_1^2+b\lambda_1+\frac{a^2}{8}-\frac{a^2b}{2}\right)=(\lambda_1-b)(\lambda_1-\lambda_-)(\lambda_1-\lambda_+)\text{ with }|\lambda_1|\le|b|.$$

In what follows, let $\omega\le a^2\left(\frac{1}{4}a^2+3b-1\right)$, i.e., $b^2\ge\frac{a^2}{2}-2a^2b$, and $b>0$. Therefore, one can see that $\lambda_\pm$ are both real and

$$\lambda_\pm=-\frac{b}{2}\pm\frac{1}{2}\sqrt{b^2-\frac{1}{2}a^2+2a^2b}, \tag{3.4}$$

with $-b<\lambda_-\le -b/2\le\lambda_+<0$ as $b<\frac{a^2}{4}$ (see Figure 1).

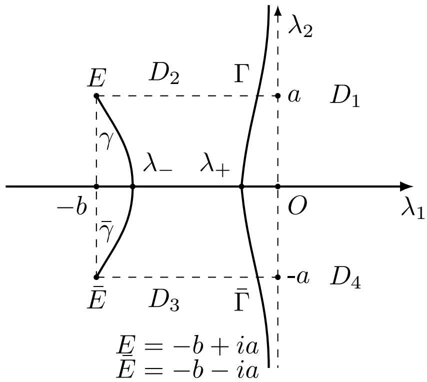

FIGURE 1. The domains $\{D_j\}_1^4$ for the case $\omega < a^2\left(\frac{1}{4}a^2 + 3b - 1\right)$, $a > 0$, $0 < b < a^2/4$.

For such case, Σ includes the real axis $\mathbb{R}$, the contour $\Gamma \cup \bar{\Gamma}$, and the finite arc $\gamma \cup \bar{\gamma}$ with the endpoints $E = -b + ia$, $\bar{E} = -b - ia$, which are both the branch points

$$\Sigma = \mathbb{R} \cup \Gamma \cup \bar{\Gamma} \cup \gamma \cup \bar{\gamma}, \tag{3.5}$$

where $\Gamma = \{\lambda \in \mathbb{C} | \lambda_1 = B, |\lambda_2| \leq D, \text{Im } \lambda_2 > 0\}$, $D^2 = \frac{a^4}{4} - a^2 b$, $\lambda_1 = \text{Re } \lambda$ and $\lambda_2 = \text{Im } \lambda$.

From above, one can see that the domains $\{D_j\}_1^4$ are given by

$$\begin{cases} D_1 := \{\lambda \in \mathbb{C} | \text{Im } \lambda > 0, \text{Im } \Omega(\lambda) > 0\}, \ D_2 := \{\lambda \in \mathbb{C} | \text{Im } \lambda > 0, \text{Im } \Omega(\lambda) < 0\}, \\ D_3 := \{\lambda \in \mathbb{C} | \text{Im } \lambda < 0, \text{Im } \Omega(\lambda) > 0\}, \ D_4 := \{\lambda \in \mathbb{C} | \text{Im } \lambda < 0, \text{Im } \Omega(\lambda) < 0\}. \end{cases} \tag{3.6}$$

From the definition of the domains $\{D_j\}_1^4$, one can further introduce $D_\pm$ as follows:

$$D_+ := D_1 \cup D_3 = \{\lambda \in \mathbb{C} | \text{Im } \Omega(\lambda) > 0\}, \ D_- := D_2 \cup D_4 = \{\lambda \in \mathbb{C} | \text{Im } \Omega(\lambda) < 0\}. \tag{3.7}$$

Then one can see that a partition of the complex k-plane $\mathbb{C}$ is of the form

$$\mathbb{C} = D_1 \cup D_2 \cup D_3 \cup D_4 \cup \Sigma. \tag{3.8}$$

4. The basic Riemann-Hilbert problem

Suppose that the spectral functions $a(k)$, $b(k)$ and $A(k)$, $B(k)$ are defined in a similar way (see [10,11]) by using initial function $q(x, 0)$ and boundary data $q(0, t)$, $q_x(0, t)$. Then, we can define the following Riemann-Hilbert problem RH_{xt}:

$$N_+(x, t; \lambda) = N_-(x; t, \lambda) J_N(x, t; \lambda), \tag{4.1}$$

where the functions $N_+(x, t; \lambda)$ and $N_-(x, t; \lambda)$ are the limiting values of the function $N(x, t; z)$ from the left and right sides of Σ as $z \to k$, respectively. The Riemann-Hilbert problem RH_{xt} can be introduced on the complex λ-plane $\mathbb{C}$ with the oriented contour

$$\Sigma = \mathbb{R} \cup \Gamma \cup \bar{\Gamma} \cup \gamma \cup \bar{\gamma}. \tag{4.2}$$

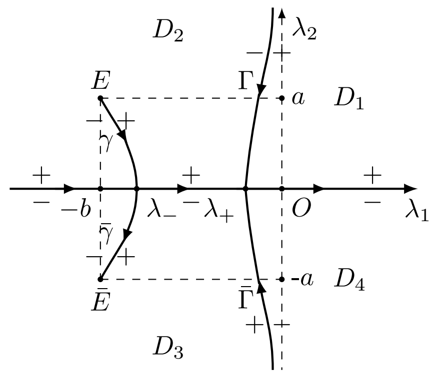

FIGURE 2. The oriented contour Σ for the case $\omega < a^2\left(\frac{1}{4}a^2+3b-1\right)$, $a>0$, $0<b<a^2/4$.

