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QUASISYMMETRIC EXTENSION ON THE REAL LINE

VYRON VELLIS

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Abstract. We give a geometric characterization of the sets $E \subset \mathbb{R}$ for which every quasisymmetric embedding $f : E \to \mathbb{R}^n$ extends to a quasisymmetric embedding $f : \mathbb{R} \to \mathbb{R}^N$ for some $N \geq n$.

1. Introduction

Suppose that E is a subset of a metric space X and f is a quasisymmetric embedding of E into some metric space Y. When is it possible to extend f to a quasisymmetric embedding of X into Y' for some metric space Y' containing Y? Questions related to quasisymmetric extensions have been considered by Beurling and Ahlfors [3], Ahlfors [1,2], Carleson [4], Tukia and Väisälä [11] and Kovalev and Onninen [7].

Tukia and Väisälä [12] showed that for $M = \mathbb{R}^p, \mathbb{S}^p$, any quasisymmetric mapping $f : M \to \mathbb{R}^n$, with $n > p$, extends to a quasisymmetric homeomorphism of $\mathbb{R}^n$ when f is locally close to a similarity. Later, Väisälä [14] extended this result to all compact, co-dimension 1, C^1 or piecewise linear manifolds M in $\mathbb{R}^n$.

In this article we are concerned with the case $X = \mathbb{R}$ and $Y = \mathbb{R}^n$. Specifically, given a set $E \subset \mathbb{R}$ and a quasisymmetric embedding f of E into $\mathbb{R}^n$, we ask when is it possible to extend f to a quasisymmetric embedding of $\mathbb{R}$ into $\mathbb{R}^N$ for some $N \geq n$? While any bi-Lipschitz embedding of a compact set $E \subset \mathbb{R}$ into $\mathbb{R}^n$ extends to a bi-Lipschitz embedding of $\mathbb{R}$ into $\mathbb{R}^N$ for some $N \geq n$ [5], the same is not true for quasisymmetric embeddings. In fact, there exist $E \subset \mathbb{R}$ and a quasisymmetric embedding $f : E \to \mathbb{R}$ that cannot be extended to a quasisymmetric embedding $F : \mathbb{R} \to \mathbb{R}^N$ for any N; see e.g. [6, p. 89]. Thus, more regularity for sets E should be assumed.

Following Trotsenko and Väisälä [10], a metric space X is termed *M-relatively connected* for some $M > 1$ if, for any point $x \in X$ and any $r > 0$ with $\overline{B}(x,r) \neq X$, either $\overline{B}(x,r) = \{x\}$ or $\overline{B}(x,r) \setminus B(x,r/M) \neq \emptyset$. A metric space X is called relatively connected if it is M-relatively connected for some $M \geq 1$.

With this terminology, our main theorem is stated as follows.

Theorem 1.1. *If $E \subset \mathbb{R}$ is M-relatively connected and $f : E \to \mathbb{R}^n$ is η-quasisymmetric, then f extends to an η'-quasisymmetric embedding $F : \mathbb{R} \to \mathbb{R}^{n+n_0}$ where n_0 depends only on M and η while η' depends only on M, η and n.*

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On the other hand, it follows from a theorem of Trotsenko and Väisälä [10] that if $E \subset \mathbb{R}$ is not relatively connected, then there exists a quasisymmetric mapping $f : E \to \mathbb{R}$ that admits no quasisymmetric extension $F : \mathbb{R} \to \mathbb{R}^N$ for any $N \geq 1$; see Corollary 2.5.

A subset E of a metric space X is said to have the *quasisymmetric extension property in* X if every quasisymmetric mapping $f : E \to X$ that can be extended homeomorphically in X can also be extended quasisymmetrically in X. The question of characterizing such sets E, given a space X, poses formidable difficulties due to the topological complexity of X. For instance, $\mathbb{S}^1$ and $\mathbb{R}$ have the quasisymmetric extension property in $\mathbb{R}^2$ [1], but it is unknown whether $\mathbb{S}^n$ or $\mathbb{R}^n$ has this property in $\mathbb{R}^{n+1}$ when $n \geq 2$.

The sets $E \subset \mathbb{R}$ that have the quasisymmetric extension property in $\mathbb{R}$ are characterized by the relative connectedness.

Theorem 1.2. *A set $E \subset \mathbb{R}$ has the quasisymmetrc extension property in $\mathbb{R}$ if and only if it is relatively connected.*

The arguments used in the proof of Theorem 1.2 apply verbatim in the case $X = \mathbb{S}^1$ and $E \subset \mathbb{S}^1$. Thus, if X is quasisymmetric homeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$, then a set $E \subset X$ has the quasisymmetric extension property in X if and only if E is relatively connected.

In dimensions $n \geq 2$, however, Theorem 1.2 fails even for small sets such as the Cantor sets. In Section 5 we show that for each $n \geq 2$, there exists a relatively connected Cantor set $E \subset \mathbb{R}^n$ and a bi-Lipschitz mapping $f : E \to \mathbb{R}^n$ which admits a homeomorphic extension in $\mathbb{R}^n$ but not a quasisymmetric extension in $\mathbb{R}^n$; see Remark 5.2.

