

\mathbb{Z}_2 -ORBIFOLD CONSTRUCTION ASSOCIATED WITH (-1) -ISOMETRY AND UNIQUENESS OF HOLOMORPHIC VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

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ABSTRACT. The vertex operator algebra structure of a strongly regular holomorphic vertex operator algebra V of central charge 24 is proved to be uniquely determined by the Lie algebra structure of its weight one space V_1 if V_1 is a Lie algebra of the type $A_{1,4}^{12}$, $B_{2,2}^6$, $B_{3,2}^4$, $B_{4,2}^3$, $B_{6,2}^2$, $B_{12,2}$, $D_{4,2}^2 B_{2,1}^4$, $D_{8,2} B_{4,1}^2$, $A_{3,2}^4 A_{1,1}^4$, $D_{5,2}^2 A_{3,1}^2$, $D_{9,2} A_{7,1}$, $C_{4,1}^4$, or $D_{6,2} B_{3,1}^2 C_{4,1}$.

1. INTRODUCTION

The classification of holomorphic vertex operator algebras (VOAs) of central charge 24 is one of the important problems in the theory of VOAs and conformal field theories. In 1993, Schellekens [Sch] obtained a partial classification of holomorphic VOAs of central charge 24 and showed that there are 71 possible Lie algebra structures for the weight one spaces of holomorphic VOAs of central charge 24 (see also [EMS]). Recently, holomorphic VOAs of central charge 24 corresponding to all 71 Lie algebras in Schellekens's list have been explicitly constructed (see [EMS], [LS16], [LLin] and [SS]). To finish the classification of holomorphic VOAs of central charge 24, it remains to show that there is a unique holomorphic VOA of central charge 24 corresponding to each Lie algebra in Schellekens's list. Motivated by the fact that the unimodular lattices of rank 24 (Niemeier lattices) are determined by their root systems, it is believed that the following conjecture is true.

Conjecture 1.1. *The VOA structure of a strongly regular holomorphic VOA V of central charge 24 is uniquely determined by its weight one Lie algebra V_1 .*

Until now, Conjecture 1.1 has been verified in the following cases: (i) the weight one Lie algebras of the 24 Niemeier lattice VOAs (24 cases) [DM04b]; (ii) $A_{1,2}^{16}$ and $E_{8,2} B_{8,1}$ [LS1]; (iii) $E_{6,3} G_{2,1}^3$, $A_{2,3}^6$, and $A_{5,3} D_{4,3} A_{1,1}^3$ [LS4]; (iv) $A_{8,3} A_{2,1}^2$ [LLin]. In this paper, we will consider 13 other Lie algebras in Schellekens's list. More precisely, we will prove the following result.

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Theorem 1.2. *The structure of a strongly regular holomorphic VOA V of central charge 24 is uniquely determined by its weight one Lie algebra V_1 if V_1 has the type*

$$A_{1,4}^{12}, \quad B_{2,2}^6, \quad B_{3,2}^4, \quad B_{4,2}^3, \quad B_{6,2}^2, \quad B_{12,2}, \quad D_{4,2}^2 B_{2,1}^4, \quad D_{8,2} B_{4,1}^2, \\ A_{3,2}^4 A_{1,1}^4, \quad D_{5,2}^2 A_{3,1}^2, \quad D_{9,2} A_{7,1}, \quad C_{4,1}^4, \quad \text{or} \quad D_{6,2} B_{3,1}^2 C_{4,1}.$$

The holomorphic VOAs in Theorem 1.2 can be obtained by applying \mathbb{Z}_2 -orbifold construction to Niemeier lattice VOAs and lifts of the (-1) -isometry of the lattices [DGM]. To apply the “reverse orbifold construction” method proposed in [LS4], there are two key steps. The first step is to find an appropriate semisimple element $h \in V_1$ such that the VOA obtained by applying \mathbb{Z}_2 -orbifold construction to V and the inner automorphism σ_h is isomorphic to a Niemeier lattice VOA V_N (see Lemma 4.7). Since σ_h acts trivially on the weight one subspace V_1 in our cases, the non-trivial part is to show that σ_h has order 2 on V (cf. Lemma 4.5). We also show a technical lemma (see Lemma 4.3), which helps us to determine the lowest conformal weight of the irreducible twisted module and greatly reduces the amount of calculations in our cases. The second main step is to show that any order 2 automorphism μ of the Niemeier lattice VOA satisfying $(V_N)_1^\mu \cong (V_1)^{\sigma_h}$ is conjugate to θ (cf. Eq. (2.2)). Although such kinds of results are not easy to show in general, we manage to find an efficient way for proving them in our cases (see Theorems 3.5 and 3.6).

The following is the organization of the paper. In Section 2, we recall some facts about orbifold construction associated with inner automorphisms and reverse orbifold construction. We also prove several lemmas which will be used to determine the lowest conformal weights of twisted modules. In Section 3, we determine the conjugacy class of the automorphism θ of the Niemeier lattice VOA. In Section 4, we determine the appropriate semisimple element $h \in V_1$ and then prove our main theorem.

2. PRELIMARIES

2.1. Basic facts about VOAs. In this subsection, we recall some basic facts about VOAs from [DM04a, DM04b, FLM]. A VOA V is called *strongly regular* if V is self-dual, rational, C_2 -cofinite and of CFT-type (cf. [DM04a, DM04b]). We call a VOA V a *holomorphic* VOA if V is rational and has a unique irreducible module up to isomorphisms.

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a strongly regular VOA. Here V_n is the subspace of V of conformal weight $n \in \mathbb{Z}_{\geq 0}$. It then follows that the weight one space V_1 is a Lie algebra with respect to the bracket $[u, v] = u_{(0)}v$ for any $u, v \in V_1$, where $u_{(n)} : V \rightarrow V$ denotes the n -th product of u in V for each $n \in \mathbb{Z}$ (see [DM04b]). Moreover, for any simple Lie subalgebra $\mathfrak{s} \subset V_1$, the subVOA of V generated by \mathfrak{s} is isomorphic to the affine VOA $L_{\mathfrak{s}}(k, 0)$ for some positive integer k [DM04b]. We then call \mathfrak{s} a simple Lie subalgebra of V_1 with level k and write $\mathfrak{s} = \mathfrak{s}_k \subset V_1$. Assume further that V is a holomorphic VOA of central charge 24; we then have the following result.

Proposition 2.1 ([DM04a, Theorem 3, (1.1)]). *Let V be a holomorphic VOA of central charge 24. If the Lie algebra V_1 is neither $\{0\}$ nor abelian, then V_1 is semisimple and the conformal vectors of V and the subVOA generated by V_1 are the same. If V_1 is semisimple, then for any simple ideal \mathfrak{s} of V_1 with the level $k \in \mathbb{Z}_{>0}$,*

the identity $h^\vee/k = (\dim V_1 - 24)/24$ holds, where h^\vee is the dual Coxeter number of \mathfrak{s} .

It is also known [DM04b] that there exists a unique symmetric invariant bilinear form $\langle \cdot | \cdot \rangle$ on V such that $\langle \mathbf{1} | \mathbf{1} \rangle = -1$, where $\mathbf{1}$ is the vacuum vector of V . Furthermore, for each simple Lie subalgebra \mathfrak{s} of V_1 with the level k , we have $\langle \cdot | \cdot \rangle|_{\mathfrak{s}} = k(\cdot | \cdot)_{\mathfrak{s}}$, where $(\cdot | \cdot)_{\mathfrak{s}}$ denotes the normalized Killing form of \mathfrak{s} (see [LS16]).

Let R be a sub VOA of V . We consider the commutant $\text{Com}_V(R)$ of R in V , that is, $\text{Com}_V(R) = \{v \in V \mid w_{(n)}v = 0, w \in R, n \geq 0\}$.

Lemma 2.2 ([KM15, Theorem 2]). *Suppose that both R and $\text{Com}_V(R)$ are strongly regular VOAs and satisfy $\text{Com}_V(\text{Com}_V(R)) = R$. Then any irreducible R -module is embedded in some irreducible V -module as an R -submodule.*

We also need the following result.