For the contour Σ (see Figure 2), the jump matrix $J_N(x,t;\lambda)$ can be written in the following six different expressions:

(4.3)

$$J_N(x,t;\lambda)=\begin{cases}\begin{pmatrix}1+\lambda|\hat{\rho}(\lambda)|^2 & \overline{\hat{\rho}(\bar{\lambda})}e^{-2i[\lambda x+(\Omega(\lambda)-\omega)t]}\\ \lambda\hat{\rho}(\lambda)e^{2i[\lambda x+(\Omega(k)-\omega)t]} & 1\end{pmatrix}, & \lambda\in(-\infty,\lambda_+),\\ \begin{pmatrix}1+\lambda|\hat{\rho}(\lambda)|^2 & \overline{\hat{r}(\bar{\lambda})}e^{-2i[\lambda x+(\Omega(\lambda)-\omega)t]}\\ \lambda\hat{r}(\lambda)e^{2i[\lambda x+(\Omega(\lambda)-\omega)t]} & 1\end{pmatrix}, & \lambda\in(\lambda_+,+\infty),\\ \begin{pmatrix}1 & 0\\ -\lambda\hat{c}(\lambda)e^{2i[\lambda x+(\Omega(\lambda)-\omega)t]} & 1\end{pmatrix}, & \lambda\in\Gamma,\\ \begin{pmatrix}1 & -\overline{\hat{c}(\bar{\lambda})}e^{-2i[\lambda x+(\Omega(\lambda)-\omega)t]}\\ 0 & 1\end{pmatrix}, & \lambda\in\bar{\Gamma},\\ \begin{pmatrix}e^{2i\Omega^+\lambda t} & 0\\ \lambda\hat{f}(\lambda)e^{2i[\lambda x-\omega t]} & e^{-2i\Omega^+kt}\end{pmatrix}, & \lambda\in\gamma,\\ \begin{pmatrix}e^{2i\Omega^+\lambda t} & \overline{\hat{f}(\bar{\lambda})}e^{-2i[\lambda x-\omega t]}\\ 0 & e^{-2i\Omega^+\lambda t}\end{pmatrix}, & \lambda\in\bar{\gamma},\end{cases}$$

where the scattering date $\hat{\rho}(\lambda)=\frac{\rho(k)}{k}$, $\hat{r}(\lambda)=\frac{r(k)}{k}$, $\hat{c}(\lambda)=\frac{c(k)}{k}$ and $\hat{f}(\lambda)=\frac{f(k)}{k}$ with $c(k):=\frac{T_{21}(k)}{T_{11}(k)}-\frac{\overline{b(\bar{k})}}{a(k)}$, $r(k):=\frac{\overline{b(\bar{k})}}{a(k)}$, $\rho(k):=c(k)+r(k)$, $f(\lambda):=c_-(\lambda)-c_+(\lambda):=\frac{-ie^{-i\delta}}{T_{11}^-(k)T_{11}^+(k)}$, and

$$\begin{aligned}T_{11}(k)&=\overline{T_{22}(\bar{k})}=a(k)\overline{A(\bar{k})}+b(k)\overline{B(\bar{k})},\\ T_{12}(k)&=-\overline{T_{21}(\bar{k})}=a(k)B(k)-b(k)A(k).\end{aligned}\tag{4.4}$$

The residue conditions at these zeros λ_j, z_j, $\bar{z}_j$ and $\bar{\lambda}_j$ are determined by

$$\left\{\begin{aligned}
&\operatorname{Res}_{\lambda=\lambda_j}[M(x,t;\lambda)]_1 = \frac{e^{2i\{\lambda_j x+[\Omega(\lambda_j)-\omega]t\}}}{\dot{a}(\lambda_j)b(\lambda_j)}[M(x,t;\lambda_j)]_2,\ \lambda_j\in D_1,\\
&\operatorname{Res}_{\lambda=z_j}[M(x,t;\lambda)]_1 = \operatorname{Res}_{\lambda=z_j}c(k)e^{2i\{z_j x+[\Omega(z_j)-\omega]t\}}[M(x,t;z_j)]_2,\ z_j\in D_2,\\
&\operatorname{Res}_{\lambda=\bar{z}_j}[M(x,t;\lambda)]_2 = \operatorname{Res}_{\lambda=\bar{z}_j}\overline{c(\bar{\lambda})}e^{-2i\{\bar{z}_j x+[\Omega(\bar{z}_j)-\omega]t\}}\overline{a(\bar{\lambda}_j)b(\bar{\lambda}_j)}[M(x,t;\bar{z}_j)]_1,\\
&\hspace{10cm}\bar{z}_j\in D_3,\\
&\operatorname{Res}_{\lambda=\bar{\lambda}_j}[M(x,t;\lambda)]_2 = -\frac{e^{2i\left\{\bar{\lambda}_j x+[\Omega(\bar{\lambda}_j)-\omega]t\right\}}}{\overline{\dot{a}(\bar{\lambda}_j)}\,\overline{b(\bar{\lambda}_j)}}[M(x,t;\bar{\lambda}_j)]_1,\ \bar{\lambda}_j\in D_4.
\end{aligned}\right. \tag{4.5}$$

Considering the off-diagonal elements of the spectral problems, we can derive the solution $q(x,t)$ of the IBV problems. Then, we have the following Riemann-Hilbert problem:

Theorem 4.1. *Let $q(x,t)$ be a solution of* (1.3a)-(1.3d) *with sufficient smoothness and decays as $x\to\infty$. Then $q(x,t)$ can be reformulated by the initial value $q_0(x)$ and boundary values $\{g_0(t),g_1(t)\}$, which are defined by*

$$\begin{aligned}
&q_0(x) = q(x,t=0),\\
&g_0(t) = q(x=0,t) = ae^{2i\omega t+i\delta}+v_0(t),\\
&g_1(t) = q_x(x=0,t) = 2iabe^{2i\omega t+i\delta}+v_1(t),\\
&q_0(0) = g_0(0) = ae^{i\delta},\ q_{0x}(0) = g_1(0) = 2iabe^{i\delta},
\end{aligned} \tag{4.6}$$

where $v_0(t)=0$, $v_1(t)=0$ for the Dirichlet IBV problem, and $v_0(t),v_1(t)\in\mathcal{S}(\mathbb{R}_+)$ for the Neumann IBV problem.

Assume that the functions $q_0(x)\in\mathcal{S}(\mathbb{R}_+)$, $g_0(t)$ and $g_1(t)$ satisfying the spectral functions $\{a(\lambda),b(\lambda),A(\lambda),B(\lambda)\}$ admit the global relation

$$b(\lambda)A(\lambda)-a(\lambda)B(\lambda)=0,\ \lambda\in D_1, \tag{4.7}$$

where the domain D_1 is defined in (3.6).