2. Preliminaries

In the following, given an open bounded interval $I = (a, b) \subset \mathbb{R}$, we denote by $|I|$ its length $b - a$; if $I = \emptyset$, then $|I| = 0$. As usual, $a \vee b$ and $a \wedge b$ denote the maximum and minimum, respectively, of two real numbers a and b. Finally, for two points $x, y \in \mathbb{R}^n$, we denote by $[x, y]$ the line segment in $\mathbb{R}^n$ with endpoints x and y.

2.1. **Mappings.** A homeomorphism $f\colon (X, d) \to (Y, d')$ between two metric spaces is called *L-bi-Lipschitz* for some $L > 1$ if both f and f^{-1} are L-Lipschitz.

A mapping $f\colon (X, d) \to (Y, d')$ is called η-quasisymmetric if there exists a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ such that for any $x, a, b \in X$ with $x \neq b$ we have

$$\frac{d'(f(x), f(a))}{d'(f(x), f(b))} \leq \eta\left(\frac{d(x, a)}{d(x, b)}\right).$$

It is a simple consequence of the definition that the composition of a similarity mapping of $\mathbb{R}^n$ and an η-quasisymmetric mapping between sets of $\mathbb{R}^n$ is η-quasisymmetric.

If f is η-quasisymmetric with $\eta(t) = C(t^{\alpha} \vee t^{1/\alpha})$ for some $\alpha \in (0, 1]$ and $C > 0$, then f is termed *power quasisymmetric*, and we say that f is (C, α)-quasisymmetric. An important property of power quasisymmetric mappings is that they are bi-Hölder continuous on bounded sets [6, Corollary 11.5].

Lemma 2.1. *Suppose that (X,d) is a bounded metric space and $f:(X,d)\to(Y,d')$ is (C,α)-quasisymmetric. There exists $C'>1$ depending only on C, α,* $\operatorname{diam} X$ *and* $\operatorname{diam} f(X)$ *such that for all $x,y\in E$,*

$$(C')^{-1}d(x,y)^{1/\alpha}\le d'(f(x),f(y))\le C'd(x,y)^{\alpha}.$$

For doubling connected metric spaces it is known that the quasisymmetric condition is equivalent to a weaker (but simpler) condition known in literature as *weak quasisymmetry.*

Lemma 2.2 ([6, Theorem 10.19])**.** *Let $I\subset\mathbb{R}$ be an interval and $f:I\to\mathbb{R}^n$ be an embedding for which there exists $H\ge 1$ such that for all $x,y,z\in I$,*

$$|x-y|\le|x-z| \text{ implies } |f(x)-f(y)|\le H|f(x)-f(z)|. \tag{2.1}$$

Then f is η-quasisymmetric with η depending only on H and n.

The next lemma is an immediate corollary to Lemma 2.2.

Lemma 2.3. *Let I_1,I_2 be open bounded intervals and $f:I_1\cup I_2\to\mathbb{R}$ be an embedding. Suppose that there exists $C>1$ such that $|I|/|J|<C$ for all $I,J\in\{I_1,I_2,I_1\cap I_2\}$. If $f|I_1$ and $f|I_2$ are η-quasisymmetric, then $f|(I_1\cup I_2)$ is η'-quasisymmetric for some η' depending on η and C.*

Proof. If $I_1\subset I_2$ or $I_2\subset I_1$ there is nothing to prove. Suppose that $I_1=(a_1,b_1)$, $I_2=(a_2,b_2)$ with $a_1<a_2<b_1<b_2$ and denote by m the center of $I_1\cap I_2$. We show that $f|(I_1\cup I_2)$ satisfies (2.1). Let $x,y,z\in I_1\cup I_2$ with $|x-y|\le|x-z|$. Since $f|I_j$ is monotone for each $j=1,2$, $f|I_1\cup I_2$ is monotone, and we may assume that either $y<x<z$ or $z<x<y$. Assume the first; the second case is identical.

If all three points are in the same I_j there is nothing to prove. Hence, we may assume that $y\le a_2$ and $z\ge b_1$.

If $x\le m$, then $|f(x)-f(y)|\le\eta(\frac{|x-y|}{|x-b_1|})|f(x)-f(z)|\le\eta(2C)|f(x)-f(z)|$.

If $x\ge m$, then

$$\begin{aligned}|f(x)-f(y)|&=|f(x)-f(a_2)|+|f(a_2)-f(y)|\le|f(x)-f(a_2)|\left(1+\frac{|f(a_2)-f(a_1)|}{|f(a_2)-f(m)|}\right)\\&\le|f(x)-f(a_2)|\left(1+\eta\left(\frac{|a_1-a_2|}{|a_2-m|}\right)\right)\le(1+\eta(2C))|f(x)-f(a_2)|\\&\le(1+\eta(2C))\eta\left(\frac{|x-a_2|}{|x-z|}\right)|f(x)-f(z)|\le(1+\eta(2C))\eta(1)|f(x)-f(z)|,\end{aligned}$$

where for the last inequality we used $|x-a_2|\le|x-y|\le|x-z|$. □

2.2. **Relatively connected sets.** Relatively connected sets were first introduced by Trotsenko and Väisälä [10] in the study of spaces for which every quasisymmetric mapping is power quasisymmetric. The definition given in [10] is equivalent to the one in Section 1 quantitatively [10, Theorem 4.11].