Lemma 2.3 ([HKL, Theorem 3.5]). *Let T be a C_2 -cofinite, simple VOA of CFT-type and S a full sub VOA of T . Assume that S is strongly regular and that the lowest conformal weight of any irreducible S -module is positive except for the vacuum module of S . Then T is rational.*

2.2. Orbifold construction associated with inner automorphisms. In this subsection, we recall from [LS16] some formulas about orbifold construction of holomorphic VOAs of central charge 24.

Let V be a strongly regular holomorphic VOA of central charge 24 and g an automorphism of V of prime order p . We then know that there is a unique g^r -twisted V -module $V^T(g^r)$ for each $1 \leq r \leq p-1$ [DLM00, Theorem 10.3]. Moreover, the fixed point subspace V^g of V with respect to g is a sub VOA of V . The weight n subspace of V^g coincides with $V_n^g = V_n \cap V^g$ ($n \geq 0$). We say that the pair (V, g) satisfies the *orbifold condition* if there exists a unique simple VOA \tilde{V} such that V^g is embedded in \tilde{V} and $\tilde{V} \cong V^g \oplus \bigoplus_{r=1}^{p-1} V^T(g^r)_{\mathbb{Z}}$ as a V^g -module, where $V^T(g^r)_{\mathbb{Z}}$ is the subspace of $V^T(g^r)$ of integral conformal weights (cf. [EMS]). If (V, g) satisfies the orbifold condition, the VOA \tilde{V} which satisfies the above assumptions is strongly regular and holomorphic. We refer to \tilde{V} as the VOA obtained by applying the \mathbb{Z}_p -orbifold construction to V and g , and we denote the VOA \tilde{V} by $\tilde{V}(g)$.

Suppose that the Lie algebra V_1 is semisimple. Then, V_1 is isomorphic to $\mathfrak{g} = \mathfrak{g}_{(1),k_1} \oplus \cdots \oplus \mathfrak{g}_{(t),k_t}$ for some simple ideals $\mathfrak{g}_{(1)}, \dots, \mathfrak{g}_{(t)}$ with levels $k_1, \dots, k_t \in \mathbb{Z}_{>0}$, respectively. Fix a Cartan subalgebra \mathfrak{h} of V_1 , and let h be a semisimple element in \mathfrak{h} such that:

- (i) $\text{Spec}(h_{(0)}) \subset (1/2)\mathbb{Z}$ and $\text{Spec}(h_{(0)}) \not\subset \mathbb{Z}$;
- (ii) $\langle h | h \rangle \in \mathbb{Z}$;
- (iii) the lowest conformal weight of $V^{(h)}$ is positive.

Then the inner automorphism $\sigma_h := \exp(-2\pi\sqrt{-1}h_{(0)})$ of V is of order 2.

Theorem 2.4 ([LS16]). *Let V and h be as above. Then (V, σ_h) satisfies the orbifold condition.*

Moreover, we have the following result.

Proposition 2.5 ([Mo] and [LS16]). *Let V , h be as above. Then we have*

$$\dim V_1 + \dim \tilde{V}(\sigma_h)_1 = 3 \dim V_1^{\sigma_h} + 24(1 - \dim V^T(\sigma_h)_{1/2}).$$

In particular, if $V_1^{\sigma_h} = V_1$ and $V^T(\sigma_h)_{1/2} = 0$, then we have

$$(2.1) \quad \dim \tilde{V}(\sigma_h)_1 = 2 \dim V_1 + 24.$$

2.3. Reverse orbifold construction of holomorphic VOAs. In this subsection, we recall from [LS4] the method called “reverse orbifold construction”. Let V be a strongly regular holomorphic VOA and let g be an automorphism of V of prime order p such that (V, g) satisfies the orbifold condition. Let

$$W = \tilde{V}(g) = V^g \oplus \bigoplus_{r=1}^{p-1} V^T(g^r)_{\mathbb{Z}}$$

be the VOA obtained by applying the \mathbb{Z}_p -orbifold construction to V and g . Define an automorphism $a = a_{V,g}$ of W by $a|_{V^g} = 1$ and $a|_{V^T(g^r)_{\mathbb{Z}}} = e^{2\pi\sqrt{-1}r/p}$ ($1 \leq r \leq p-1$). It then follows that the pair (W, a) satisfies the orbifold condition and $\tilde{W}(a) \cong V$ (see [EMS]).

Let \mathfrak{g} be a semisimple Lie algebra and let h be a semisimple element of \mathfrak{g} . Assume that there exists a strongly regular holomorphic VOA U such that for any strongly regular holomorphic VOA V satisfying $V_1 \cong \mathfrak{g}$, the following conditions hold:

- (a) σ_h has prime order p on V and the pair (V, σ_h) satisfies the orbifold condition;
- (b) $\tilde{V}(\sigma_h) \cong U$;
- (c) for any automorphism g of U of order p , if $U_1^g \cong \mathfrak{g}^{\sigma_h}$, then g is conjugate to the automorphism a_{V, σ_h} of $\tilde{V}(\sigma_h) \cong U$ in $\text{Aut}(U)$.

Then we have the following result which was essentially obtained in [LS4].

Theorem 2.6. *The structure of a strongly regular holomorphic VOA V such that $V_1 \cong \mathfrak{g}$ is unique up to isomorphisms.*

Proof. Let V and W be strongly regular holomorphic VOAs such that $V_1 \cong \mathfrak{g} \cong W_1$. By condition (b), we see that $\tilde{V}(\sigma_h) \cong U \cong \tilde{W}(\sigma_h)$. Let a and b be the automorphisms of U induced from the automorphisms a_{V, σ_h} and a_{W, σ_h} of $\tilde{V}(\sigma_h)$ and $\tilde{W}(\sigma_h)$, respectively. It then follows from (c) that a is conjugate to b . By applying the \mathbb{Z}_p -orbifold construction to (U, a) and (U, b) , we see that $V \cong W$. \square

Remark 2.7. Although using condition (c) in Theorem 2.6 is sufficient for our purpose in this paper, it is usually too strong. Note that there exist automorphisms g and h of a lattice VOA V_L such that $(V_L)_1^g \cong (V_L)_1^h$ as Lie algebras but g and h are not conjugate in $\text{Aut}(V_L)$ (see for example [LS3, p. 1583]). In this situation, we may replace (c) by a weaker condition

- (c') any automorphism g of U of order p such that the pair (U, g) satisfies the orbifold condition and $\tilde{U}(g)_1 \cong \mathfrak{g}$ is conjugate to $a_{U, g}$.

Then Theorem 2.6 still holds.

In Section 4, we will study the case that V is a holomorphic VOA of central charge 24 with the Lie algebra $V_1 = \mathfrak{g}$, where \mathfrak{g} is one of Lie algebras in Table 1. In this case, the VOA U will be the lattice VOA V_N associated with some Niemeier lattice N . To verify condition (b) in this case, we will need the following result, which can be deduced from [DM04b, Theorem 3].

Proposition 2.8. *Let N be a Niemeier lattice and let U be a strongly regular holomorphic VOA of central charge 24 such that $U_1 \cong (V_N)_1$. Then the vertex operator algebra U is isomorphic to the lattice VOA V_N .*

We next recall some facts about automorphisms of lattice VOAs. Let L be a positive definite even lattice. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and view \mathfrak{h} as an abelian Lie algebra equipped with a non-degenerate symmetric bilinear form. Let $\widehat{\mathfrak{h}}$ be the corresponding Heisenberg Lie algebra and let $M(1)$ be the highest weight module of $\widehat{\mathfrak{h}}$ with the highest weight 0 (cf. [FLM]). Then the lattice VOA V_L associated to L is defined on the vector space $M(1) \otimes \mathbb{C}[L]$, where $\mathbb{C}[L] = \text{span} \{e^x | x \in L\}$ (cf. [FLM]). In particular, V_L is spanned by the vectors of the form $h^1(-n_1) \cdots h^k(-n_k) \otimes e^x$, where $h^1, \dots, h^k \in \mathfrak{h}$, $x \in L$; and n_1, \dots, n_k are positive integers (cf. [FLM]). Let $\theta : V_L \rightarrow V_L$ be the linear map determined by

$$(2.2) \quad h^1(-n_1) \cdots h^k(-n_k) \otimes e^x \mapsto (-1)^k h^1(-n_1) \cdots h^k(-n_k) \otimes e^{-x}.$$

It was proved in [FLM] that θ is an automorphism of V_L of order 2. Notice that θ is a lift of the (-1) -isometry of L (cf. [DGH]).