Then the solution $q(x,t)$ of (1.3a) *with IBV problems* (1.3b)-(1.3d) *is given by*

$$q(x,t) = 2i\lim_{\lambda\to\infty}(\lambda N(x,t;\lambda))_{12} \tag{4.8}$$

where $N(x,t;\lambda)$ admits the following 2×2 matrix Riemann-Hilbert problem: Given the spectral functions $\rho(\lambda)$, $r(\lambda)$, $f(\lambda)$, $c(\lambda)=c^-(\lambda)-c^+(\lambda)$, and the contour Σ, find a 2×2 matrix-value function $N(x,t;\lambda)$ such that

(i) *$N(x,t;\lambda)$ is sectionally meromorphic in $\lambda\in\mathbb{C}\backslash\Sigma$ or $\lambda\in\mathscr{R}\backslash\Sigma$, where $\mathscr{R}$ is the Riemann of genus zero.*

(ii) *Its first column $[N(x,t;\lambda)]_1$ admits simple poles at $\lambda_j\in D_1$ and $z_j\in D_2$; the second column $[N(x,t;\lambda)]_2$ has simple poles at $\bar{z}_j\in D_3$ and $\bar{\lambda}_j\in D_4$. Here the domains $\{D_j\}_1^4$ are determined in* (3.6). *The associated residues admit the relations* (4.5).

(iii) *The boundary value $N_\pm(x,t;\lambda)$ at Σ admits the jump condition*

$$N_+(x,t;\lambda) = N_-(x,t;\lambda)J_N(x,t;\lambda),\ \lambda\in\Sigma, \tag{4.9}$$

where the jump matrix $J(x,t;\lambda)$ can be defined in terms of the spectral functions by (4.3).

(iv) *Behavior at* ∞ *is*

$$N(x,t;\lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \quad as\ \lambda \to \infty. \tag{4.10}$$

(v) $\det N(x,t;\lambda) = 1$.

Proof. From the spectral analysis of (2.1a) and (2.2a), it only remains to prove (4.8) and (4.9), which follow from the large λ asymptotics of the eigenfunctions. □

5. Long-time asymptotic analysis

In order to analyze the long-time asymptotic behavior of the solution $q(x,t)$, let the spectral functions $\rho(\lambda)$, $r(\lambda)$, $c(\lambda)$ and $f(\lambda)$ admit the following properties:

(a) The function $c(\lambda)$ satisfies analytic continuation across the cut $\gamma \cup \bar{\gamma}$ connecting E and $\bar{E}$ on the second sheet of the Riemann surface of the function $X(\lambda)$.

(b) The function $f(\lambda)$ satisfies the following expansion at $\lambda = E = -b + ia$:

$$f(\lambda) = \sum_{j=0}^{\infty} c_j (\lambda - E)^{\frac{2j+1}{2}}. \tag{5.1}$$

(c) The discrete spectrum of the problem is empty; i.e., (i) $a(\lambda)$ does not vanish in D_1; (ii) $T_{11}(\lambda) = a(\lambda)\overline{A(\bar{\lambda})} + b(\lambda)\overline{B(\bar{\lambda})}$ does not vanish in D_2.

In what follows, in order to describe the long-time behavior of the solution $q(x,t)$ of the IBV problem, in this section some different asymptotic formulae are derived by the following three theorems in different regions of the first quarter of the xt-plane; see Figure 3.

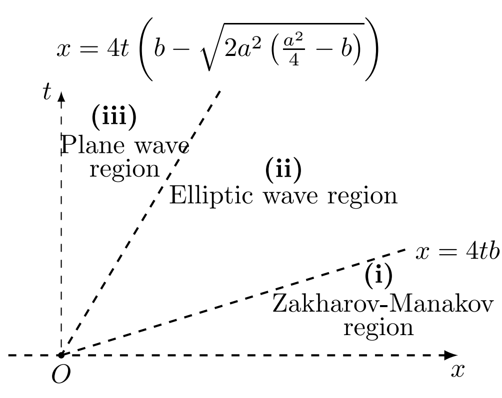

FIGURE 3. The different regions of the (x,t)-quarter-plane under the condition $2b^2 + a^2 b - \frac{1}{4}a^4 + \omega = 0$, $0 < b < a^2/4$.

Theorem 5.1 (The Zakharov-Manakov region, $x > 4tb$)**.** *Let all conditions of Theorem* 4.1 *and assumption* (c) *be satisfied.*

In the region $x > 4tb$*, the asymptotics, as* $t \to +\infty$*, of the solution* $q(x,t)$ *for the IBV problems of* (1.3a)-(1.3d) *takes the form of the Zakharov-Manakov type* *formula*

$$q(x,t)=\frac{1}{\sqrt{t}}\widetilde{q_{asy}}(\lambda_0)e^{i\frac{x^2}{4t}-i\nu(\lambda_0)\log t}+O\left(\frac{\log t}{t}\right), \tag{5.2}$$

with

$$\begin{aligned}
&|\widetilde{q_{asy}}(\lambda_0)|^2=\frac{\nu(\lambda_0)}{2}=-\frac{1}{4\pi}\log\left(1-\lambda_0|\rho(\lambda_0)|^2\right),\\
&\arg\widetilde{q_{asy}}(\lambda_0)=\arg\Gamma(i\nu(\lambda_0))-\arg\rho(\lambda_0)-3\nu(\lambda_0)\log 2-\frac{\pi}{4}\\
&\quad+\frac{1}{\pi}\int_{-\infty}^{\lambda_0}\log(|\lambda-\lambda_0|)d\log(1-\lambda|\rho(\lambda)|^2),
\end{aligned} \tag{5.3}$$

where $\nu(\lambda_0)=-\frac{1}{2\pi}\log(1-\lambda_0|\rho(\lambda_0)|^2)$, $\rho(\lambda)=\frac{r(k)}{k}$, *and* $\Gamma(\cdot)$ *implies Euler's gamma-function, and* $\lambda_0=-\frac{x}{4t}$.

