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THE SPEED OF RELAXATION FOR DIFFUSION WITH DRIFT SATISFYING EXPONENTIAL DECAY OF CORRELATIONS

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Abstract. We study the convergence speed in the L^2-norm of the diffusion semigroup toward its equilibrium when the underlying flow satisfies decay of correlation. Our result is an extension of the main theorem given by Constantin, Kiselev, Ryzhik and Zlatoš (2008). Our proof is based on Weyl asymptotic law for the eigenvalues of the Laplace operator, Sobolev imbedding and some assumption on decay of correlation for the underlying flow.

1. Introduction

Let $(\mathbb{M}, g)$ be a d-dimensional compact Riemannian manifold without boundary. The Riemannian metric g yields a volume measure on $\mathbb{M}$ denoted by $\mathrm{vol}_{\mathbb{M}}$, a Laplace operator denoted by Δ and a covariant derivative denoted by ∇. Moreover it also yields a notion of divergence for C^1-vector fields (see [2] for an introduction to those notions). For a divergence free vector field u and some $A \in \mathbb{R}$ we consider the solution $\phi^A(t)$ of the parabolic partial differential equation

$$\begin{cases} \frac{d}{dt}\phi^A(t) = Au \cdot \nabla\phi^A(t) + \Delta\phi^A(t), \\ \phi^A(0) = \phi_0. \end{cases} \tag{1.1}$$

We are interested in the asymptotic behavior of the solutions $\phi^A(t)$ when ϕ_0 satisfies $\int_{\mathbb{M}} \phi_0 d\mathrm{vol}_{\mathbb{M}} = 0$. It is well known that

$$\left\|\phi^A(t)\right\| \leq K_A e^{-\rho_A t} \left\|\phi_0\right\|,$$

where ρ_A is the spectral gap of the operator $L_A = \Delta + Au \cdot \nabla$ and K_A is some positive constant. Here and in the following we denote $\|\cdot\|$ the usual L^2-norm on $\mathbb{M}$ with respect to $\mathrm{vol}_{\mathbb{M}}$. Therefore, if A is fixed, then $\left\|\phi^A(t)\right\| \to 0$ as $t \to \infty$. A natural question is to study what happens if the times t is fixed and A tends to infinity. Franke, Hwang, Pai and Sheu proved in [7] that

$$\lim_{|A|\to\infty} \rho_A = \inf\left\{\frac{1}{2}\int |\nabla\phi|^2 \, d\mathrm{vol}_{\mathbb{M}}, \|\phi\| = 1, \phi \text{ is eigenfunction of } u\cdot\nabla \text{ in } H^1\right\}.$$

It follows that ρ_A diverges to infinity as $|A| \to \infty$, if and only if, the anti-symmetric operator $u \cdot \nabla$ has no eigenfunctions satisfying H^1-regularity. However, we do not

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have any control on K_A as $|A| \to \infty$. Constantin, Kiselev, Ryzhik and Zlatoš proved in [3], that when t is fixed $\|\phi^A(t)\| \to 0$ as $|A| \to \infty$, if and only if, $u \cdot \nabla$ has no eigenfunction in H^1. They call a vector field u having this property relaxation enhancing. In particular, this property is satisfied when the volume preserving flow $(\Phi_t)_{t\in\mathbb{R}}$ which is generated by the evolution equation $\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x))$, $\Phi_0(x) = x$ is weakly mixing (see [10] for a definition). In this article we will make the following decay of correlation assumption on the flow $(\Phi_t)_{t\in\mathbb{R}}$:

Assumption 1.1 (Decay of correlation)**.** *We suppose that for some $\kappa > 0$, there exist two positive constants C_1, C_2 such that for all $f_1, f_2 \in \mathcal{C}^\kappa(\mathbb{M})$ and all $t > 0$, we have*

$$|\langle f_1, f_2 \circ \Phi_t\rangle - \langle f_1, 1\rangle\langle 1, f_2\rangle| \leq C_1 e^{-C_2 t} \|f_1\|_{\mathcal{C}^\kappa} \|f_2\|_{\mathcal{C}^\kappa}.$$

Results on decay of correlation for Anosov flows on compact manifolds were proved for $\kappa = 5$ by Dolgopyat in [5]. Our main result in this paper is as follows:

Theorem 1.2. *Let $(\phi^A(t))_{t\geq 0}$ be the solution of* (1.1) *with $\|\phi_0\| = 1$. If $(\Phi_t)_{t\in\mathbb{R}}$ satisfies Assumption* 1.1*, then for any $t > 0$ there exist three constants $A_t, \Theta_t, \Xi > 0$ such that*

$$\left\|\phi^A(t)\right\| < \exp\left[-\Theta_t(\ln(\Xi A))^{\frac{2}{3d+2\kappa+2}}\right] \quad \text{for all } A > A_t.$$

Theorem 1.2 provides an answer to the question of how close the diffusion is to its equilibrium as A grows. It thus determines the speed of the relaxation phenomenon. The essential ingredients for the proof are Assumption 1.1 and Weyl asymptotic law on the eigenvalues of the Laplace operator. The constants Θ_t and A_t depend on the constants in those statements and will be made explicit in the the proof of the main result. In particular those constants become more explicit if we consider the problem on the torus $\mathbb{T}^2 = [0,1]^2$ (see Section 4).

For some fixed real valued function U defined on $\mathbb{R}^n$, Hwang, Hwang-Ma and Sheu proved in [9] that among the vector fields satisfying $\mathrm{div}(ue^{-U}) = 0$, the zero vector field yields the smallest spectral gap for the family of diffusion operators $L_u = \Delta - \nabla U \cdot \nabla + u \cdot \nabla$. This means that the convergence toward the equilibrium is slowest for the reversible diffusion generated by the self-adjoint operator $L = \Delta - \nabla U \cdot \nabla$. This has some consequence in Markov Monte Carlo Methods, where usually reversible diffusions are used to approximate a given probability distribution (see Geman, Hwang [8]). It was then suggested in [9] to perturb the self-adjoint generator by adding some anti-symmetric operator. However, it is then important to measure the improvement made through this device. For this it might be important to understand the relaxation speed in the result of [3]. Our result generalizes to diffusions generated by L_u as long as the unperturbed self-adjoint operator L has discrete spectrum and as information on the asymptotic of its eigenvalues is available.

The paper is organized as follows. In Section 2, we present some known results on eigenvalue distributions that will be needed in the proof of our main theorem. We also prove a result connected to the RAGE theorem, which stands for Ruelle, Amrein, Georgescu and Enss (see [4]). Proposition 2.5 will play a central role in the proof of our main result, since it relates the convergence speed in RAGE theorem with the eigenvalues of the Laplacian and the decay of correlation assumption. Our main result, Theorem 3.1, is restated in an equivalent form and proved in Section 3. In the last section, we consider the relaxation speed on the torus.

2. Preliminaries

On the compact manifold $\mathbb{M}$ the operator $-\Delta$ is a self-adjoint positive definite operator with discrete spectrum, which is composed of non-negative eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$. Let us denote by $\mathcal{N}(x) = \sum_{\lambda_j \leq x} 1$ the number of eigenvalues, counted with multiplicity, smaller or equal to x. We need the following classical results. The detailed proofs can be found in the references.

Proposition 2.1 (Corollary 2.5 [6])**.** *As $x \to +\infty$, we have*

$$\mathcal{N}(x) = (2\pi)^{-d}\omega_d \mathrm{vol}_{\mathbb{M}}(\mathbb{M}) x^{\frac{d}{2}} + O(x^{\frac{d-1}{2}}), \tag{2.1}$$

where ω_d is the volume of the unit disk in $\mathbb{R}^d$.

For simplicity of notation, we will denote $\Omega_d := (2\pi)^{-d}\omega_d \mathrm{vol}_{\mathbb{M}}(\mathbb{M})$. For more information on the $O(x^{(d-1)/2})$-function, one can also consult [11]. The following corollary is an immediate consequence.

Corollary 2.2. *There exists a constant $C_3 > 0$ such that for all $x \geq (2C_3)^2/\Omega_d^2$, we have*

$$\frac{\Omega_d}{2} x^{\frac{d}{2}} \leq \left(\Omega_d - C_3 x^{-\frac{1}{2}}\right) x^{\frac{d}{2}} \leq \mathcal{N}(x) \leq \left(\Omega_d + C_3 x^{-\frac{1}{2}}\right) x^{\frac{d}{2}} \leq \frac{3}{2}\Omega_d x^{\frac{d}{2}}. \tag{2.2}$$