Relative connectedness is a weak form of the well known notion of uniform perfectness. A metric space X is *c-uniformly perfect* for some $c>1$ if for all $x\in X$, $\overline{B}(x,r)\ne X$ implies $\overline{B}(x,r)\setminus B(x,r/c)\ne\emptyset$. The difference between the two notions is that relatively connected sets allow isolated points. In particular, if E is c-uniformly perfect, then it is M-relatively connected for all $M>c$, and if E is M-relatively connected and has no isolated points, then it is $(2M+1)$-uniformly perfect [10, Theorem 4.13].

The connection between relative connectedness and power quasisymmetric mappings is illustrated in the following theorem from [10].

Theorem 2.4 ([10, Theorem 6.20]). *A subset E of a metric space X is relatively connected if and only if every quasisymmetric map $f : E \to X$ is power quasisymmetric.*

The necessity of relative connectedness for extensions of quasisymmetric mappings on $\mathbb{R}$ follows now as a corollary.

Corollary 2.5. *If $E \subset \mathbb{R}$ is not relatively connected, then there exists a monotone quasisymmetric mapping $f : E \to \mathbb{R}$ such that, for every metric space Y containing the Euclidean line $\mathbb{R}$, there exists no quasisymmetric extension $F : \mathbb{R} \to Y$ of f.*

Proof. By [10, Theorem 6.6], there exists a quasisymmetric mapping $f : E \to \mathbb{R}$ that is not power quasisymmetric. A close inspection of its proof reveals, moreover, that the mapping f is increasing. Now let Y be a metric space containing the Euclidean line $\mathbb{R}$. If there was a quasisymmetric extension $F : \mathbb{R} \to Y$, then, by Theorem 2.4, F would be power quasisymmetric. Thus, f would be power quasisymmetric, which is a contradiction. □

2.3. **Relative distance.** Let E, F be two compact sets in a metric space (X, d) both of which contain at least two points. The *relative distance* of E and F is defined to be the quantity

$$d^*(E,F) = \frac{\operatorname{dist}(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F}$$

where $\operatorname{dist}(E,F) = \min\{d(x,y) : x \in E, y \in F\}$.

Note that if $E, F \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity, then $d^*(f(E), f(F)) = d^*(E,F)$. In general, if $f : E \cup F \to Y$ is η-quasisymmetric, then

$$\frac{1}{2}\phi(d^*(E,F)) \leq d^*(f(E), f(F)) \leq \eta(2d^*(E,F)) \tag{2.2}$$

where $\phi(t) = (\eta(t^{-1}))^{-1}$; see for example [13, p. 532].

The following remark ties together the notions of uniform perfectness in $\mathbb{R}$ and relative distance of sets in $\mathbb{R}$.

Remark 2.6. A closed set $E \subset \mathbb{R}$ is c-uniformly perfect for some $c \geq 1$ if and only if there exists $C > 0$ such that for all bounded components I, J of $\mathbb{R} \setminus E$, $d^*(I,J) \geq C$. The constants c and C are quantitatively related.

3. Quasisymmetric extension on $\mathbb{R}$

Suppose that $E \subset \mathbb{R}$ is relatively connected and $f : E \to \mathbb{R}^n$ is quasisymmetric. If E is a singleton, then trivially f admits a quasisymmetric extension. Moreover, since quasisymmetric functions have a quasisymmetric extension to the closure of their domains, we may assume that E is closed.

In Section 3.1 we construct a quasisymmetric extension $f_0 : E_0 \to \mathbb{R}^m$ of f, where $E \subset E_0 \subset \mathbb{R}$ is a uniformly perfect set with no lower or upper bound and m is either n or $n+1$. In Section 3.2, for some $n_0 \in \mathbb{N}$ depending only on M and η, we construct a homeomorphic extension $F_0 : \mathbb{R} \to \mathbb{R}^{n+n_0}$ of f_0. Finally, in Section 3.3 we construct a quasisymmetric extension $F : \mathbb{R} \to F_0(\mathbb{R}) \subset \mathbb{R}^{n+n_0}$ of f_0.

For the rest, $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$ and, for each $i = 1, \dots, n$, $\mathbf{e}_i$ denotes the vector in $\mathbb{R}^n$ whose i-th coordinate is 1 and the rest are 0.

3.1. **Two preliminary extensions.** Throughout this section we assume that E is an M-relatively connected closed set and f is an η-quasisymmetric embedding of E into $\mathbb{R}^n$ with $\eta = C(t^\alpha \vee t^{1/\alpha})$.