Proof. In what follows, we can follow the technique of asymptotic analysis proposed for the first time in [6] to study the asymptotic behavior of the Riemann-Hilbert problem (4.9) in the region $x>4tb$. We introduce the first transformation

$$N^{(1)}(x,t,\lambda)=N(x,t,\lambda)\delta^{\sigma_3}(\lambda), \tag{5.4}$$

with

$$\delta(\lambda)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\lambda_0}\frac{\log(1-\lambda'|\rho(\lambda')|^2)}{\lambda'-\lambda}d\lambda'\right\}, \tag{5.5}$$

where $\lambda_0=-\frac{x}{4t}$. Next, the new jump matrix $J_N^{(1)}(x,t,\lambda)$ is analytically extended from Σ. It yields the second transformation

$$N^{(2)}(x,t,\lambda)=N^{(1)}(x,t,\lambda)G(\lambda), \tag{5.6}$$

with $G(\lambda)$ defined by

$$G(\lambda)=\begin{cases}\begin{pmatrix}1 & \overline{\hat{r}(\bar{\lambda})}\delta^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_1,\ \arg(\lambda-\lambda_0)\in(0,\pi/4),\\ \begin{pmatrix}1 & 0\\ -\lambda\hat{r}(\lambda)\delta^{-2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_4,\ \arg(\lambda-\lambda_0)\in(7\pi/4,2\pi),\end{cases}\tag{5.7a}$$

$$G(\lambda)=\begin{cases}\begin{pmatrix}1 & 0\\ -\lambda\delta^{-2}\hat{c}(\lambda)e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_1,\ \arg(\lambda-\lambda_0)\in(0,\pi/4),\\ \begin{pmatrix}1 & \delta^2\overline{\hat{c}(\bar{\lambda})}e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_4,\ \arg(\lambda-\lambda_0)\in(3\pi/2,7\pi/4),\end{cases}\tag{5.7b}$$

$$G(\lambda)=\begin{cases}\begin{pmatrix}1 & \overline{\hat{\rho}(\bar{\lambda})}\delta^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_2,\ \arg(\lambda-\lambda_0)\in(0,\pi/4),\\ \begin{pmatrix}1 & 0\\ -\lambda\hat{\rho}(\lambda)\delta^{-2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_3,\ \arg(\lambda-\lambda_0)\in(7\pi/4,2\pi),\end{cases}\tag{5.7c}$$

$$G(\lambda)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \begin{matrix}\lambda\in D_2,\ \arg(\lambda-\lambda_0)\in(\pi/4,3\pi/4),\\ \lambda\in D_3,\ \arg(\lambda-\lambda_0)\in(5\pi/4,7\pi/4),\end{matrix}\tag{5.7d}$$

$$G(\lambda)=\begin{cases}\begin{pmatrix}1 & 0\\ \lambda\hat{\rho}'(\lambda)\delta^{-2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_2,\ \arg(\lambda-\lambda_0)\in(3\pi/4,\pi),\\ \begin{pmatrix}1 & -\overline{\hat{\rho}'(\bar{\lambda})}\delta^2e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & \lambda\in D_3,\ \arg(\lambda-\lambda_0)\in(\pi,5\pi/4),\end{cases}\tag{5.7e}$$

where $\hat{r}(\lambda)$, $\hat{c}(\lambda)$, $\hat{\rho}(\lambda)$ and $\hat{\rho}'(\lambda)$ are suitable analytic approximations of the functions $r(\lambda)$, $c(\lambda)$, $\rho(\lambda)$, $\rho'(\lambda)=\rho(\lambda)/(1+|\rho(\lambda)|^2)$. From above, we have the final transformation

$$N^{(2)}(x,t,\lambda)=Z(x,t,\lambda)N^{asy}(x,t,\lambda),\tag{5.8}$$

where the function $N^{asy}(x,t,\lambda)$ solves the model problem explicitly presented in terms of parabolic cylinder functions, whereas $Z(x,t,\lambda)$ can be estimated:

$$Z(x,t,\lambda)=\mathbb{I}+O\left(\frac{\log t}{t^{\frac{1}{2}}}\right).\tag{5.9}$$

Then, the asymptotic result of $N^{asy}(x,t,\lambda)$ yields the Zakharov-Manakov wave (5.2). □

Theorem 5.2 (Plane wave region, $0\le x<4t\left(b-\sqrt{2a^2\left(\frac{a^2}{4}-b\right)}\right)$). *Let all conditions of Theorem* 4.1 *and assumptions* (a), (b), (c) *be satisfied.*

Then in the region $0\le x<4t\left(b-\sqrt{2a^2\left(\frac{a^2}{4}-b\right)}\right)$, *the asymptotics, as* $t\to+\infty$, *of the solution* $q(x,t)$ *for the IBV problems of* (1.3a)-(1.3d) *is described by a plane wave:*

$$q(x,t)=ae^{i\delta}e^{2i[\omega t+bx-\phi(\xi)]}+O\left(t^{-1/2}\right),\ \text{as } t\to+\infty,\tag{5.10}$$

where

$$\phi(\xi)=\frac{1}{2\pi}\left\{\int_{-\infty}^{\mu_-(\xi)}\log\left[1+\lambda|\hat{\rho}(\lambda)|^2\right]\frac{d\lambda}{X(\lambda)}+\int_{\gamma_g\cup\bar{\gamma}_g}\log[h(s)e^{i\delta}\delta^{-2}(s,\xi)]\frac{ds}{X_+(s)}\right\},$$

(5.11)

$$\mu_-(\xi)=-\frac{b}{2}-\xi-\sqrt{\left(\xi-\frac{b}{2}\right)^2-\frac{a^2e^{2i\delta}}{2}\left(\frac{a^2e^{2i\delta}}{4}-b\right)},\ \gamma_g\cup\bar{\gamma}_g=\gamma\cup\bar{\gamma},\ \xi=\frac{x}{4t}.$$

Proof. For the region $0\le x<4t\left(b-\sqrt{2a^2\left(\frac{a^2}{4}-b\right)}\right)$, we take the g-function as follows:

$$g(x,t,\lambda)=xX(\lambda)+t\Omega(\lambda), \tag{5.12}$$

where the functions $\Omega(\lambda)$ and $X(\lambda)$ are determined by (2.5d) and (2.5e). The g-function admits the following asymptotic behavior:

$$g(\lambda;\xi)=t\left(2\lambda^2+4\xi\lambda+g(\infty;\xi)\right)+O\left(\frac{1}{\lambda}\right),\ \lambda\to\infty,\ \text{with } g(\infty;\xi)=t(\omega+4\xi B).$$