Corollary 2.3. *For any $x > \max\left\{1, \frac{(C_3+1)^2}{\Omega_d^2}\right\}$, we have $\mathcal{N}(9x) - \mathcal{N}(x) \geq 1$.*

Proof. By Corollary 2.2, we have for all $x > 1$ with $x > (C_3+1)^2/\Omega_d^2$,

$$\begin{aligned}
\mathcal{N}(9x) - \mathcal{N}(x) &\geq \left(\Omega_d - C_3(9x)^{-\frac{1}{2}}\right)(9x)^{\frac{d}{2}} - \left(\Omega_d + C_3 x^{-\frac{1}{2}}\right) x^{\frac{d}{2}} \\
&= \Omega_d x^{\frac{d}{2}}(3^d - 1) - C_3 x^{\frac{d-1}{2}}(3^{d-1} + 1) \\
&\geq x^{\frac{d-1}{2}}(3^{d-1} + 1)(\Omega_d x^{\frac{1}{2}} - C_3) \geq 1.
\end{aligned}$$

□

We denote the eigenfunctions of the operator $-\Delta$ associated to the eigenvalues $\lambda_1, \lambda_2, \dots$ by $\varphi_1, \varphi_2, \dots$. They form some orthogonal basis for the Hilbert space

$$H := \left\{ f \in L^2(\mathbb{M}, \mathrm{vol}_{\mathbb{M}}) : \int_{\mathbb{M}} f d\mathrm{vol}_{\mathbb{M}} = 0 \right\}.$$

Let us also denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $\varphi_1, \varphi_2, \dots, \varphi_N$. The Sobolev space H^m associated with $-\Delta$ is formed by all vectors $\psi = \sum_{j=1}^{\infty} c_j \varphi_j \in H$ satisfying

$$\|\psi\|_{H^m}^2 = \sum_{j=1}^{\infty} \lambda_j^m |c_j|^2 < \infty.$$

The relation between the norms $\|.\|_{\mathcal{C}^\kappa}$ and $\|.\|_{H^m}$ is given through the following result.

Proposition 2.4 (Sobolev imbedding [1])**.** *There exists a constant $C_4 > 0$ such that for all $n \geq 1$*

$$\|\varphi_n\|_{\mathcal{C}^\kappa} \leq C_4 \|\varphi_n\|_{H^{\frac{d}{2}+\kappa+1}} = C_4 \lambda_n^{\frac{d+2\kappa+2}{4}}.$$

We now present the following proposition which is central for the proof of our main result.

Proposition 2.5. *Under Assumption* 1.1 *one has for any* $N, T > 0$ *and for any function* $f \in H$ *with* $\|f\| = 1$ *that*

$$\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2\,dt \le \frac{\sqrt{2C_1}C_4}{\sqrt{TC_2}} N\lambda_N^{\frac{d+2\kappa+2}{4}}.$$

Remark 2.6. Proposition 2.5 gives an explicit expression for a constant in Lemma 3.2 from [3]. This lemma states that for any $N, \xi > 0$ and any compact set $K \subset \{f \in H, \|f\| = 1\}$, there exists $T(N,\xi,K)$ such that $\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2\,dt \le \xi$ for all $T \ge T(N,\xi,K)$ and all $f \in K$. According to Proposition 2.5 the explicit choice

$$T(N,\xi) = \frac{2C_1C_4^2N^2\lambda_N^{\frac{d+2\kappa+2}{2}}}{\xi^2 C_2}$$

implies $\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2\,dt \le \xi$ for all $T \ge T(N,\xi)$. It therefore turns out that the constant in Lemma 3.2 from [3] can be chosen not to depend on K.

Proof of Proposition 2.5. The proof follows the proof of the RAGE theorem from the book of Cycon, Froese, Kirsch and Simon (see [4]). We use our assumption on decay of correlation and the explicit expression for the projection operator P_N to obtain an inequality from the proof presented there. For all $f \in H$ we have the decomposition $f = \sum_{k=1}^\infty \langle\varphi_k, f\rangle\,\varphi_k$. By the above notation, we have

$$P_N f = \sum_{k=1}^N \langle\varphi_k, f\rangle\,\varphi_k.$$

Let us define $Q(T)f = \frac{1}{T}\int_0^T (P_N(f\circ\Phi_t))\circ\Phi_{-t}dt$. Thus we have