To determine the conjugacy class of the automorphism θ , we also need the following result.

Proposition 2.9 ([DGH, Theorem D.6]). *Any lifts of the (-1) -isometry of L are conjugate under $\text{Aut}(V_L)$.*

3. UNIQUENESS OF AUTOMORPHISMS OF LIE ALGEBRAS

3.1. Automorphisms of Lie algebras. In this subsection, we recall some facts about automorphisms of simple Lie algebras from [Hel78] and [DGM]. Let \mathfrak{s} be a finite-dimensional simple Lie algebra of rank n with a Cartan subalgebra \mathfrak{h} of \mathfrak{s} . For automorphisms g and g' of \mathfrak{s} , we write $g \sim g'$ if g is conjugate to g' in $\text{Aut}(\mathfrak{s})$. Let $[x]$ denote the maximum integer less than or equal to a real number x . The following proposition can be found in [Hel78, Theorem 6.1, TABLE II and pp. 513–515].

Proposition 3.1. *Let σ_1 and σ_2 be automorphisms of \mathfrak{s} of order 2. Then $\sigma_1 \sim \sigma_2$ if and only if the Lie algebra \mathfrak{s}^{σ_1} is isomorphic to \mathfrak{s}^{σ_2} . Moreover, if σ is an automorphism of \mathfrak{s} of order 2 and \mathfrak{s}^σ is semisimple, then the fixed point Lie algebra \mathfrak{s}^σ is given by the following:*

- (1) $(A_{2n})^\sigma \cong B_n$ ($n \geq 1$), (2) $(A_{2n+1})^\sigma \cong C_{n+1}$ or D_{n+1} ($n \geq 2$), (3) $(B_n)^\sigma \cong B_{n-p} \oplus D_p$ ($n \geq 3$, $2 \leq p \leq n$), (4) $(C_n)^\sigma \cong C_p \oplus C_{n-p}$ ($n \geq 2$, $1 \leq p \leq [n/2]$), (5) $(D_n)^\sigma \cong D_p \oplus D_{n-p}$ ($n \geq 4$, $2 \leq p \leq [n/2]$) and $(D_n)^\sigma \cong B_p \oplus B_{n-p-1}$ ($n \geq 3$, $0 \leq p \leq [(n-1)/2]$), (6) $(E_6)^\sigma \cong F_4$, C_4 or $A_1 \oplus A_5$, (7) $(E_7)^\sigma \cong A_7$ or $A_1 \oplus D_6$, (8) $(E_8)^\sigma \cong D_8$ or $A_1 \oplus E_7$, (9) $(F_4)^\sigma \cong B_4$ or $A_1 \oplus C_3$, (10) $(G_2)^\sigma \cong A_1 \oplus A_1$.

Let $\theta \in \text{Aut}(\mathfrak{s})$ be a lift of the (-1) -automorphism of \mathfrak{h} . Then we have

Proposition 3.2 (cf. [DGM]). *If \mathfrak{s} is simply laced, then the fixed point Lie algebra \mathfrak{s}^θ is given by the following:*

- (1) $(A_{2n})^\theta \cong B_n$, (2) $(A_{2n+1})^\theta \cong D_{n+1}$, (3) $(D_{2n})^\theta \cong D_n^2$, (4) $(D_{2n+1})^\theta \cong B_n^2$, (5) $(E_6)^\theta \cong C_4$, (6) $(E_7)^\theta \cong A_7$, (7) $(E_8)^\theta \cong D_8$.

3.2. Uniqueness of automorphisms of Lie algebras. In this subsection, we will prove that the conjugacy class of the automorphism θ of the Niemeier lattice VOA is uniquely determined by the Lie algebra structure of the fixed-point weight one subspace. This will be used to verify condition (c) in Subsection 2.3 in the proof of Theorem 1.2.

Let V be a strongly regular simple VOA and let g be an automorphism of V of order 2. Assume that the Lie algebra V_1 is semisimple and let $V_1 = \bigoplus_{i=1}^p \mathfrak{s}_{(i), \ell_i}$ be the decomposition of V_1 into the sum of simple ideals $\mathfrak{s}_{(1)}, \dots, \mathfrak{s}_{(p)}$ with levels ℓ_1, \dots, ℓ_p , respectively. Then g acts on $\{\mathfrak{s}_{(i)} \mid 1 \leq i \leq p\}$ as a permutation. Without loss of generality, we may assume that there exists a non-negative integer q such that $2q \leq p$, $g(\mathfrak{s}_{(i)}) = \mathfrak{s}_{(i+q)}$ if $1 \leq i \leq q$, $g(\mathfrak{s}_{(i)}) = \mathfrak{s}_{(i-q)}$ if $q+1 \leq i \leq 2q$, and $g(\mathfrak{s}_{(i)}) = \mathfrak{s}_{(i)}$ if $2q+1 \leq i \leq p$. The following result can be established by the same argument as in [LS4].

Proposition 3.3 (cf. [LS4, Proposition 3.7]). *The fixed point Lie algebra V_1^g is isomorphic to a sum of ideals*

$$\bigoplus_{i=1}^q (\mathfrak{s}_{(i)} \oplus \mathfrak{s}_{(i+q)})^g \oplus \bigoplus_{i=2q+1}^p \mathfrak{s}_{(i)}^g.$$

For any $1 \leq i \leq q$, we have $\ell_i = \ell_{i+q}$. In addition, the Lie subalgebra $(\mathfrak{s}_{(i)} \oplus \mathfrak{s}_{(i+q)})^g$ is a simple ideal of V_1^g isomorphic to $\mathfrak{s}_{(i)}$ and its level is $2\ell_i$.

Let \mathfrak{f} be a semisimple Lie algebra with the decomposition $\mathfrak{f} = \bigoplus_{i=1}^t \mathfrak{f}_{(j)}$ into simple ideals. Let σ be an automorphism of \mathfrak{f} of order 2 and $\mathfrak{f}^\sigma = \bigoplus_{i=1}^s \mathfrak{g}_{(i)}$ the decomposition of \mathfrak{f}^σ into simple ideals.

Corollary 3.4. *If $\mathfrak{f}_{(j)} \not\cong \mathfrak{g}_{(i)}$ for all $1 \leq j \leq t$ and $1 \leq i \leq s$, then \mathfrak{f}^σ decomposes into the sum of ideals $\mathfrak{f}^\sigma = \bigoplus_{j=1}^t \mathfrak{f}_{(j)}^\sigma$.*

Proof. Let $V = \bigotimes_{j=1}^t L_{\mathfrak{f}_j}(1, 0)$. It then follows that σ induces an order 2 automorphism g of V such that $g|_{V_1} = \sigma$. Since V is a strongly regular VOA and $V_1 \cong \mathfrak{f}$, we can get the result by Proposition 3.3. \square

We are now ready to describe our main result in this subsection. We consider a pair of Lie algebras $(\mathfrak{g}, \mathfrak{f})$ as in Table 1.

TABLE 1. Lie algebras $(\mathfrak{g}, \mathfrak{f})$.

Cases	\mathfrak{g}	\mathfrak{f}	
(A) $(n 12)$	$B_{n,2}^{12/n}$	$A_{2n,1}^{12/n}$	$(X_1 = D_4, X_2 = D_6, X_4 = E_7)$
(B) $(n 4)$	$D_{2n,2}^{4/n} B_{n,1}^{8/n}$	$A_{4n-1,1}^{4/n} D_{2n+1,1}^{4/n}$	
(C) $(n 4)$	$D_{2n+1,2}^{4/n} A_{2n-1,1}^{4/n}$	$A_{4n+1,1}^{4/n} X_{n,1}$	
(D)	$C_{4,1}^4$	$E_{6,1}^4$	
(E)	$D_{6,2} B_{3,1}^2 C_{4,1}$	$A_{11,1} D_{7,1} E_{6,1}$	

Here $n|N$ denotes the condition that the positive integer n divides N . We also use the identifications $D_{2,k} = A_{1,k}^2$, $D_{3,k} = A_{3,k}$, and $B_{1,k} = A_{1,2k}$.