We make the following three transforms:

$$N^{(1)}(x,t,\lambda)=e^{-itg(\infty;\xi)\sigma_3}N(x,t,\lambda)e^{-i[\lambda x+4\lambda^2t-g(\lambda)]\sigma_3}, \tag{5.13}$$

$$N^{(2)}(x,t,\lambda)=N^{(1)}(x,t,\lambda)\delta^{-\sigma_3}(\lambda), \tag{5.14}$$

$$N^{(3)}(x,t,\lambda)=N^{(2)}(x,t,\lambda)G(\lambda), \tag{5.15}$$

where δ is introduced in (5.5), and $G(\lambda)$ is introduced similarly to (5.7a), with $t\theta(k)$ instead of $g(\lambda)$. Next, we make a fourth transformation:

(5.16)

$$N^{(4)}(x,t,\lambda)=\begin{cases}F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi), & \lambda \text{ outside the lenses},\\ F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi)\hat{N}_{\text{low}}(\lambda), & \lambda \text{ inside the lower right lens},\\ F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi)N_{\text{low}}^{-1}(\lambda), & \lambda \text{ inside the lower left lens},\\ F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi)\hat{N}_{\text{up}}(\lambda), & \lambda \text{ inside the upper right lens},\\ F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi)N_{\text{up}}^{-1}(\lambda), & \lambda \text{ inside the upper left lens},\end{cases}$$

where $F(\lambda)=\exp\left\{\frac{X(\lambda)}{2\pi i}\int_{\gamma_g\cup\bar{\gamma}_g}\log\left[h(s)e^{i\delta}\delta^{-2}(s)\right]\frac{ds}{(s-\lambda)X_+(s)}\right\}$, $F(\infty)=e^{i\phi(\xi)}$ with $\phi(\xi)=\frac{1}{2\pi}\int_{\gamma_g\cup\bar{\gamma}_g}\log\left[h(s)e^{i\delta}\delta^{-2}(s,\xi)\right]\frac{ds}{X_+(s)}$, and $\hat{N}_{\text{low}}(\lambda)=\begin{pmatrix}1 & 0\\ \hat{N}_{\text{low}}^{(21)} & 1\end{pmatrix}$, $N_{\text{low}}(\lambda)=\begin{pmatrix}1 & 0\\ N_{\text{low}}^{(21)} & 1\end{pmatrix}$, $\hat{N}_{\text{up}}(\lambda)=\begin{pmatrix}1 & \hat{N}_{\text{up}}^{(12)}\\ 0 & 1\end{pmatrix}$, $N_{\text{up}}(\lambda)=\begin{pmatrix}1 & N_{\text{up}}^{(12)}\\ 0 & 1\end{pmatrix}$, with

$$\hat{N}_{\text{low}}^{(21)} = \lambda\delta^{-2}(\lambda)F_-^2(\lambda)\left[\hat{\rho}'_-(\lambda) - f^{-1}(\lambda)\right]e^{2ig_+(\lambda)},$$
$$N_{\text{low}}^{(21)} = -\lambda\delta^{-2}(\lambda)F_+^2(\lambda)\left[\hat{\rho}'_+(\lambda) + f^{-1}(\lambda)\right]e^{-2ig_+(\lambda)},$$
$$\hat{N}_{\text{up}}^{(12)} = -\delta^2(\lambda)F_-^2(\lambda)\left[\overline{\hat{\rho}'_-(\bar{\lambda})} - \overline{f^{-1}(\bar{\lambda})}\right]e^{-2ig_+(\lambda)},$$
$$\text{and } N_{\text{up}}^{(12)} = \delta^2(\lambda)F_+^2(\lambda)\left[\overline{\hat{\rho}'_+(\bar{\lambda})} + \overline{f^{-1}(\bar{\lambda})}\right]e^{2ig_+(\lambda)}.$$

Then, we have the model problem

$$N^{(4)}(x,t,\lambda = \left(\mathbb{I} + O(t^{-1/2})\right)N^{\text{mod}}(x,t,\lambda), \tag{5.17}$$

where $N^{\text{mod}}(x,t,\lambda)$ satisfies the following zero-gap model problem RH^{mod}:

$$N_+^{\text{mod}}(x,t,\lambda) = N_-^{\text{mod}}(x,t,\lambda)J_N^{\text{mod}},\ \ \lambda \in \gamma_g \cup \bar{\gamma}_g, \tag{5.18}$$

with constant jump matrix

$$J_N^{\text{mod}} = \begin{pmatrix} 0 & -ie^{i\delta} \\ -ie^{-i\delta} & 0 \end{pmatrix}. \tag{5.19}$$

From above, $q(x,t)$ in terms of the solution of the basic Riemann-Hilbert problem can be derived by $q(x,t) = 2i(N^{\text{mod}})_{12}(x,t)$, which yields the plane wave (5.10). □

Remark 5.3. For the structure of the Dirichlet to Neumann map, equation (5.10) satisfies the assumption (1.8a)-(1.8b). For the case of $x = 0$, the restrictions (a) and (b) are not needed to further consider the Riemann-Hilbert data for researching the asymptotic approach. In fact, this case yields $\gamma = \gamma_g$, which implies that it is not needed to deform the contour $\gamma_g \cup \bar{\gamma}_g$.