$$Q(T)(f) = \sum_{k=1}^N \frac{1}{T}\int_0^T \langle\varphi_k\circ\Phi_{-t}, f\rangle\,\varphi_k\circ\Phi_{-t}dt,$$

and therefore

$$\begin{aligned}
Q(T)Q(T)(f) &= \sum_{k=1}^N \frac{1}{T}\int_0^T \langle\varphi_k\circ\Phi_{-t}, Q(T)(f)\rangle\,\varphi_k\circ\Phi_{-t}dt\\
&= \sum_{k=1}^N\sum_{j=1}^N \frac{1}{T^2}\int_0^T\int_0^T \langle\varphi_k\circ\Phi_{-t}, \varphi_j\circ\Phi_{-s}\rangle\,\langle\varphi_j\circ\Phi_{-s}, f\rangle\,\varphi_k\circ\Phi_{-t}dsdt.
\end{aligned}$$

It follows that

$$\|Q(T)\|^2 \le \sum_{k=1}^N\sum_{j=1}^N \frac{1}{T^2}\int_0^T\int_0^T |\langle\varphi_k\circ\Phi_{-t}, \varphi_j\circ\Phi_{-s}\rangle|\,dsdt\,\|\varphi_k\|\,\|\varphi_j\|$$

$$= \sum_{k=1}^N\sum_{j=1}^N \frac{1}{T^2}\int_0^T\int_0^T |\langle\varphi_k, \varphi_j\circ\Phi_{t-s}\rangle|\,dsdt. \tag{2.3}$$

By Assumption 1.1, there exist positive constants C_1, C_2 such that

$$|\langle\varphi_k, \varphi_j\circ\Phi_{t-s}\rangle| \le C_1e^{-C_2|t-s|}\,\|\varphi_k\|_{\mathcal{C}^\kappa}\,\|\varphi_j\|_{\mathcal{C}^\kappa}. \tag{2.4}$$

By Proposition 2.4, for all n, we have

$$\|\varphi_n\|_{\mathcal{C}^\kappa} \le C_4\,\|\varphi_n\|_{H^{\frac{d}{2}+\kappa+1}} = C_4\lambda_n^{\frac{d+2\kappa+2}{4}}. \tag{2.5}$$

From (2.3), (2.4) and (2.5) we obtain

$$\|Q(T)\|^2 \leq \sum_{k=1}^{N}\sum_{j=1}^{N}\frac{1}{T^2}\int_0^T\int_0^T C_1 e^{-C_2|t-s|}C_4^2\lambda_k^{\frac{d+2\kappa+2}{4}}\lambda_j^{\frac{d+2\kappa+2}{4}}dsdt$$

$$= \frac{C_1C_4^2}{T^2}\left(\sum_{k=1}^{N}\lambda_k^{\frac{d+2\kappa+2}{4}}\right)^2\int_0^T\int_0^T e^{-C_2|t-s|}dsdt. \tag{2.6}$$

Moreover, one has

$$\int_0^T\int_0^T e^{-C_2|t-s|}dsdt = 2\frac{C_2T+e^{-C_2T}-1}{C_2^2} < \frac{2T}{C_2}. \tag{2.7}$$

It is obvious that $\sum_{k=1}^{N}\lambda_k^{\frac{d+2\kappa+2}{4}} \leq N\lambda_N^{\frac{d+2\kappa+2}{4}}$. Combining with (2.6) and (2.7) we obtain

$$\|Q(T)\|^2 \leq \frac{2C_1C_4^2}{TC_2}N^2\lambda_N^{\frac{d+2\kappa+2}{2}}. \tag{2.8}$$

One has for all f with $\|f\|^2=1$,

$$\begin{aligned}\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2\,dt &= \frac{1}{T}\int_0^T \langle f,(P_N(f\circ\Phi_t))\circ\Phi_{-t}\rangle\,dt\\ &= \left\langle f,\frac{1}{T}\int_0^T (P_N(f\circ\Phi_t))\circ\Phi_{-t}dt\right\rangle\\ &\leq \|Q(T)f\|\,\|f\| \leq \|Q(T)\|\,. \end{aligned}\tag{2.9}$$

Combination of (2.8) and (2.9) gives

$$\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2\,dt \leq \frac{\sqrt{2C_1}C_4}{\sqrt{TC_2}}N\lambda_N^{\frac{d+2\kappa+2}{4}}.$$

□

We also need the following classical statement on the Lipschitz norm of the flow:

Proposition 2.7. *For all $t\in\mathbb{R}$ one has that $\|\Phi_t\|_{\mathrm{Lip}} \leq e^{\|u\|_{\mathrm{Lip}}|t|}$.*

Proof. See [3], p. 661. □

3. Main result and proofs

Following the approach from [3], we prefer to work with the rescaled solution $\phi^A(t)=\phi^\epsilon(t/\epsilon)$, which satisfies the following equation:

$$\begin{cases}\frac{d}{ds}\phi^\epsilon(s)=(u\cdot\nabla+\epsilon\Delta)\phi^\epsilon(s),\\ \phi^\epsilon(0)=\phi_0.\end{cases}\tag{3.1}$$

The following theorem is then equivalent to our main result, Theorem 1.2.

Theorem 3.1. *Let $(\phi^\epsilon(s))_{s\geq 0}$ be the solution of (3.1) with $\|\phi_0\|=1$. For any $\tau>0$ there exist constants A_τ,Θ_τ and Ξ such that*

$$\left\|\phi^\epsilon\left(\frac{\tau}{\epsilon}\right)\right\| < \exp\left[-\Theta_\tau\left(\ln\left(\Xi\frac{1}{\epsilon}\right)\right)^{\frac{2}{3d+2\kappa+2}}\right] \quad \textit{for all } \epsilon<\frac{1}{A_\tau}.$$

Remark 3.2. Some explicit expressions for the constants Θ_τ and A_τ are given later in Remark 3.3.

Proof. Since our proof relies strongly on the proof of Theorem 1.4 from the paper of Constantin, Kiselev, Ryzhik and Zlatoš (see [3]) we have to introduce some concepts and notation used there. They prove that, for any given τ and δ, there exists an $\epsilon_0(\delta)$ such that for all $\epsilon < \epsilon_0(\delta)$, one has $\|\phi^\epsilon(\tau/\epsilon)\| < \delta$.

Our purpose here is to make the constants involved in this statement explicit; that means to better understand the relation between ϵ and δ when τ is fixed. We will produce some explicit function $\epsilon^{\mathrm{expl}}(\delta)$ with $\epsilon^{\mathrm{expl}}(\delta) < \epsilon_0(\delta)$. It then follows that

$$\left\|\phi^{\epsilon^{\mathrm{expl}}(\delta)}(\tau/\epsilon^{\mathrm{expl}}(\delta))\right\| \leq \delta, \text{ for all } \delta.$$

The function $\epsilon^{\mathrm{expl}}(\delta)$ has some explicit inverse function $\delta^{\mathrm{expl}}(\epsilon)$ and it will then follow that

$$\|\phi^\epsilon(\tau/\epsilon)\| < \delta^{\mathrm{expl}}(\epsilon), \text{ for all } \epsilon \text{ sufficiently small.}$$

We will first briefly explain how the constant $\epsilon(\delta)$ is constructed in [3]. Note that some of the constructions presented there are simplified by the fact that Assumption 1.1 rules out point spectrum for the operator $u \cdot \nabla$ (see [10] p. 45 and p. 48). They construct $\epsilon_0(\delta)$ as follows:

$$\epsilon_0(\delta) = \min\left\{\frac{\tau}{2\tau_1(\delta)}, \frac{1}{20\lambda_N(\delta)\int_0^{\tau_1(\delta)} B^2(t)dt}\right\},$$

where $\lambda_N(\delta)$ is a suitable eigenvalue $\lambda_{N(\delta)}$ satisfying $e^{-\lambda_{N(\delta)}\tau/80} < \delta$ and with $B(t) = d^2\|\Phi_t\|_{\mathrm{Lip}}$ (see proofs of Theorem 1.2 in [3] and Theorem 2.2.2 in [12]). Without loss of generality, we assume that $\lambda_{N(\delta)+1} > \lambda_{N(\delta)}$. Moreover, according to [3] $\tau_1(\delta) = T(N(\delta), 1/20, K)$ where $T(N, \xi, K)$ is a constant satisfying

$$\frac{1}{T}\int_0^T \|P_N(f\circ\Phi_t)\|^2 dt < \xi, \text{ for all } T > T(N,\xi,K) \text{ and all } f \in K,$$

where K is a suitably chosen compact subset of the set $S := \{f \in H : \|f\| = 1\}$. However we saw in Remark 2.6 that the constant $T(N,\xi,K)$ can be chosen independently from K. We therefore drop the K in the notation.