Suppose that E is bounded from above or bounded from below. Then one of the following cases applies.

Case 1. Suppose that E has a lower bound but no upper bound. Applying suitable similarities we may assume that $1 \in E$, $\min E = 0$ and $f(0) = \mathbf{0}$. Let $C_0 = \max\{2, 1/\eta^{-1}(1/2)\}$. Set $a_0 = 0$ and, by relative connectedness, there exists a sequence $\{a_k\}_{k\in\mathbb{N}} \subset E$ with $a_1 = 1$ and $a_k/a_{k-1} \in [C_0, MC_0]$. Set $\tilde{E} = E \cup \{-a_k\}_{k\in\mathbb{N}}$ and $\tilde{f} : \tilde{E} \to \mathbb{R}^{n+1}$ with $\tilde{f}|E = f \times \{0\}$ and $\tilde{f}(-a_k) = \{\mathbf{0}\} \times \{-|f(a_k)|\}$.

Case 2. Suppose that E has an upper bound but no lower bound. Applying suitable similarities we may assume that $1 \in E$, $\max E = 0$ and $f(0) = \mathbf{0}$. We define $\tilde{E}$ and $\tilde{f}$ similarly to Case 1.

Case 3. Suppose that E is bounded. Applying suitable similarities, we may assume that $\min E = 0$, $\max E = 1$, $\max_{x\in E} |f(x)| = 1$ and $\operatorname{diam} f(E) = 1$. For any $k \in \mathbb{Z}$ define $\tilde{E}_k = \{2k + x : x \in E\}$, $\tilde{E} = \bigcup_{k\in\mathbb{Z}} \tilde{E}_k$ and $\tilde{f} : \tilde{E} \to \mathbb{R}^n$ with $\tilde{f}(2k+x) = 2k\mathbf{e}_1 + f(x)$. A similar extension in the case $n = 1$ has been considered by Lehto and Virtanen in [8, II.7.2].

Lemma 3.1. *In each case, $\tilde{E}$ is an $\tilde{M}$-relatively connected closed set and $\tilde{f}$ is $\tilde{\eta}$-quasisymmetric with $\tilde{M}$ and $\tilde{\eta}$ depending only on M and η.*

Proof. We prove the lemma only for Case 1 and Case 3; the proof for Case 2 is similar to that of Case 1.

Case 1. Note first that $\{-a_n\}_{n\in\mathbb{N}}$ is M_1-relatively connected for some M_1 depending only on M and η. Let $x \in \tilde{E}$ and $r > 0$ such that $\overline{B}(x, r) \cap \tilde{E} \neq \{x\}$. If $x \in E$, then $\overline{B}(x, r) \cap E \neq \{x\}$ and $(\overline{B}(x, r) \setminus B(x, r/M)) \cap \tilde{E} \neq \emptyset$. If $x = -a_n$, $n \geq 1$, then $\overline{B}(x, r) \cap \{-a_n\}_{n\in\mathbb{N}} \neq \{x\}$ and $(\overline{B}(x, r) \setminus B(x, r/M_1)) \cap \tilde{E} \neq \emptyset$. Thus, $\tilde{E}$ is $(M \vee M_1)$-relatively connected.

For the quasisymmetry of $\tilde{f}$, note first that $\tilde{f}$ restricted on $\{-a_n\}_{n\in\mathbb{N}}$ is $C\eta$-quasisymmetric for some $C > 1$ depending only on η. Let $x, y, z \in \tilde{E}$. If all three of them are in E or in $\tilde{E} \setminus E$, then the quasisymmetry of $\tilde{f}$ follows trivially.

Assume first that $x, z \in E$ and $y = -a_n$ for some $a_n \in E$. Then, $|\tilde{f}(y)| = |f(a_n)|$, $|y| = |a_n|$ and

$$\begin{aligned}\frac{|\tilde{f}(x) - \tilde{f}(y)|}{|\tilde{f}(x) - \tilde{f}(z)|} &\leq 2\frac{|f(x)|}{|f(x) - f(z)|} + \frac{|f(x) - f(a_n)|}{|f(x) - f(z)|} \\ &\leq 2\eta\left(\frac{|x|}{|x - z|}\right) + \eta\left(\frac{|x - a_n|}{|x - z|}\right) \leq 3\eta\left(\frac{|x - y|}{|x - z|}\right).\end{aligned}$$

We work similarly if $x, z \in \{-a_n\}_{n\in\mathbb{N}}$ and $y \in E$.