The following is the main result of this subsection.

Theorem 3.5. *Let $(\mathfrak{g}, \mathfrak{f})$ be a pair of semisimple Lie algebras listed in Table 1. Then any automorphism σ of \mathfrak{f} of order 2 such that $\mathfrak{f}^\sigma \cong \mathfrak{g}$ is conjugate to θ in $\text{Aut}(\mathfrak{f})$.*

Proof. By using Corollary 3.4, we see that $\mathfrak{f}^\sigma = \bigoplus_{j=1}^t \mathfrak{f}_{(j)}^\sigma$. We prove the assertion using case-by-case analysis.

(A) Let $(\mathfrak{g}, \mathfrak{f}) = (B_{n,2}^{12/n}, A_{2n,1}^{12/n})$, where $n \in \mathbb{Z}_{>0}$ and $n|12$. It then follows by Propositions 3.1 and 3.2 that $A_{2n}^\sigma \cong B_n$ and σ is conjugate to θ .

(B) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{2n,2}^{4/n} B_{n,1}^{8/n}, A_{4n-1,1}^{4/n} D_{2n+1,1}^{4/n})$, where $n \in \mathbb{Z}_{>0}$ and $n|4$. If $n = 1$, it follows that $\mathfrak{g} = A_{1,2}^{16}$ and $\mathfrak{f} = D_3^8 = A_3^8$. By Propositions 3.1 and 3.2, we have $\mathfrak{f}^\sigma \cong A_{1,2}^{16}$ and $\sigma \sim \theta$. Suppose that $n = 2$ or 4. By Proposition 3.1, we see that $D_{2n+1}^\sigma \cong B_n^2$ and that $\sigma|_{D_{2n+1}}$ is unique up to conjugate. Therefore, $(A_{4n-1}^\sigma)^{4/n} \cong D_{2n}^{4/n}$, and $\sigma|_{A_{4n-1}}$ is unique up to conjugate. Finally, we have $\sigma \sim \theta$ by Proposition 3.2.

(C) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{2n+1,2}^{4/n} A_{2n-1,1}^{4/n}, A_{4n+1,1}^{4/n} X_{n,1})$, where $n \in \mathbb{Z}_{>0}$, $n|4$, $X_1 = D_4$, $X_2 = D_6$, and $X_4 = E_7$. It then follows that $A_{4n+1}^\sigma \cong D_{2n+1}$ and $\sigma|_{A_{4n+1}} \sim \theta$. Moreover, $X_n^\sigma \cong A_{2n-1}^{4/n}$, and $\sigma \sim \theta$ by Propositions 3.1 and 3.2.

(D) Let $(\mathfrak{g}, \mathfrak{f}) = (C_{4,1}^4, E_{6,1}^4)$. In a similar way, we see that $E_6^\sigma = C_4$ and $\sigma|_{E_6} \sim \theta$. Hence, σ is conjugate to θ .

(E) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{6,2} B_{3,1}^2 C_{4,1}, A_{11,1} D_{7,1} E_{6,1})$. By Proposition 3.1, we see that $A_{11}^\sigma \cong D_6$ and $\sigma|_{A_{11}}$ is conjugate to θ . Similarly, $E_6^\sigma \cong C_4$ and $\sigma|_{E_6} \sim \theta$. Therefore, we have $D_7^\sigma \cong B_3^2$, and hence $\sigma|_{D_7} \sim \theta$. Thus, σ is conjugate to θ . \square

Combining Proposition 2.9 and Theorem 3.5, we immediately obtain the second main result in this subsection.

Theorem 3.6. *Let $(\mathfrak{g}, \mathfrak{f})$ be one of the pairs of semisimple Lie algebras listed in Table 1, let $N(\mathfrak{f})$ be a Niemeier lattice such that $(V_{N(\mathfrak{f})})_1 = \mathfrak{f}$, and let θ be the automorphism of $V_{N(\mathfrak{f})}$ defined above. Then any automorphism μ of $V_{N(\mathfrak{f})}$ of order 2 such that $(V_{N(\mathfrak{f})})_1^\mu \cong \mathfrak{g}$ is conjugate to θ under $\text{Aut}(V_{N(\mathfrak{f})})$.*

4. UNIQUENESS OF HOLOMORPHIC VOAS OF CENTRAL CHARGE 24

4.1. Conformal weights of twisted modules of affine VOAs. To apply the “reverse orbifold construction” method on a holomorphic VOA V , we need to choose an appropriate semisimple element $h \in V_1$. One of the restrictions on h concerns the conformal weights of the irreducible σ_h -twisted V -modules. In this subsection, we will prove some results about conformal weights of σ_h -twisted V -modules.

First, we recall some facts about simple Lie algebras. Let \mathfrak{s} be a finite-dimensional simple Lie algebra and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{s} with the simple roots $\alpha_1, \dots, \alpha_n$ and fundamental weights $\varpi_1, \dots, \varpi_n$ labelled as in [Bou]. The highest root of \mathfrak{s} is denoted by θ_0 . Let $(\cdot|\cdot)$ be the normalized Killing form of \mathfrak{s} so that $(\alpha|\alpha) = 2$ for any long root α . We identify \mathfrak{h} and \mathfrak{h}^* via $(\cdot|\cdot)$. A vector $v \in \mathfrak{s}$ has \mathfrak{s} -weight $\lambda \in \mathfrak{h}$ if $[x, v] = (x|\lambda)v$ for any $x \in \mathfrak{h}$, where $[\cdot, \cdot]$ is the Lie bracket of \mathfrak{s} . The set of the dominant integral weights of \mathfrak{s} is denoted by $P^+(\mathfrak{s})$. For any positive integer k , we denote by $P^+(\mathfrak{s}, k) = \{\lambda \in P^+(\mathfrak{s}) \mid (\lambda|\theta_0) \leq k\}$ the set of all dominant integral weights of \mathfrak{s} with the level k .

For a dominant integral weight λ of \mathfrak{s} , let $L(\lambda)$ be the irreducible \mathfrak{s} -module with highest weight λ . We denote by $\Pi(\lambda)$ the set of all weights of $L(\lambda)$. Let i be a node of the Dynkin diagram of \mathfrak{s} .

Lemma 4.1. (1) If \mathfrak{s} is of type D_{2n} , then $\min\{(\varpi_i|\mu) \mid \mu \in \Pi(\lambda)\} = -(\varpi_i|\lambda)$.
 (2) If i is fixed by any diagram automorphism of the Dynkin diagram of \mathfrak{s} , then $\min\{(\varpi_i|\mu) \mid \mu \in \Pi(\lambda)\} = -(\varpi_i|\lambda)$.

Proof. Let w_0 be the longest element of the Weyl group of \mathfrak{s} . Since the lowest weight of $L(\lambda)$ is $w_0(\lambda)$ and ϖ_i is a dominant weight, it follows that $\min\{(\varpi_i|\mu) \mid \mu \in \Pi(\lambda)\} = (\varpi_i|w_0(\lambda))$. In the case that \mathfrak{s} is of type D_{2n} , it is known that w_0 is equal to -1 (see [Hum90]), which shows (1). Suppose that i is fixed by any diagram automorphism of \mathfrak{s} . We see that the automorphism $-w_0$ is (the standard lift of) a diagram automorphism as it permutes positive simple roots of \mathfrak{s} and preserves the inner product. Since w_0 is an involution, it follows that $(\varpi_i|w_0(\lambda)) = (w_0(\varpi_i)|\lambda) = -(\varpi_i|\lambda)$. Thus, we obtain (2), as desired. \square

We next recall some facts about affine VOAs. Let k be a positive integer and let $L_{\mathfrak{s}}(k, 0)$ be the affine VOA associated with \mathfrak{s} and with level k . It is known [FZ92] that $L_{\mathfrak{s}}(k, 0)$ is a strongly regular VOA and the set of all irreducible modules over $L_{\mathfrak{s}}(k, 0)$ up to isomorphisms is given by $\{L_{\mathfrak{s}}(k, \lambda) \mid \lambda \in P^+(\mathfrak{s}, k)\}$, where $L_{\mathfrak{s}}(k, \lambda)$ is the irreducible $L_{\mathfrak{s}}(k, 0)$ -module of \mathfrak{s} -weight λ .