If we can find explicit functions $\tau_1^{\mathrm{expl}}(\delta)$ and $\lambda_N^{\mathrm{expl}}(\delta)$ satisfying $\tau_1^{\mathrm{expl}}(\delta) > \tau_1(\delta)$ and $\lambda_N^{\mathrm{expl}}(\delta) > \lambda_N(\delta)$, then the following function will be an explicit lower bound for $\epsilon_0(\delta)$:

$$\epsilon_0^{\mathrm{expl}}(\delta) := \min\left\{\frac{\tau}{2\tau_1^{\mathrm{expl}}(\delta)}, \frac{1}{20\lambda_N^{\mathrm{expl}}(\delta)\int_0^{\tau_1^{\mathrm{expl}}(\delta)} B^2(t)dt}\right\}. \tag{3.2}$$

We choose the function $\lambda_N^{\mathrm{expl}}(\delta) := -720\ln(\delta)/\tau$. From Corollary 2.3, we have

$$\mathcal{N}(\lambda_N^{\mathrm{expl}}(\delta)) - \mathcal{N}\left(\frac{-80\ln(\delta)}{\tau}\right) \geq 1, \tag{3.3}$$

for all δ satisfying

$$\frac{-80\ln(\delta)}{\tau} > \max\left\{1, \frac{(C_3+1)^2}{\Omega_d^2}\right\}. \tag{3.4}$$

This is equivalent to the existence of an eigenvalue $\lambda_{N(\delta)}$ satisfying

$$\frac{-80\ln(\delta)}{\tau} < \lambda_{N(\delta)} < \frac{-720\ln(\delta)}{\tau} = \lambda_N^{\mathrm{expl}}(\delta). \tag{3.5}$$

Since we assumed $\lambda_{N(\delta)+1} > \lambda_{N(\delta)}$, we have by Corollary 2.2 that

$$N(\delta) = \mathcal{N}(\lambda_{N(\delta)}) \le \frac{3}{2}\Omega_d \lambda_{N(\delta)}^{\frac{d}{2}}$$

for all δ satisfying

$$\frac{-80\ln(\delta)}{\tau} \ge \frac{(2C_3)^2}{\Omega_d^2}. \tag{3.6}$$

From Remark 2.6, we obtain

$$\begin{aligned}\tau_1(\delta) = T\left(N(\delta), \frac{1}{20}\right) &= \frac{800 C_1 C_4^2 N(\delta)^2 \lambda_{N(\delta)}^{d/2+\kappa+1}}{C_2} \\ &\le \frac{1800 C_1 C_4^2 \Omega_d^2 \lambda_{N(\delta)}^{3d/2+\kappa+1}}{C_2} \\ &\le C_5 \left(\frac{-80\ln(\delta)}{\tau}\right)^{3d/2+\kappa+1} =: \tau_1^{\mathrm{expl}}(\delta),\end{aligned}$$

where

$$C_5 := 9^{3d/2+\kappa+1}\Omega_d^2 \frac{1800 C_1 C_4^2}{C_2}.$$

Thus the function $\epsilon_0^{\mathrm{expl}}(\delta)$ given in (3.2) yields a lower bound for $\epsilon_0(\delta)$. However the minimum in the expression of $\epsilon_0^{\mathrm{expl}}(\delta)$ is not suitable for the computation of an explicit inverse. We therefore introduce the function

$$\epsilon_1^{\mathrm{expl}}(\delta) := \frac{\|u\|_{\mathrm{Lip}}}{10 d^4 \lambda_N^{\mathrm{expl}}(\delta) \exp[2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)]} \tag{3.7}$$

for all δ satisfying the relations (3.4), (3.6) and

$$-80\ln(\delta) \ge \frac{1}{90 d^4}. \tag{3.8}$$

We then have by (3.8) and the definition of $\lambda_N^{\mathrm{expl}}(\delta)$ in (3.5)

$$\begin{aligned}\frac{\tau}{2\tau_1^{\mathrm{expl}}(\delta)} &= \frac{d^4(-7200\ln(\delta))\|u\|_{\mathrm{Lip}}}{10 d^4 \frac{-720\ln(\delta)}{\tau} 2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)} \\ &> \frac{\|u\|_{\mathrm{Lip}}}{10 d^4 \lambda_N^{\mathrm{expl}}(\delta)[2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)]} \\ &> \frac{\|u\|_{\mathrm{Lip}}}{10 d^4 \lambda_N^{\mathrm{expl}}(\delta) \exp[2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)]} = \epsilon_1^{\mathrm{expl}}(\delta).\end{aligned}$$