Assume now that $z \in E$ and $y, x \notin E$. Let n_0 be the smallest integer n such that $a_n \geq z$ and set $\overline{z} = -a_{n_0}$. Then, there exist constants $C_1, C_2 > 1$ depending only on M, C and α such that

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|\tilde{f}(x)-\tilde{f}(z)|} \leq C_1 \min\left\{\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|\tilde{f}(x)-\tilde{f}(\overline{z})|}, \frac{|\tilde{f}(x)-\tilde{f}(y)|}{|\tilde{f}(x)|}\right\}$$
$$\leq C_1 \min\left\{\eta\left(\frac{|x-y|}{|x-\overline{z}|}\right), \eta\left(\frac{|x-y|}{|x|}\right)\right\} \leq C_2\eta\left(\frac{|x-y|}{|x-z|}\right).$$

We work similarly if $z \in \{-a_n\}_{n\in\mathbb{N}}$ and $x, y \in E$.

Case 3. We first show that $\tilde{E}$ is M'-relatively connected with $M' = 8M$. Let $x \in \tilde{E}$ and $r > 0$ such that $\overline{B}(x,r) \cap \tilde{E} \neq \{x\}$. Since $\tilde{E}$ is unbounded, $\tilde{E} \setminus \overline{B}(x,r) \neq \emptyset$. By periodicity of $\tilde{E}$, we may assume that $x \in E$. If $r \geq 4$, then $\overline{B}(x,r) \setminus B(x,r/2)$ contains an interval of length 2 and therefore it contains points of $\tilde{E}$. Suppose now that $r < 4$. Then, $\tilde{E} \cap B(x,r/8) \subset E$ and $E \setminus \overline{B}(x,r/8) \neq \emptyset$. If $E \cap \overline{B}(x,r/8) = \{x\}$, then $\tilde{E} \cap \overline{B}(x,r/8) = \{x\}$ and the relative connectedness is satisfied with $M' = 8$. If $E \cap \overline{B}(x,r/8) \neq \{x\}$, then, by the relative connectedness of E, $E \cap (\overline{B}(x,r) \setminus B(x,r/(8M))) \neq \emptyset$.

We now show the second claim. Recall that by Theorem 2.4 f is power quasi-symmetric. Let $y, x, z \in \tilde{E}$ and assume $y \in \tilde{E}_{n_1}$, $x \in \tilde{E}_{n_2}$ and $z \in \tilde{E}_{n_3}$ with $n_1, n_2, n_3 \in \mathbb{Z}$. If $n_1 = n_2 = n_3$ the claim follows trivially. If n_1, n_2, n_3 are all different, then

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|\tilde{f}(x)-\tilde{f}(z)|} \leq \frac{2|n_2-n_1|+1}{2|n_3-n_2|-1} \leq 9\frac{|n_2-n_1|}{|n_3-n_2|+2} \leq 9\frac{|x-y|}{|x-z|}.$$

If $n_1 = n_2 \neq n_3$, then the second inequality in Lemma 2.1 gives

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|\tilde{f}(x)-\tilde{f}(z)|} \leq C'\frac{|x-y|^\alpha}{|n_3-n_2|} \leq 3C'\left(\frac{|x-y|}{|x-z|}\right)^\alpha.$$

The remaining case $n_1 \neq n_2 = n_3$ is treated similarly using the first inequality of Lemma 2.1. □

By Lemma 3.1 we may assume for the rest that E is a relatively connected closed set with no upper or lower bound. Hence, all components of $\mathbb{R} \setminus E$ are bounded open intervals.

For the second extension, we treat the case when E has isolated points. For each isolated point $x \in E$ let $\pi(x) \in E$ be the closest point of $E \setminus \{x\}$ to x and define

$$E_x = \overline{B}(x, |x-\pi(x)|/10)$$

and $f_x : E_x \to \mathbb{R}^n$ with

$$f_x(y) = f(x) + \frac{1}{\eta(1)}\frac{|f(x)-f(\pi(x))|}{|x-\pi(x)|}(y-x)\mathbf{e}_1.$$

If x is an accumulation point of E, then set $E_x = \{x\}$ and $f_x : \{x\} \to \mathbb{R}$ with $f_x(x) = f(x)$. Finally, set $\hat{E} = \bigcup_{x\in E} E_x$ and $\hat{f} \colon \hat{E} \to \mathbb{R}$ with $\hat{f}|E_x = f_x$. Similar extensions also appear in a paper of Semmes [9, Section 2].

Remark 3.2. Suppose that $x \in E$ is an isolated point. Then,

$$4 \leq d^*(E_x, \hat{E} \setminus E_x) \leq 5 \quad \text{and} \quad 3 \leq d^*(\hat{f}(E_x), \hat{f}(\hat{E} \setminus E_x)) \leq 5\eta(1).$$

The first claim of Remark 3.2 is clear. For the upper bound of the second claim note that $\operatorname{dist}(\hat{f}(E_x), \hat{f}(\hat{E} \setminus E_x)) \le |f(x) - f(\pi(x))| \le 5\eta(1) \operatorname{diam} \hat{f}(E_x)$. For the lower bound, take points $x' \in E_x$ and $y' \in \hat{E} \setminus E_x$ and assume that $y' \in E_y$. Then,

$$\frac{|\hat{f}(x') - \hat{f}(x)|}{|\hat{f}(x') - \hat{f}(y')|} \le \frac{1}{10\eta(1)} \eta\left(\frac{|x - \pi(x)|}{|x - y|}\right) \frac{|\hat{f}(x) - \hat{f}(y)|}{\frac{4}{5}|\hat{f}(x) - \hat{f}(y)|} \le \frac{1}{8}. \tag{3.1}$$

Thus, if x' is an endpoint of E_x, (3.1) yields $\operatorname{dist}(\hat{f}(x'), \hat{f}(\hat{E} \setminus E_x)) \ge 4 \operatorname{diam} \hat{f}(E_x)$. Hence, $\operatorname{dist}(\hat{f}(E_x), \hat{f}(\hat{E} \setminus E_x)) \ge 3 \operatorname{diam} \hat{f}(E_x)$ and the lower bound follows.