Consider the VOA $W = \bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, 0)$, where k_1, \dots, k_t are positive integers. Then any irreducible W -module is isomorphic to $\bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, \lambda_i)$ with $\lambda_i \in P^+(\mathfrak{g}_{(i)}, k_i)$ for each $1 \leq i \leq t$. Let $h = (h_1, h_2, \dots, h_t)$ be a semisimple element of W_1 such that $(h|\alpha) \geq -1$ for any root α of W_1 and the spectrum $\text{Spec}(h_{(0)})$ of $h_{(0)} : W \rightarrow W$ is contained in $(1/T)\mathbb{Z}$ for some positive integer T . Then we know that σ_h is an inner automorphism of W such that $\sigma_h^T = 1$. Moreover, for each W -module M , it is proved in [Li96] that $(M^{(h)}, Y_M^{(h)}(\cdot, z)) := (M, Y_M(\Delta(h, z)\cdot, z))$ is a σ_h -twisted W -module, where $Y_M(\cdot, z)$ is the vertex operator map of M and $\Delta(h, z) = z^{h_{(0)}} \exp\left(\sum_{n=1}^{\infty} \frac{h_{(n)}}{-n} (-z)^{-n}\right)$.

Lemma 4.2 ([LS4, Lemma 2.7]). Set $\mathbf{P}_{\mathfrak{g}} = P^+(\mathfrak{g}_{(1)}, k_1) \times \dots \times P^+(\mathfrak{g}_{(t)}, k_t)$ and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_t)$ be an element of $\mathbf{P}_{\mathfrak{g}}$. Then the lowest conformal weight of $(\bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, \lambda_i))^{(h)}$ is equal to $w(\boldsymbol{\lambda}) = \ell(\boldsymbol{\lambda}) + \sum_{i=1}^t \min\{(h_i|\mu) \mid \mu \in \Pi(\lambda_i)\} + \langle h|h \rangle/2$, where $\ell(\boldsymbol{\lambda})$ is the lowest conformal weight of $\bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, \lambda_i)$ and $\Pi(\lambda_i)$ is the set of all weights of the irreducible $\mathfrak{g}_{(i)}$ -module $L(\lambda_i)$ with the highest weight λ_i .

We now let V be a strongly regular, holomorphic VOA such that V_1 is semisimple. Let $h = (h_1, h_2, \dots, h_t)$ be a semisimple element of V_1 such that $(h|\alpha) \geq -1$ for any root α of V_1 and the spectrum $\text{Spec}(h_{(0)})$ of $h_{(0)} : V \rightarrow V$ is contained in $(1/T)\mathbb{Z}$ for some positive integer T . Then we know that σ_h is an inner automorphism of V of finite order. Assume that $V_1 = \mathfrak{g} \cong \mathfrak{g}_{(1)} \oplus \dots \oplus \mathfrak{g}_{(t)}$ for some simple Lie algebras $\mathfrak{g}_{(1)}, \dots, \mathfrak{g}_{(t)}$. For each $\boldsymbol{\lambda} \in \mathbf{P}_{\mathfrak{g}}$, set $d(\boldsymbol{\lambda}) = w(\boldsymbol{\lambda}) - \ell(\boldsymbol{\lambda})$. Write $\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{P}_{\mathfrak{g}}$ and $L(\boldsymbol{\lambda}) = \bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, \lambda_i)$. We then have the following lemma.

Lemma 4.3. Assume $d(\boldsymbol{\lambda}) > -3/2$ for any $\boldsymbol{\lambda} \in \mathbf{P}_{\mathfrak{g}}$ and $d(\mathbf{0}) > 1/2$. Then the lowest conformal weight of $V^T(\sigma_h)$ is greater than $1/2$. In particular, $V^T(\sigma_h)_{1/2} = 0$.

Proof. Note that the sub VOA of V generated by V_1 is isomorphic to $\bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, 0)$ for some positive integers k_1, \dots, k_t . Thus, V viewed as a $\bigotimes_{i=1}^t L_{\mathfrak{g}_{(i)}}(k_i, 0)$ -module

has the decomposition $V \cong \bigoplus_{j=0}^n L(\boldsymbol{\lambda}^{(j)})$, where $n \geq 0$, $\boldsymbol{\lambda}^{(0)}, \dots, \boldsymbol{\lambda}^{(n)} \in \mathbf{P}_{\mathfrak{g}}$ and $\boldsymbol{\lambda}^{(j)} = \mathbf{0}$ if and only if $j = 0$. It then follows that $V^T(\sigma_h) = \bigoplus_{j=0}^n L(\boldsymbol{\lambda}^{(j)})^{(h)}$ (see [Li96]). Therefore, it suffices to show $w(\boldsymbol{\lambda}^{(j)}) > 1/2$ for all $0 \leq j \leq n$. By assumption, $d(\mathbf{0}) > 1/2$ and $\ell(\mathbf{0}) = 0$; hence, we have $w(\mathbf{0}) > 1/2$. Assume that $0 < j \leq n$. Since $V_1 \cong \mathfrak{g}$ and $\dim V_0 = 1$, we have $\ell(\boldsymbol{\lambda}^{(j)}) \geq 2$. It follows immediately from the assumption $d(\boldsymbol{\lambda}^{(j)}) > -3/2$ that $w(\boldsymbol{\lambda}^{(j)}) > 1/2$. The proof is complete. \square

4.2. Orbifold construction of holomorphic VOAs. In this subsection, we begin to prove Theorem 1.2. To make the statement of Theorem 1.2 more precise, we will prove the following theorem.

Theorem 4.4. *Let $(\mathfrak{g}, \mathfrak{f})$ be a pair of Lie algebras listed in Table 1. Let V be a strongly regular holomorphic VOA of central charge 24 such that V_1 is isomorphic to \mathfrak{g} . Then V is isomorphic to the VOA $\tilde{V}_{N(\mathfrak{f})}(\theta)$, where $N(\mathfrak{f})$ is the Niemeier lattice such that $(V_{N(\mathfrak{f})})_1 = \mathfrak{f}$ and θ is the automorphism of the lattice VOA $V_{N(\mathfrak{f})}$ defined as in (2.2).*

Note that Theorem 1.2 follows immediately from Theorem 4.4. We will prove Theorem 4.4 after several lemmas. Our idea is to apply the “reverse orbifold construction” method on the holomorphic VOA V . We start by choosing an appropriate semisimple element $h \in \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_{(1),k_1} \oplus \dots \oplus \mathfrak{g}_{(t),k_t}$ be a semisimple Lie algebra listed, where the $\mathfrak{g}_{(i),k_i}$ ’s are arranged in the same order as in Table 1. For example, if $\mathfrak{g} = D_{6,2}B_{3,1}^2C_{4,1}$, then $\mathfrak{g}_{(1)} = D_6$, $\mathfrak{g}_{(2)} = B_3$, $\mathfrak{g}_{(3)} = B_3$, and $\mathfrak{g}_{(4)} = C_4$.

TABLE 2. Choice of h .