Theorem 5.4 (Modulated elliptic wave region, $4t\left(b - \sqrt{2a^2\left(\frac{a^2}{4} - b\right)}\right) < x < 4tb$)**.** *Let all conditions of Theorem* 4.1 *and assumptions* (a), (b), (c) *be satisfied.*

In the region $4t\left(b - \sqrt{2a^2\left(\frac{a^2}{4} - b\right)}\right) < x < 4tb$*, the asymptotics, as* $t \to +\infty$*, of the solution* $q(x,t)$ *for the initial value problem of* (1.3a)-(1.3d) *is described by a modulated elliptic wave:*

$$\begin{aligned} q(x,t) = {} & \left[\sqrt{a^2\left(\frac{a^2}{4} - b\right)} + Im\ d(\xi)\right] \\ & \times e^{i\delta}\frac{\theta_3\left[\frac{B_g t}{2\pi} + \frac{B_\omega \Delta}{2\pi} - \mathcal{U}_-(\xi) + \frac{\tau}{2} + \frac{1}{2}\right]\theta_3\left[\mathcal{U}_+(\xi) - \frac{\tau}{2} - \frac{1}{2}\right]}{\theta_3\left[\frac{B_g t}{2\pi} + \frac{B_\omega \Delta}{2\pi} - \mathcal{U}_+(\xi) + \frac{\tau}{2} + \frac{1}{2}\right]\theta_3\left[\mathcal{U}_-(\xi) - \frac{\tau}{2} - \frac{1}{2}\right]} \\ & \times e^{2ig(\infty,\xi) - 2i\phi(\xi)} + O\left(t^{-\frac{1}{2}}\right),\ \ t \to +\infty, \end{aligned} \tag{5.20}$$

where B_g, Δ, *and* B_ω *are functions with respect to the variables* $\xi = \frac{x}{4t}$*, respectively, and* $\mathcal{U}_\pm(\xi) = U_0 \pm U(\infty)$ *with* $U(\lambda) = \frac{1}{c}\int_E^\lambda \frac{d\lambda'}{\omega(\lambda')}$ *and* $E = -b + iD$*. Moreover,*

$$\theta_3(z) = \sum_{z \in \mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z} \tag{5.21}$$

is the theta function of invariant $\tau = \tau(\xi)$ *introduced by* $\tau = \frac{2}{c}\int_E^d \frac{d\lambda'}{\omega(\lambda')}$. *The function* $g = g(\infty, \xi)$ *is given by*

$$g(\infty,\xi) = t\left(2\left(\int_E^\infty + \int_{\bar{E}}^\infty\right)\right)\left[(\lambda' - \mu(\xi))\sqrt{\frac{(\lambda' - d(\xi))(\lambda' - \bar{d}(\xi))}{(\lambda' - E)(\lambda' - \bar{E})}} - (\lambda' + \xi)\right] d\lambda' \tag{5.22}$$
$$+ 2t\left(\frac{a^4}{4} - a^2 b - b^2 + 2\xi b\right),$$

where $d(\xi) = d_1 + id_2$, $c_1 = -(b+\xi)d_1 + b\xi + \frac{1}{2}\left(d_2^2 + D^2\right)$, $c_2 = \xi - d_1 + b$, $c_0 \doteqdot c_0(\xi, d_1, d_2) = -\frac{\int_{\bar{d}}^d \left(\lambda^3 + c_2\lambda^2 + c_1\lambda\right)\frac{d\lambda}{\omega(\lambda)}}{\int_{\bar{d}}^d \frac{d\lambda}{\omega(\lambda)}} \in \mathbb{R}$, *with* $d_1 = d_1(\xi) = -\mu - \xi - b$, $d_2 = d_2(\xi) = \sqrt{D^2 - 2(b+\mu)(\xi+\mu)}$, *and* $\mu(\xi) = \frac{\mathcal{I}_1\left(-\mu-\xi-b, \sqrt{D^2 - 2(b+\mu)(\xi+\mu)}\right)}{\mathcal{I}_0\left(-\mu-\xi-b, \sqrt{D^2 - 2(b+\mu)(\xi+\mu)}\right)}$, *with*

$$\mathcal{I}_0(d_1, d_2) = \int_{d_1 - id_2}^{d_1 + id_2} \sqrt{\frac{(\lambda' - d_1)^2 + d_2}{(\lambda' + b)^2} + D^2 d\lambda'},$$
$$\mathcal{I}_1(d_1, d_2) = \int_{d_1 - id_2}^{d_1 + id_2} \lambda' \sqrt{\frac{(\lambda' - d_1)^2 + d_2}{(\lambda' + b)^2} + D^2 d\lambda'}.$$

The phase shift $\phi(\xi)$ *is determined as follows:*

$$\phi(\xi) = \frac{1}{2\pi}\int_{\Gamma_d \cup \Gamma_{\bar{d}}} \frac{[\lambda' + e_1(\xi) - \omega_\infty(\xi)]\log\left[h(\lambda')e^{i\delta}\delta^{-2}(\lambda', \xi)\right]}{\left[(\lambda' - E)(\lambda' - \bar{E})(\lambda' - d(\xi))(\lambda' - \bar{d}(\xi))\right]^{\frac{1}{2}}} d\lambda', \tag{5.23}$$

where $h(\lambda)$, $\delta(\lambda, \xi)$ *are given by*

$$h(\lambda) = \begin{cases} i\lambda\hat{f}(\lambda), \ \lambda \in \gamma_d, \\ -i\lambda\overline{\hat{f}^{-1}(\bar{\lambda})}, \ \lambda \in \overline{\gamma_d}, \end{cases} \quad \text{with } \gamma_0 \cup \bar{\gamma}_0 = \gamma_d \cup \overline{\gamma_d} \cup [d, \bar{d}], \tag{5.24}$$

$$\delta(\lambda, \xi) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\mu(\xi)} \frac{\log\left[1 + \lambda'|r(\lambda') + c(\lambda')|^2\right]}{\lambda' - \lambda} d\lambda'\right\}, \tag{5.25}$$

and $e_1(\xi) = -\frac{E + \bar{E} + d + \bar{d}}{2}$, $\omega_\infty = \int_E^\infty \left[\frac{z^2 + e_1 z + e_0}{\omega(z)} - 1\right] dz - E$.