Lemma 3.3. *The set $\hat{E}$ is closed and c-uniformly perfect, and $\hat{f} : \hat{E} \to \mathbb{R}^n$ is $\hat{\eta}$-quasisymmetric where c depends only on M and $\hat{\eta}$ depends only on η.*

Proof. Clearly, $E_x \cap E_y = \emptyset$ for $x, y \in E$ with $x \ne y$. To see that $\hat{E}$ is closed, take $y \in \overline{\hat{E}}$. If $y \in \overline{\hat{E}} \setminus E$, then $y \in \overline{E_x}$ for some $x \in E$ and, thus, $y \in \hat{E}$.

Since $\hat{E}$ has no isolated points, we only need to show that $\hat{E}$ is M'-relatively connected for some M' depending on M. Take $x \in \hat{E}$ and $r > 0$. From the unboundedness of $\hat{E}$ and the fact that $\hat{E}$ has no isolated points, we have $\{x\} \subsetneq \overline{B}(x, r) \cap \hat{E} \subsetneq \hat{E}$. If $x \in E$ is not isolated in E, then

$$\emptyset \ne E \cap (\overline{B}(x, r) \setminus B(x, r/M)) \subset \hat{E} \cap (\overline{B}(x, r) \setminus B(x, r/M)).$$

Suppose $x \in E_z$ for some isolated point z in E. If $r > 2M \operatorname{dist}(z, E \setminus \{z\})$, then $\emptyset \ne (E \setminus \{z\}) \cap \overline{B}(z, r/2) \subset \hat{E} \cap \overline{B}(x, r)$. Therefore,

$$\emptyset \ne E \cap (\overline{B}(z, r/2) \setminus B(z, (2M)^{-1} r)) \subset \hat{E} \cap (\overline{B}(x, r) \setminus B(x, (4M)^{-1} r)).$$

If $r \le 2M \operatorname{dist}(z, E \setminus \{z\})$, then $(20M)^{-1} r \le \frac{1}{10} \operatorname{dist}(z, E \setminus \{z\})$ and

$$\emptyset \ne E_z \cap (\overline{B}(x, r) \setminus B(x, (20M)^{-1} r)) \subset \hat{E} \cap (\overline{B}(x, r) \setminus B(x, (20M)^{-1} r)).$$

It remains to show that $\hat{f}$ is quasisymmetric. Let $x, y, z \in \hat{E}$ be three distinct points with $x \in E_{x'}$, $y \in E_{y'}$ and $z \in E_{z'}$ for some $x', y', z' \in E$. If $x' = y' = z'$, then x, y, z are in an interval where $\hat{f}$ is a similarity.

If $x' \ne z'$ and $x' = y'$, then, by Remark 3.2, the prerequisites of Lemma 2.29 in [9] are satisfied for $A = E \setminus \{x'\}$, $A^* = E \cup E_{x'}$ and $H = \hat{f}|A^*$, and $\hat{f}|E \cup E_{x'}$ is η'-quasisymmetric for some η' depending only on η. Hence,

$$\frac{|\hat{f}(x) - \hat{f}(y)|}{|\hat{f}(x) - \hat{f}(z)|} \le C_1 \frac{|\hat{f}(x) - \hat{f}(y)|}{|\hat{f}(x) - \hat{f}(z')|} \le C_1 \eta'\left(\frac{|x - y|}{|x - z'|}\right) \le C_1 \eta'\left(C_2 \frac{|x - y|}{|x - z|}\right)$$

for some $C_1, C_2 > 1$ depending only on η. Similarly for $x' = z' \ne y'$. If x', y', z' are distinct, then by Remark 3.2,

$$\frac{|\hat{f}(x) - \hat{f}(y)|}{|\hat{f}(x) - \hat{f}(z)|} \le C_3 \frac{|\hat{f}(x') - \hat{f}(y')|}{|\hat{f}(x') - \hat{f}(z')|} \le C_3 \eta\left(\frac{|x' - y'|}{|x' - z'|}\right) \le C_3 \eta\left(C_4 \frac{|x - y|}{|x - z|}\right)$$

for some constants $C_3, C_4 > 1$ depending only on η. Thus, $\hat{f}$ is quasisymmetric. $\square$

3.2. **Bridges.** By Lemma 3.1 and Lemma 3.3, we may assume that E is a closed c-uniformly perfect set such that every component of $\mathbb{R} \setminus E$ is a bounded open interval, and $f : E \to \mathbb{R}^n$ is an η-quasisymmetric embedding.