Cases	\mathfrak{g}	h
(A) $(n 12)$	$B_{n,2}^{12/n}$	$(\varpi_1, 0, \dots, 0)$ ($12/n - 1$ times 0’s)
(B) $(n 4)$	$D_{2n,2}^{4/n}B_{n,1}^{8/n}$	$(\varpi_1, 0, \dots, 0)$ ($12/n - 1$ times 0’s)
(C) $(n 4)$	$D_{2n+1,2}^{4/n}A_{2n-1,1}^{4/n}$	$(0, \dots, 0, \varpi_n, \dots, \varpi_n)$ ($4/n$ times 0’s and ϖ_n ’s)
(D)	$C_{4,1}^4$	$(\varpi_4, 0, 0, 0)$
(E)	$D_{6,2}B_{3,1}^2C_{4,1}$	$(\varpi_1, 0, 0, 0)$

Lemma 4.5. *Let V and $(\mathfrak{g}, \mathfrak{f})$ be as above and let h be the semisimple element of \mathfrak{g} defined as in Table 2, where ϖ_1 of D_2 means the weight (ϖ_1, ϖ_1) of A_1^2 . Then h satisfies $\langle h|h \rangle = 2$, $(h|\boldsymbol{\lambda}) \in \frac{1}{2}\mathbb{Z}$, $d(\mathbf{0}) = 1$, and $d(\boldsymbol{\lambda}) > -3/2$ for any $\boldsymbol{\lambda} \in \mathbf{P}_{\mathfrak{g}}$ such that $\boldsymbol{\lambda} \neq \mathbf{0}$. Moreover, σ_h is an automorphism of V of order 2 such that $\mathfrak{g}^{\sigma_h} = \mathfrak{g}$.*

Proof. Since h is a sum of fundamental weights corresponding to long roots, it follows immediately that $\mathfrak{g}^{\sigma_h} = \mathfrak{g}$. By direct calculations, it is also easy to verify that $\langle h|h \rangle = 2$ and $(h|\lambda) \in \frac{1}{2}\mathbb{Z}$ for all $\lambda \in P^+(\mathfrak{g}_{(i)}, k)$ ($1 \leq i \leq t$). Moreover, for any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_t) \in \mathbf{P}_{\mathfrak{g}}$, we have $d(\boldsymbol{\lambda}) = -\sum_{i=1}^{t/2} (\varpi_n|\lambda_{t/2+i}) + 1$ if \mathfrak{g} is in case (C), and $d(\boldsymbol{\lambda}) = -(h|\lambda_1) + 1$ otherwise, by using Lemma 4.1. It is now straightforward to show that $d(\mathbf{0}) = 1$ and $d(\boldsymbol{\lambda}) > -3/2$ for any $\boldsymbol{\lambda} \in \mathbf{P}_{\mathfrak{g}}$ such that $\boldsymbol{\lambda} \neq \mathbf{0}$.

Finally, we show that the order of σ_h is 2. Set $\mathfrak{r} = \mathfrak{g}_{(t/2+1),k_{t/2+1}} \oplus \dots \oplus \mathfrak{g}_{(t),k_t}$ and $\mathfrak{s} = \mathfrak{g}_{(1),k_1} \oplus \dots \oplus \mathfrak{g}_{(t/2),k_{t/2}}$ when \mathfrak{g} is in case (C), and set $\mathfrak{r} = \mathfrak{g}_{(1),k_1}$ and $\mathfrak{s} = \mathfrak{g}_{(2),k_2} \oplus \dots \oplus \mathfrak{g}_{(t),k_t}$ otherwise. Then h belongs to \mathfrak{r} . Let R and S be the

sub VOAs of V generated by \mathfrak{r} and \mathfrak{s} , respectively. It follows that R and S are strongly regular. We divide the proof into 3 parts: (1°) $\mathfrak{g} \neq B_{12,2}$ and $D_{8,2}B_{4,1}^2$, (2°) $\mathfrak{g} = B_{12,2}$, and (3°) $\mathfrak{g} = D_{8,2}B_{4,1}^2$.

(1°) Suppose that $\mathfrak{g} \neq B_{12,2}$ and $D_{8,2}B_{4,1}^2$. It then follows that the lowest conformal weight of any irreducible R -module does not belong to $\mathbb{Z}_{\geq 2}$. Therefore, we have $\text{Com}_V(\text{Com}_V(R)) = R$. Since $T = \text{Com}_V(R)$ is an extension of S , the VOA T is C_2 -cofinite and of CFT-type. Since the commutant of a rational simple subVOA in a rational simple VOA is also simple ([ACKL]), by Lemma 2.3, T is rational. By applying Lemma 2.2 to R and T , we see that all the irreducible modules of R must appear in V . Since there exists an irreducible R -module of \mathfrak{r} -weight λ such that $(\lambda|h) \in 1/2 + \mathbb{Z}$, the order of σ_h is 2.

(2°) Suppose that $\mathfrak{g} = B_{12,2}$. Then as a module of $R \cong L_{B_{12}}(2, 0)$, V decomposes as

$$V \cong L_{B_{12}}(2, 0) \oplus \bigoplus_{i=1}^{12} a_i L_{B_{12}}(2, \varpi_i) \oplus \bigoplus_{i,j \in \{1,12\}, i \leq j} b_{ij} L_{B_{12}}(2, \varpi_i + \varpi_j)$$

with non-negative integers a_i ($1 \leq i \leq 12$) and b_{ij} ($i, j \in \{1, 12\}$, $i \leq j$). By computing the lowest conformal weights of the irreducible R -modules, we see that

$$(4.1) \quad \dim V_2 = \dim(L_{B_{12}}(2, 0))_2 + a_5 \dim L(\varpi_5) + b_{1,12} \dim L(\varpi_1 + \varpi_{12}).$$

Here, $L(\lambda)$ is the irreducible B_{12} -module of highest weight λ . Since V is a holomorphic VOA of central charge 24, the character of V coincides with $j(\tau) - 744 + \dim B_{12}$, where $j(\tau)$ is the j -function. Therefore, we have $\dim V_2 = 196884$. We also have $\dim(L_{B_{12}}(2, 0))_2 = 45450$, $\dim L(\varpi_5) = 53130$, and $\dim L(\varpi_1 + \varpi_{12}) = 98304$. It then follows by (4.1) that $a_5 = b_{1,12} = 1$. Since $(\varpi_1 + \varpi_{12}|h) = 3/2$, we see that the order of σ_h is 2.

(3°) Suppose that $\mathfrak{g} = D_{8,2}B_{4,1}^2$. The set of all irreducible $R = L_{D_8}(2, 0)$ -modules M such that the lowest conformal weight of M belongs to $\mathbb{Z}_{\geq 2}$ consists of $L_{D_8}(2, 2\varpi_7)$ and $L_{D_8}(2, 2\varpi_8)$. They are self-dual simple current modules such that $L_{D_8}(2, 2\varpi_7) \boxtimes L_{D_8}(2, 2\varpi_8) \cong L_{D_8}(2, 2\varpi_1)$, where \boxtimes denotes the fusion product of R -modules. Therefore, we see that either (i) $\text{Com}_V(\text{Com}_V(R)) = R$ or (ii) $\text{Com}_V(\text{Com}_V(R)) = R \oplus L_{D_8}(2, 2\varpi_i)$ ($i = 7, 8$) holds. If (i) holds, then by a similar argument to (1°) above, we see that $L_{D_8}(2, \varpi_8)$ is a summand in the decomposition of V as an R -module. Since $(\varpi_8|h) = 1/2$, the order of σ_h is 2. Suppose that (ii) holds with $i = 7$ or 8 . It then follows by the theory of simple current extensions that the irreducible $\text{Com}_V(\text{Com}_V(R))$ -modules are given by $R \oplus L_{D_8}(2, 2\varpi_i)$, $L_{D_8}(2, 2\varpi_1) \oplus L_{D_8}(2, 2\varpi_j)$, $L_{D_8}(2, \varpi_i)^\pm$, $L_{D_8}(2, \varpi_1 + \varpi_j)^\pm$, $L_{D_8}(2, \varpi_2) \oplus L_{D_8}(2, \varpi_6)$, $L_{D_8}(2, \varpi_4)^\pm$, where j satisfies $\{i, j\} = \{7, 8\}$. In particular, $L_{D_8}(2, \varpi_i)^+ \subset V$ as a module of $\text{Com}_V(\text{Com}_V(R))$. By a similar argument as in (1°), we see that both $\text{Com}_V(R)$ and $\text{Com}_V(\text{Com}_V(R))$ are regular. Since $(h|\varpi_i) = 1/2$, by applying Lemma 2.2 to $\text{Com}_V(\text{Com}_V(R))$ and $\text{Com}_V(R)$, we have shown that the order of σ_h is 2. \square

Lemma 4.6. *Let V , $(\mathfrak{g}, \mathfrak{f})$, and h be as in Lemma 4.5. Then (V, σ_h) satisfies the orbifold condition, and $V^T(\sigma_h)_{\frac{1}{2}} = 0$.*

Proof. By Lemmas 4.5 and 4.3, h satisfies conditions (i), (ii) and (iii) in Subsection 2.2. Hence, by Theorem 2.4, (V, σ_h) satisfies the orbifold condition. Moreover, we have $V^T(\sigma_h)_{1/2} = 0$ by Lemma 4.3. \square

By Lemmas 4.5 and 4.6, we have the strongly regular holomorphic VOA $\tilde{V}(\sigma_h)$ of central charge 24.