Proof. In the region $4t\left(b - \sqrt{2a^2\left(\frac{a^2}{4} - b\right)}\right) < x < 4tb$, by considering the sum of two Abelian integrals, one can rewrite $g(\lambda, \xi)$ and $g(\infty, \xi)$ as follows:

$$g(\lambda, \xi) = 2\left(\int_E^\lambda + \int_{\bar{E}}^\lambda\right) \frac{\lambda'^3 + c_2\lambda'^2 + c_1\lambda' + c_0}{\sqrt{(\lambda' - E)(\lambda' - \bar{E})(\lambda' - d(\xi))(\lambda' - \bar{d}(\xi))}} d\lambda',$$
$$g(\infty, \xi) =$$
$$t\left[2\left(\int_E^\infty + \int_{\bar{E}}^\infty\right)\left(\frac{\lambda'^3 + c_2\lambda'^2 + c_1\lambda' + c_0}{\sqrt{(\lambda' - E)(\lambda' - \bar{E})(\lambda' - d(\xi))(\lambda' - \bar{d}(\xi))}} - \lambda' - \xi\right) d\lambda'\right]$$
$$+ 2t(D^2 - b^2 + 2b\xi).$$

The same transformations can be made as follows:

$$N(x,t,\lambda) \rightsquigarrow N^{(1)}(x,t,\lambda) \rightsquigarrow N^{(2)}(x,t,\lambda) \rightsquigarrow N^{(3)}(x,t,\lambda), \tag{5.26}$$

which are similar to ones for the plane wave region, but $\lambda_0 = \mu(\xi)$, where $\mu(\xi)$ is the real stationary point of $g(\lambda)$. Following in the same way, we have the following model problem:

$$N_+^{\mathrm{mod}}(x,t,\lambda) = N_-^{\mathrm{mod}}(x,t,\lambda) J_N^{\mathrm{mod}}(x,t,\lambda),\ \lambda \in \gamma_d \cup \overline{\gamma_d} \cup [d,\bar{d}], \tag{5.27a}$$

$$N^{\mathrm{mod}}(x,t,\lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right),\ \lambda \to \infty. \tag{5.27b}$$

The jump matrix reads

$$J_N^{\mathrm{mod}}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 0 & -ie^{i\delta} \\ -ie^{-i\delta} & 0 \end{pmatrix},\ \lambda \in \gamma_d \cup \overline{\gamma_d}, \\ \begin{pmatrix} e^{-itB_g - i\Delta B_\omega} & 0 \\ 0 & e^{itB_g + i\Delta B_\omega} \end{pmatrix},\ \lambda \in [d,\bar{d}], \end{cases} \tag{5.27c}$$

where B_g, Δ and B_ω are functions with respect to the variables $\xi = \frac{x}{4t}$. By using elliptic theta functions, one can solve the model problem. In order to find such a solution, let us consider the following elliptic Riemann surface of

$$\omega(\lambda) = \sqrt{(\lambda - E)(\lambda - \bar{E})(\lambda - d)(\lambda - \bar{d})},$$

where $d = d(\xi)$ and $\bar{d} = \bar{d}(\xi)$. Following the same method as in the plane wave region, one has $q(x,t) = 2i(N^{\mathrm{mod}})_{12}(x,t)$, where

$$\begin{aligned} &2i(N^{\mathrm{mod}})_{12}(x,t,\lambda) \\ &\quad = [D + d_2(\xi)]e^{i\delta} \frac{\theta_3\left[\frac{B_g t}{2\pi} + \frac{B_\omega \Delta}{2\pi} + U(\infty) - U_0 - \frac{\tau}{2} - \frac{1}{2}\right]\theta_3[U(\infty) + U_0 + \frac{\tau}{2} + \frac{1}{2}]}{\theta_3\left[\frac{B_g t}{2\pi} + \frac{B_\omega \Delta}{2\pi} - U(\infty) + U_0 + \frac{\tau}{2} + \frac{1}{2}\right]\theta_3[U(\infty) + U_0 + \frac{\tau}{2} + \frac{1}{2}]}, \end{aligned}$$

which yields the modulated elliptic wave (5.20). □

Remark 5.5. In this paper, we study the long-time asymptotics of the solution to the GI equation (1.2) with time-periodic boundary condition. For $\xi = \xi_0$, one has $\mathrm{Im}\, d(\xi_0) = 0$. Then $g^{\mathrm{elliptic}}(\infty, \xi_0) = g^{\mathrm{planewave}}(\infty, \xi_0)$, and $\theta(\cdot, \xi_0) = 1$. It provides matching at the interface between the plane wave (5.10) and the elliptic wave (5.20) as $\xi = \xi_0$.

Suppose that $\xi_0 = 0$, i.e., $b^2 = \frac{a^4}{2} - 2a^2 b$. Then the plane wave region disappears, and the asymptotic behavior of the solution is only described by elliptic functions with modulated parameter.

Acknowledgments

The first author appreciates the hospitality of the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, where the work was done.

References

[1] R. Beals and R. R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. Pure Appl. Math. **37** (1984), no. 1, 39–90, DOI 10.1002/cpa.3160370105. MR728266

[2] Anne Boutet de Monvel, Alexander Its, and Vladimir Kotlyarov, *Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line*, Comm. Math. Phys. **290** (2009), no. 2, 479–522, DOI 10.1007/s00220-009-0848-7. MR2525628

[3] Anne Boutet de Monvel, Alexander Its, and Vladimir Kotlyarov, *Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **345** (2007), no. 11, 615–620, DOI 10.1016/j.crma.2007.10.018. MR2371477

[4] Anne Boutet de Monvel and Vladimir Kotlyarov, *The focusing nonlinear Schrödinger equation on the quarter plane with time-periodic boundary condition: a Riemann-Hilbert approach*, J. Inst. Math. Jussieu **6** (2007), no. 4, 579–611, DOI 10.1017/S1474748007000151. MR2337309

[5] Robert Buckingham and Stephanos Venakides, *Long-time asymptotics of the nonlinear Schrödinger equation shock problem*, Comm. Pure Appl. Math. **60** (2007), no. 9, 1349–1414, DOI 10.1002/cpa.20179. MR2337507

[6] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math. (2) **137** (1993), no. 2, 295–368, DOI 10.2307/2946540. MR1207209

[7] P. A. Deift, A. R. It·s, and X. Zhou, *Long-time asymptotics for integrable nonlinear wave equations*, Important developments in soliton theory, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993, pp. 181–204. MR1280475

[8] P. Deift, S. Venakides, and X. Zhou, *The collisionless shock region for the long-time behavior of solutions of the KdV equation*, Comm. Pure Appl. Math. **47** (1994), no. 2, 199–206, DOI 10.1002/cpa.3160470204. MR1263128