In this section, for each component I of $\mathbb{R} \setminus E$, we construct a path in a higher dimensional space $\mathbb{R}^N$, $N \geq n$, connecting the images of the endpoints of I. The union of these paths along with $f(E)$ gives a homeomorphic extension $F : \mathbb{R} \to \mathbb{R}^N$.

For two points $x, y \in \mathbb{R}^n \subset \mathbb{R}^k$ let $T_k(x, y)$ be the equilateral triangle which contains the line segment $[x, y]$ and lies on the 2-dimensional plane defined by the points x, y and $\mathbf{e}_k$. The *bridge* of x and y in dimension k, denoted by $\mathcal{B}_k(x, y)$, is the closure of $T_k(a, b) \setminus [x, y]$.

Remark 3.4. If $z, a, b \in \mathbb{R}^n$ with $|z - a| \leq |z - b|$, then, for all $x \in \mathcal{B}_k(a, b)$, $|z - x| \geq C^{-1}(|z - a| + |x - a|)$ for some universal $C > 1$.

Remark 3.5. Each bridge $\mathcal{B}_k(x, y)$ is 4-bi-Lipschitz equivalent to a closed interval of $\mathbb{R}$ of length $|x - y|$.

Using Remark 3.4 and triangle inequality, it is easy to verify that the relative distance of two bridges $\mathcal{B}_k(x_1, y_1)$ and $\mathcal{B}_m(x_2, y_2)$, with $k \neq m$, is comparable to the relative distance of the sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$.

Remark 3.6. Let $n, m_1, m_2 \in \mathbb{N}$ with $n < m_1 \leq m_2$ and let $x_1, y_1, x_2, y_2 \in \mathbb{R}^n$. There exists a universal $C_1 > 0$ such that

$$d^*(\mathcal{B}_{m_2}(x_1, y_1), \mathcal{B}_{m_1}(x_2, y_2)) \leq C_1 d^*(\{x_1, y_1\}, \{x_2, y_2\}).$$

On the other hand, there exist universal constants $d_0 > 0$ and $C_2 > 0$ such that $d^*(\{x_1, y_1\}, \{x_2, y_2\}) \geq d_0$ implies that

$$d^*(\{x_1, y_1\}, \{x_2, y_2\}) \leq C_2 d^*(\mathcal{B}_{m_2}(x_1, y_1), \mathcal{B}_{m_1}(x_2, y_2)).$$

For each component I of $\mathbb{R} \setminus E$ we denote by a_I, b_I the endpoints of I with $a_I < b_I$ and by m_I the center of I. We also write $\mathcal{B}_k(I) = \mathcal{B}_k(f(a_I), f(b_I))$ where $k > n$. In general, two bridges $\mathcal{B}_k(I)$ and $\mathcal{B}_k(I')$, with $I \neq I'$, may intersect. Therefore, more dimensions are needed to make sure that such an intersection will never happen. The next lemma allows us to use only a finite number of dimensions for this purpose.

Lemma 3.7. *Let $d > 0$. If $I_1, \dots, I_k$ are mutually disjoint closed intervals in $\mathbb{R}$ with $d^*(I_i, I_j) \leq d$ for all $i, j = 1, \dots, k$, $i \neq j$, then $k \leq 2d + 3$.*

Proof. We may assume that if $i \notin \{1, k\}$, $x \in I_1$, $y \in I_i$ and $z \in I_k$, then $x < y < z$. Furthermore, applying a similarity we may assume that $\operatorname{dist}(I_1, I_k) = 1$.

Since $d^*(I_1, I_k) \leq d$, we have $\operatorname{diam} I_1 \wedge \operatorname{diam} I_k \geq d^{-1}$. Since the intervals $I_2, \dots, I_{k-1}$ are between I_1 and I_k, there exists at least one $j \in \{2, \dots, k-1\}$ such that $\operatorname{diam} I_j \leq \operatorname{dist}(I_1, I_k)/(k-2) = (k-2)^{-1}$. Thus, $\operatorname{dist}(I_1, I_j) \vee \operatorname{dist}(I_k, I_j) \geq \frac{1}{2}(1 - \frac{1}{k-2})$. If $\operatorname{diam} I_j \geq d^{-1}$, then $k \leq d + 2$. Otherwise,

$$d \geq d^*(I_1, I_j) \vee d^*(I_k, I_j) \geq \frac{\operatorname{dist}(I_1, I_j) \vee \operatorname{dist}(I_k, I_j)}{d^{-1} \wedge \operatorname{diam} I_j} \geq \frac{1}{2}(k - 3). \qquad \square$$