Lemma 4.7. *Let V , $(\mathfrak{g}, \mathfrak{f})$, and h be as in Lemma 4.5. Then the holomorphic VOA $\tilde{V}(\sigma_h)$ is isomorphic to $V_{N(\mathfrak{f})}$.*

Proof. From now on, we set $W = \tilde{V}(\sigma_h)$. Then since the Lie algebra W_1 is semisimple by Proposition 2.1, we have the decomposition $W_1 = \bigoplus_{i=1}^r \mathfrak{s}_{(i), \ell_i}$ of W_1 into simple ideals, where $r \in \mathbb{Z}_{>0}$ and $\mathfrak{s}_{(i)}$ is a simple ideal of W_1 with the level $\ell_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq r$). Let $a = a_{V, \sigma_h}$ be the automorphism of W defined in Subsection 2.3, which is of order 2. It then follows from $\mathfrak{g}^{\sigma_h} = \mathfrak{g}$ that $W_1^a \cong \mathfrak{g}$. We give a case-by-case analysis to show the assertion.

(A) Let $(\mathfrak{g}, \mathfrak{f}) = (B_{n,2}^{12/n}, A_{2n,1}^{12/n})$, where $n|12$. Since $\dim \mathfrak{g} = 12(2n+1)$, it follows by (2.1) that W_1 has dimension $48(n+1)$. Let i be an element of $\{1, \dots, r\}$. Then, by Proposition 2.1, we have $h_i^\vee/\ell_i = (\dim W_1 - 24)/24 = 2n+1$, where h_i^\vee denotes the dual Coxeter number of $\mathfrak{s}_{(i)}$. Since ℓ_i is a positive integer, h_i^\vee is divisible by $2n+1$. We now prove the assertion for each n .

(A.1) **Case of $n = 1$.** Since $\mathfrak{g} = A_{1,4}^{12}$, by applying Propositions 3.3 and 3.1 to W_1 and $a \in \text{Aut}(W_1)$, we see that $\mathfrak{s}_{(i)}$ is of type $A_1, A_2, B_3, C_2, D_4, D_3$, or G_2 . As $3|h_i^\vee$, we have $\mathfrak{s}_{(i)} \cong A_2, C_2$, or D_4 . Since $\dim \mathfrak{g} = 96$, W_1 is of type $D_{4,2}^2 C_{2,1}^4, D_{4,2} A_{2,1} C_{2,1}^6, D_{4,2} A_{2,1}^6 C_{2,1}^2, A_{2,1}^2 C_{2,1}^8, A_{2,1}^7 C_{2,1}^4$, or $A_{2,1}^{12}$. Since $\mathfrak{g} = A_1^{12}$, it follows by Proposition 3.1 that $\mathfrak{s}_{(i)}^a \cong A_1^4, A_1^2$, and A_1 if $\mathfrak{s}_{(i)} \cong D_4, C_2$, and A_2 , respectively. As the Lie rank of \mathfrak{g} is 12, we have $W_1 \cong A_{2,1}^{12}$. It then follows by Proposition 2.8 that W is isomorphic to $V_{N(\mathfrak{f})}$.

(A.2) **Case of $n = 2$.** Similarly, by Propositions 3.3 and 3.1, we have $\mathfrak{s}_{(i)} \cong B_2, A_4, C_4$, or D_5 . Since $5|h_i^\vee$, it follows that $\mathfrak{s}_{(i)} \cong A_4$ or C_4 . Therefore, $W_1 = C_{4,1}^4, C_{4,1}^2 A_{4,1}^3$, or $A_{4,1}^6$. Since $\mathfrak{g} = B_2^6$, we have $\mathfrak{s}_{(i)}^a \cong B_2^2$ if $\mathfrak{s}_{(i)} \cong C_4$ and $\mathfrak{s}_{(i)}^a \cong B_2$ if $\mathfrak{s}_{(i)} \cong A_4$. By using $\text{rank}(\mathfrak{g}) = 6$, we see that $\tilde{W}_1 \cong A_{4,1}^6$. As a result, W is isomorphic to $V_{N(\mathfrak{f})}$.

(A.3) **Case of $n \geq 3$, $n|12$.** Since $\mathfrak{g} = B_n^{12/n}$, it follows by Propositions 3.3 and 3.1 that $\mathfrak{s}_{(i)} = B_n, A_{2n}$, or D_{2n+1} . As $(2n+1)|h_i^\vee$, we have $\mathfrak{s}_{(i)} = A_{2n}$, which forces that $W_1 \cong A_{2n,1}^{12/n}$. Hence, $W \cong V_{N(\mathfrak{f})}$.

By combining (A.1)–(A.3), we see that $W \cong V_{N(\mathfrak{f})}$ for each $n|12$.

(B) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{2n,2}^{4/n} B_{n,1}^{8/n}, A_{4n-1,1}^{4/n} D_{2n+1,1}^{4/n})$, where $n|4$. It follows by (2.1) that W_1 has dimension $96n+24$. Then $h_i^\vee/\ell_i = (\dim W_1 - 24)/24 = 4n$.

(B.1) **Case of $n = 1$.** Since $4|h_i^\vee$ and $\dim W_1 = 120$, it follows that $\mathfrak{s}_{(i)}$ is of type $A_3, A_7, C_3, C_7, D_5, D_7, E_6$, or G_2 . By applying Proposition 3.1 to W_1 and a , we see that $\mathfrak{s}_{(i)}$ must be A_3 or G_2 . Since $\dim W_1 = 120$, it follows that $W_1 \cong A_3^8$, and hence, $W \cong V_{N(\mathfrak{f})}$.

(B.2) **Case of $n = 2$.** We see that $\mathfrak{s}_{(i)}$ has type A_7, C_7, D_5 , or D_9 . Since $\dim W_1 = 216$, we have $W_1 = A_{7,1} D_{9,2}$ or $A_{7,1}^2 D_{5,1}^2$. Suppose that $W_1 = A_{7,1} D_{9,2}$. It then follows that $A_7^a = D_4$. Therefore, we have $D_9^a = D_4 B_2^2$, which contradicts Proposition 3.1. Hence, W_1 must have type $A_{7,1}^2 D_{5,1}^2$. As a result, W is isomorphic to $V_{N(\mathfrak{f})}$.

(B.3) **Case of $n = 4$.** Then $\mathfrak{s}_{(i)}$ has type A_{15} or D_9 , which shows that $W_1 = A_{15,1} D_{9,1}$. As a result, W is isomorphic to $V_{N(\mathfrak{f})}$.

(C) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{2n+1,2}^{4/n} A_{2n-1,1}^{4/n}, A_{4n+1,1}^{4/n} X_{n,1})$, where $n|4$. Since $\dim \mathfrak{g} = 24(2n+1)$, it follows by (2.1) that W_1 has dimension $24(4n+3)$. Then we have $h_i^\vee/\ell_i = (\dim W_1 - 24)/24 = 4n+2$.

(C.1) **Case of $n = 1$.** Since $\dim(W_1) = 168$ and $h_i^\vee/k_i = 6$, we see that $\mathfrak{s}_{(i)}$ has type $A_{11}, D_7, E_6, C_5, A_5, D_4$, or E_7 . Since $\mathfrak{g} = D_3^4 A_1^4$, it follows by Proposition 3.1 that $\mathfrak{s}_{(i)} \cong A_5$ or D_4 , which forces that $W_1 = A_{5,1}^4 D_{4,1}$ or $D_{4,1}^6$. Since $D_3 \subset \mathfrak{g}$, we see that $A_5 \subset W$, and hence W_1 has the type $A_{5,1}^4 D_{4,1}$. Therefore, $W \cong V_{N(\mathfrak{f})}$.