[9] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), no. 11, 1335–1425, DOI 10.1002/(SICI)1097-0312(199911)52:11⟨1335::AID-CPA1⟩3.0.CO;2-1. MR1702716

[10] A. S. Fokas and A. R. Its, *The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation*, SIAM J. Math. Anal. **27** (1996), no. 3, 738–764, DOI 10.1137/0527040. MR1382831

[11] A. S. Fokas, A. R. Its, and L.-Y. Sung, *The nonlinear Schrödinger equation on the half-line*, Nonlinearity **18** (2005), no. 4, 1771–1822, DOI 10.1088/0951-7715/18/4/019. MR2150354

[12] Engui Fan, *Darboux transformation and soliton-like solutions for the Gerdjikov-Ivanov equation*, J. Phys. A **33** (2000), no. 39, 6925–6933, DOI 10.1088/0305-4470/33/39/308. MR1791943

[13] V. S. Gerdžikov, M. I. Ivanov, and P. P. Kuliš, *Quadratic pencils and nonlinear equations* (Russian), Teoret. Mat. Fiz. **44** (1980), no. 3, 342–357. MR596207

[14] A. R. It·s, *Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations* (Russian), Dokl. Akad. Nauk SSSR **261** (1981), no. 1, 14–18. MR636848

[15] Mourad E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, with two chapters by Walter Van Assche, with a foreword by Richard A. Askey, reprint of the 2005 original, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, 2009. MR2542683

[16] Mourad E. H. Ismail and Erik Koelink, *Spectral analysis of certain Schrödinger operators*, SIGMA Symmetry Integrability Geom. Methods Appl. **8** (2012), Paper 061, 19, DOI 10.3842/SIGMA.2012.061. MR2970767

[17] Mourad E. H. Ismail and Erik Koelink, *Spectral properties of operators using tridiagonalization*, Anal. Appl. (Singap.) **10** (2012), no. 3, 327–343, DOI 10.1142/S0219530512500157. MR2948897

[18] A. V. Kitaev and A. H. Vartanian, *Asymptotics of solutions to the modified nonlinear Schrödinger equation: solitons on a nonvanishing continuous background*, SIAM J. Math. Anal. **30** (1999), no. 4, 787–832, DOI 10.1137/S0036141098332019. MR1684726

[19] A. V. Kitaev and A. H. Vartanian, *Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: solitonless sector*, Inverse Problems **13** (1997), no. 5, 1311–1339, DOI 10.1088/0266-5611/13/5/014. MR1474371

[20] J. Lenells and A. S. Fokas, *The nonlinear Schrödinger equation with t-periodic data: I. Exact results*, Proc. A. **471** (2015), no. 2181, 20140925, 22, DOI 10.1098/rspa.2014.0925. MR3404291

[21] J. Lenells and A. S. Fokas, *The nonlinear Schrödinger equation with t-periodic data: II. Perturbative results*, Proc. A. **471** (2015), no. 2181, 20140926, 25, DOI 10.1098/rspa.2014.0926. MR3404292

[22] Jonatan Lenells, *Admissible boundary values for the defocusing nonlinear Schrödinger equation with asymptotically time-periodic data*, J. Differential Equations **259** (2015), no. 11, 5617–5639, DOI 10.1016/j.jde.2015.07.003. MR3397302

[23] S. V. Manakov, *Nonlinear Fraunhofer diffraction*, Zh. Eksp. Teor. Fiz. **65** (1973), 1392–1398.

[24] Wen-Xiu Ma and Min Chen, *Direct search for exact solutions to the nonlinear Schrödinger equation*, Appl. Math. Comput. **215** (2009), no. 8, 2835–2842, DOI 10.1016/j.amc.2009.09.024. MR2563398

[25] W.X. Ma, Soliton, positon and negaton solutions to a Schröedinger self-consistent source equation, J. Phys. Soci. Japan, **72** (2003), 3017-3019.

[26] Wen-Xiu Ma and Yuncheng You, *Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1753–1778, DOI 10.1090/S0002-9947-04-03726-2. MR2115075

[27] Shou-Fu Tian, *Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method*, J. Differential Equations **262** (2017), no. 1, 506–558, DOI 10.1016/j.jde.2016.09.033. MR3567493

[28] Shou-Fu Tian, *The mixed coupled nonlinear Schrödinger equation on the half-line via the Fokas method*, Proc. A. **472** (2016), no. 2195, 20160588, 22, DOI 10.1098/rspa.2016.0588. MR3592256

[29] S. F. Tian, A generalized derivative nonlinear Schrödinger equation with time-periodic boundary condition, submitted.

[30] S. F. Tian, Riemann-Hilbert problem to a generalized derivative nonlinear Schrödinger equation: Long-time dynamics of step-like initial data, submitted.

[31] Arthur H. Vartanian, *Higher order asymptotics of the modified non-linear Schrödinger equation*, Comm. Partial Differential Equations **25** (2000), no. 5-6, 1043–1098, DOI 10.1080/03605300008821541. MR1759802

[32] X. B. Wang, S. F. Tian, and T. T. Zhang, Characteristics of the breather and rogue waves in a (2+1)-dimensional nonlinear Schrödinger equation, Proc. Amer. Math. Soc., DOI: https://doi.org/10.1090/proc/13765.

[33] Jian Xu, Engui Fan, and Yong Chen, *Long-time asymptotic for the derivative nonlinear Schrödinger equation with step-like initial value*, Math. Phys. Anal. Geom. **16** (2013), no. 3, 253–288, DOI 10.1007/s11040-013-9132-3. MR3090832

[34] J. Xu, E. G. Fan, Long-time asymptotic for the derivative nonlinear Schrödinger equation with decaying initial value. Preprint, arXiv.1209.4245v2.

School of Mathematics and Institute of Mathematical Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China –and– Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom

Current address: School of Mathematics and Institute of Mathematical Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China

E-mail address: `sftian@cumt.edu.cn, shoufu2006@126.com`

School of Mathematics and Institute of Mathematical Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China

E-mail address: `ttzhang@cumt.edu.cn`