Moreover, we obtain with Proposition 2.7 that

$$\int_0^{\tau_1^{\mathrm{expl}}(\delta)} B^2(t)dt \le \int_0^{\tau_1^{\mathrm{expl}}(\delta)} d^4 e^{2\|u\|_{\mathrm{Lip}} t} dt < \frac{d^4 \exp[2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)]}{2\|u\|_{\mathrm{Lip}}}$$

and it then follows that

$$\frac{1}{20\lambda_N^{\mathrm{expl}}(\delta) \int_0^{\tau_1^{\mathrm{expl}}(\delta)} B^2(t)dt} > \frac{\|u\|_{\mathrm{Lip}}}{10 d^4 \lambda_N^{\mathrm{expl}}(\delta) \exp[2\|u\|_{\mathrm{Lip}} \tau_1^{\mathrm{expl}}(\delta)]} = \epsilon_1^{\mathrm{expl}}(\delta).$$

Therefore,

$$\epsilon_1^{\mathrm{expl}}(\delta) < \min\left\{\frac{\tau}{2\tau_1^{\mathrm{expl}}(\delta)}, \frac{1}{20\lambda_N^{\mathrm{expl}}(\delta)\int_0^{\tau_1^{\mathrm{expl}}(\delta)} B^2(t)dt}\right\} = \epsilon_0^{\mathrm{expl}}(\delta).$$

However the expression for $\epsilon_1^{\mathrm{expl}}(\delta)$ still contains $\lambda_N^{\mathrm{expl}}(\delta)$. In order to remove $\lambda_N^{\mathrm{expl}}(\delta)$ we note that for all δ satisfying (3.4) we have

$$\begin{aligned}\lambda_N^{\mathrm{expl}}(\delta) = \frac{-720\ln(\delta)}{\tau} &= 9\left(\frac{-80\ln(\delta)}{\tau}\right) \\ &< 9\left(\frac{-80\ln(\delta)}{\tau}\right)^{3d/2+\kappa+1} = \frac{9}{C_5}\tau_1^{\mathrm{expl}}(\delta).\end{aligned}$$

It then follows that

$$\begin{aligned}\epsilon_1^{\mathrm{expl}}(\delta) &\geq \frac{C_5\|u\|_{\mathrm{Lip}}}{90d^4\tau_1^{\mathrm{expl}}(\delta)\exp[2\|u\|_{\mathrm{Lip}}\tau_1^{\mathrm{expl}}(\delta)]} \\ &> \frac{C_5\|u\|_{\mathrm{Lip}}}{90d^4\exp[(1+2\|u\|_{\mathrm{Lip}})\tau_1^{\mathrm{expl}}(\delta)]} =: \epsilon^{\mathrm{expl}}(\delta).\end{aligned} \tag{3.9}$$

This final lower bound function can be inverted and its inverse function is given by

$$\delta^{\mathrm{expl}}(\epsilon) = \exp\left[-\frac{\tau}{80}\left(\frac{1}{C_5(1+2\|u\|_{\mathrm{Lip}})}\ln\left(\frac{C_5\|u\|_{\mathrm{Lip}}}{90d^4}\frac{1}{\epsilon}\right)\right)^{\frac{2}{3d+2\kappa+2}}\right]. \tag{3.10}$$

The proof is complete with

$$\Xi := \frac{C_5\|u\|_{\mathrm{Lip}}}{90d^4}, \qquad \Theta_\tau := \frac{\tau}{80}\left(\frac{1}{C_5(1+2\|u\|_{\mathrm{Lip}})}\right)^{\frac{2}{3d+2\kappa+2}}$$

and A_τ from the following remark. □

Remark 3.3. The relation between τ and ϵ^{expl} is deduced from (3.9) and (3.7). Adding conditions (3.4), (3.6) and (3.8) we have

$$\frac{-80\ln(\delta)}{\tau} \geq \max\left\{1, \frac{(C_3+1)^2}{\Omega_d^2}, \frac{(2C_3)^2}{\Omega_d^2}, \frac{1}{90d^4\tau}\right\}. \tag{3.11}$$