(C.2) **Case of $n = 2$.** We see that $\mathfrak{s}_{(i)}$ has type A_9, C_9, D_6, D_{11} , or E_8 , which shows that $W_1 = D_{6,1}^4$ or $A_{9,1}^2 D_{6,1}$. Since $D_5 \subset \mathfrak{g}$, we have $W_1 \cong A_{9,1}^2 D_{6,1}$, and hence, $W \cong V_{N(\mathfrak{f})}$.

(C.3) **Case of $n = 4$.** In this case, $\mathfrak{s}_{(i)}$ has type A_{17}, D_{10} , or E_7 , which forces that $W_1 = A_{17,1} E_{7,1}$ or $D_{10,1} E_{7,1}^2$. Since $\mathfrak{g} = D_9 A_7$, it follows by Proposition 3.1 that the multiplicity of the ideal E_7 in W_1 is less than 2. Therefore, W_1 has type $A_{17,1} E_{7,1}$. As a result, W is isomorphic to $V_{N(\mathfrak{f})}$.

(D) Let $(\mathfrak{g}, \mathfrak{f}) = (C_{4,1}^4, E_{6,1}^4)$. By (2.1), we know that $\dim W_1 = 312$ and $h_i^\vee/\ell_i = 12$. It follows that $\mathfrak{s}_{(i)}$ has type C_{11}, A_{11}, D_7 , or E_6 , which shows that $W_1 = A_{11,1} D_{7,1} E_{6,1}$ or $E_{6,1}^4$. Since A_{11}^a is isomorphic to D_6 or C_6 , it follows by Proposition 3.1 and Corollary 3.4 that $W_1 \not\cong A_{11,1} D_{7,1} E_{6,1}$. Thus, W_1 has type $E_{6,1}^4$. Hence, W is isomorphic to $V_{N(\mathfrak{f})}$.

(E) Let $(\mathfrak{g}, \mathfrak{f}) = (D_{6,2} B_{3,1}^2 C_{4,1}, A_{11,1} D_{7,1} E_{6,1})$. By the same argument, we have $\dim(W_1) = 312$ and $h_i^\vee/\ell_i = (\dim W_1 - 24)/24 = 12$. It follows that $\mathfrak{s}_{(i)}$ has type A_{11}, C_{11}, D_7 , or E_6 , which forces that $W_1 = E_{6,1}^4$ or $A_{11,1} D_{7,1} E_{6,1}$. Since $\mathfrak{g} = D_6 B_3^2 C_4$, it follows by Proposition 3.1 that the multiplicity of the ideal E_6 in W_1 is less than 2. Thus, W_1 has type $A_{11,1} D_{7,1} E_{6,1}$, and thus, W is isomorphic to $V_{N(\mathfrak{f})}$. \square

To summarize, we have proved that there exists a semisimple element $h \in V_1$ such that: (1) σ_h is an automorphism of V of order 2, and the pair (V, σ_h) satisfies the orbifold condition; (2) The holomorphic VOA $\tilde{V}(\sigma_h) = V^{\sigma_h} \oplus V^T(\sigma_h)_{\mathbb{Z}}$ is isomorphic to $V_{N(\mathfrak{f})}$; (3) $\mathfrak{g}^{\sigma_h} = \mathfrak{g}$. Taking $U = V_{N(\mathfrak{f})}$, we can obtain Theorem 4.4 by Theorems 2.6 and 3.6.

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REFERENCES

- [ACKL] Tomoyuki Arakawa, Thomas Creutzig, Kazuya Kawasetsu, and Andrew R. Linshaw, *Orbifolds and Cosets of Minimal W -Algebras*, Comm. Math. Phys. **355** (2017), no. 1, 339–372, DOI 10.1007/s00220-017-2901-2. MR3670736
- [Bou] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. MR1890629
- [DGM] L. Dolan, P. Goddard, and P. Montague, *Conformal field theories, representations and lattice constructions*, Comm. Math. Phys. **179** (1996), no. 1, 61–120. MR1395218
- [DGH] Chongying Dong, Robert L. Griess Jr., and Gerald Höhn, *Framed vertex operator algebras, codes and the Moonshine module*, Comm. Math. Phys. **193** (1998), no. 2, 407–448, DOI 10.1007/s002200050335. MR1618135

- [DLM00] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Comm. Math. Phys. **214** (2000), no. 1, 1–56, DOI 10.1007/s002200000242. MR1794264
- [DM04a] Chongying Dong and Geoffrey Mason, *Holomorphic vertex operator algebras of small central charge*, Pacific J. Math. **213** (2004), no. 2, 253–266, DOI 10.2140/pjm.2004.213.253. MR2036919
- [DM04b] Chongying Dong and Geoffrey Mason, *Rational vertex operator algebras and the effective central charge*, Int. Math. Res. Not. **56** (2004), 2989–3008, DOI 10.1155/S1073792804140968. MR2097833
- [EMS] J. van Ekeren, S. Möller, and N. Scheithauer, *Construction and classification of holomorphic vertex operator algebras*. arXiv:1507.08142 (2015)
- [FLM] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988. MR996026
- [FZ92] Igor B. Frenkel and Yongchang Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), no. 1, 123–168, DOI 10.1215/S0012-7094-92-06604-X. MR1159433
- [Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR514561
- [HKL] Yi-Zhi Huang, Alexander Kirillov Jr., and James Lepowsky, *Braided tensor categories and extensions of vertex operator algebras*, Comm. Math. Phys. **337** (2015), no. 3, 1143–1159, DOI 10.1007/s00220-015-2292-1. MR3339173
- [Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460
- [Ka] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219
- [KM15] Matthew Krauel and Masahiko Miyamoto, *A modular invariance property of multivariable trace functions for regular vertex operator algebras*, J. Algebra **444** (2015), 124–142, DOI 10.1016/j.jalgebra.2015.07.013. MR3406171
- [LLin] C. Lam and X. Lin, *Holomorphic vertex operator algebra of central charge 24 with Lie algebra $F_{4,6}A_{2,2}$* . arXiv:1612.08123 (2016)
- [LS1] Ching Hung Lam and Hiroki Shimakura, *Classification of holomorphic framed vertex operator algebras of central charge 24*, Amer. J. Math. **137** (2015), no. 1, 111–137, DOI 10.1353/ajm.2015.0001. MR3318088
- [LS16] Ching Hung Lam and Hiroki Shimakura, *Orbifold construction of holomorphic vertex operator algebras associated to inner automorphisms*, Comm. Math. Phys. **342** (2016), no. 3, 803–841, DOI 10.1007/s00220-015-2484-8. MR3465432
- [LS3] Ching Hung Lam and Hiroki Shimakura, *A holomorphic vertex operator algebra of central charge 24 whose weight one Lie algebra has type $A_{6,7}$* , Lett. Math. Phys. **106** (2016), no. 11, 1575–1585, DOI 10.1007/s11005-016-0883-1. MR3555415
- [LS4] C. Lam, H. Shimakura, *Reverse orbifold construction and uniqueness of holomorphic vertex operator algebras*. arXiv:1606.08979 (2016)
- [Li96] Hai-Sheng Li, *Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules*, Moonshine, the Monster, and related topics (South Hadley, MA, 1994), Contemp. Math., vol. 193, Amer. Math. Soc., Providence, RI, 1996, pp. 203–236, DOI 10.1090/conm/193/02373. MR1372724
- [Mo] P. S. Montague, *Orbifold constructions and the classification of self-dual $c = 24$ conformal field theories*, Nuclear Phys. B **428** (1994), no. 1-2, 233–258, DOI 10.1016/0550-3213(94)90201-1. MR1299260
- [SS] Daisuke Sagaki and Hiroki Shimakura, *Application of a \mathbb{Z}_3 -orbifold construction to the lattice vertex operator algebras associated to Niemeier lattices*, Trans. Amer. Math. Soc. **368** (2016), no. 3, 1621–1646, DOI 10.1090/tran/6382. MR3449220
- [Sch] A. N. Schellekens, *Meromorphic $c = 24$ conformal field theories*, Comm. Math. Phys. **153** (1993), no. 1, 159–185. MR1213740

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