It then follows that

$$\begin{aligned}\frac{1}{\epsilon^{\mathrm{expl}}(\delta)} &= \frac{90d^4\exp[(1+2\|u\|_{\mathrm{Lip}})\tau_1^{\mathrm{expl}}(\delta)]}{C_5\|u\|_{\mathrm{Lip}}} \\ &= \frac{90d^4\exp\left[(1+2\|u\|_{\mathrm{Lip}})C_5\left(\frac{-80\ln(\delta)}{\tau}\right)^{3d/2+\kappa+1}\right]}{C_5\|u\|_{\mathrm{Lip}}} \\ &\geq A_\tau\end{aligned}$$

with

$$A_\tau := \frac{90d^4}{C_5\|u\|_{\mathrm{Lip}}}\exp\left[(1+2\|u\|_{\mathrm{Lip}})C_5\max\left\{1, \frac{(C_3+1)^2}{\Omega_d^2}, \frac{(2C_3)^2}{\Omega_d^2}, \frac{1}{90d^4\tau}\right\}^{3d/2+\kappa+1}\right].$$

This condition ensures that $C_5\|u\|_{\mathrm{Lip}}/(90d^4\epsilon) > 1$; therefore the formula (3.10) is well defined.

4. The particular case of the torus

We consider the problem on the torus $\mathbb{T}^2 = [0,1]^2$. In this case, we know exactly the eigenvalues of the Laplace operator. Therefore, Corollary 2.2 and Corollary 2.3 will be simplified by Corollary 4.1. Moreover, we can give the exact value for the constant C_4 in Proposition 2.4; this is provided by Proposition 4.2. The following are the details.

Corollary 4.1. *In the case of torus $\mathbb{T}^2 = [0,1]^2$, the number of eigenvalues of the Laplace operator $-\Delta$ smaller than or equal to x is*

$$\mathcal{N}(x) = \sum_{\lambda \le x} 1 = \# \left\{ (m,n) \in \mathbb{Z}^2 : m^2 + n^2 \le \frac{x}{4\pi^2} \right\}.$$

It is easy to see that with all $x > 0$ we have $\mathcal{N}(x) \le (\sqrt{x}/\pi + 1)^2$. Furthermore,

$$\mathcal{N}((\sqrt{x} + 2\pi)^2) - \mathcal{N}(x) \ge 1.$$

Proposition 4.2. *For any eigenfunction φ_n associated to the eigenvalue λ_n and for any $\kappa > 0$, we get*

$$\|\varphi_n\|_{\mathcal{C}^\kappa} \le 2^{\kappa/2} \kappa \lambda_n^{\kappa/2}.$$

Proposition 2.5 is based on Proposition 2.4 which is improved for the case of the torus in Remark 2.6. If we use this in our proof, then we get the following improved proposition.

Proposition 4.3. *For any $N, T > 0$ and for any function f with $\|f\| = 1$, we have*

$$\frac{1}{T} \int_0^T \|P_N(f \circ \Phi_t)\|^2 \, dt \le \frac{\sqrt{2C_1}}{\sqrt{TC_2}} \kappa 2^{\kappa/2} N \lambda_N^{\kappa/2}.$$

Then we have the concrete case of Theorem 1.2.

Theorem 4.4. *If $(\Phi_t)_{t \in \mathbb{R}}$ satisfies decay of correlation in Assumption 1.1, then for any $\tau > 0$, we have*

$$\left\|\phi^A(\tau)\right\| \le \exp\left[-\frac{\tau}{80} \left(\left(\frac{C_2 \pi^4}{C_2 \pi^4 + 1600 C_1 \|u\|_{\mathrm{Lip}} \kappa^2 2^\kappa} \ln \left(\frac{\|u\|_{\mathrm{Lip}}}{10 d^4} A \right) \right)^{\frac{1}{2\kappa+4}} - 3\pi \right)^2 \right],$$

where A satisfies

$$A \ge \frac{10 d^4 \exp\left[\left(1 + \frac{1600 C_1 \|u\|_{\mathrm{Lip}} \kappa^2 2^\kappa}{C_2 \pi^4} \right) \left(\sqrt{\frac{1}{90 d^4 \tau}} + 3\pi \right)^{2\kappa+4} \right]}{\|u\|_{\mathrm{Lip}}}.$$

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