Now let $I_1, I_2, \dots$ be an enumeration of the components of $\mathbb{R} \setminus E$. By Remark 3.6 and (2.2), there exists $C_0 > 0$ so that $d^*(\overline{I_i}, \overline{I_j}) \geq C_0$ implies $d^*(\mathcal{B}_m(I_i), \mathcal{B}_m(I_j)) \geq 1$ for all $m > n$. By Lemma 3.7, there exists $n_0 \in \mathbb{N}$, depending only on c and η, such that if distinct $J_1, \dots, J_k \in \{I_1, I_2, \dots\}$ with $d^*(J_i, J_j) < C_0$ for all $i \neq j$, then $k \le n_0$. Set $N = n + n_0 + 1$. Let $\mathcal{B}_{n_{I_1}}(I_1)$ be the bridge with $n_{I_1} = n+1$. Suppose that $\mathcal{B}_{n_{I_1}}(I_1), \dots, \mathcal{B}_{n_{I_m}}(I_m)$ have been defined. Then, there exist at most n_0 indices $i_1, \dots, i_k$ in $\{1, \dots, m\}$ such that $d^*(I_{m+1}, I_{i_j}) < C_0$. Pick $n_{I_{m+1}} \in \{n+1, \dots, N\} \setminus \{n_{I_{i_1}}, \dots, n_{I_{i_k}}\}$ and define the bridge $\mathcal{B}_{n_{I_{m+1}}}(I_{m+1})$. Inductively, for each component I of $\mathbb{R} \setminus E$ we obtain a bridge $\mathcal{B}_{n_I}(I)$ with $n_I \le N$.

Corollary 3.8. *Set $I' = \{f(a_I), f(b_I)\}$ for any component $I = (a_I, b_I)$ of $\mathbb{R} \setminus E$. Then, there exist $C > 1$ depending only on c and η such that, for every two components I, J of $\mathbb{R} \setminus E$ with $I \ne J$,*

$$(C)^{-1} d^*(I', J') \le d^*(\mathcal{B}_{n_I}(I), \mathcal{B}_{n_J}(J)) \le C d^*(I', J')$$

and $C^{-1} \operatorname{dist}(I', J') \le \operatorname{dist}(\mathcal{B}_{n_I}(I), \mathcal{B}_{n_J}(J)) \le C \operatorname{dist}(I', J')$.

3.3. **Reflected sets and functions.** As before, we assume that E is a closed c-uniformly perfect set such that every component of $\mathbb{R} \setminus E$ is a bounded open interval, and $f : E \to \mathbb{R}^N$ is an η-quasisymmetric embedding with $N = n + n_0 + 1$.

Recall from Section 3.2 that, given a component $I = (a_I, b_I)$ of $\mathbb{R} \setminus E$, we denote by m_I the midpoint of I. Moreover, we denote by $m_{\mathcal{B}(I)}$ the point in $\mathcal{B}_{n_I}(I)$ such that $\mathcal{B}_{n_I}(I) = [f(a_I), m_{\mathcal{B}(I)}] \cup [f(b_I), m_{\mathcal{B}(I)}]$. Note that $[f(a_I), m_{\mathcal{B}(I)}] \cap [f(b_I), m_{\mathcal{B}(I)}] = \{m_{\mathcal{B}(I)}\}$.

Let $I = (a_I, b_I)$ be a component of $\mathbb{R} \setminus E$. We define an increasing sequence in E converging to a_I as follows. Set $\delta_0 = \min\{1/2, \eta^{-1}(1/2)\}$. Since E is uniformly perfect, there exists $a_0 \in E$, $a_0 < a_I$ with $|a_0 - a_I| \in [(2c)^{-1}|I|, 2^{-1}|I|]$. Inductively, suppose that a_k has been defined. Since E is uniformly perfect, there exists $a_{k+1} \in E \cap (a_k, a_I)$ such that

$$\frac{\delta_0}{c} \le \frac{|a_{k+1} - a_I|}{|a_k - a_I|} \le \delta_0.$$

Let $a'_0 = m_I$ and for each $k \ge 1$ let $a'_k \in (a_I, m_I)$ with $a'_k = 2a_I - a_k$. Similarly we obtain sequences $\{b_k\}_{k \ge 0} \subset E$ and $\{b'_k\}_{k \ge 0} \subset [m_I, b_I]$ for the point b_I. In the following, two intervals $[a'_{k+1}, a'_k]$ and $[a'_k, a'_{k-1}]$ are called *neighbor intervals*. Similarly, $[a'_1, m_I]$ is a neighbor of $[m_I, b'_1]$, and for each $k \in \mathbb{N}$, $[b'_{k-1}, b'_k]$ is a neighbor of $[b'_k, b'_{k+1}]$.

We define now $f_I : \overline{I} \to \mathcal{B}_{n_I}(I)$. Set $f_I(m_I) = m_{\mathcal{B}(I)}$ and for each $k \ge 1$, define $f_I(a'_k) \in [f(a_I), m_{\mathcal{B}(I)}]$ and $f_I(b'_k) \in [f(b_I), m_{\mathcal{B}(I)}]$ by

$$\frac{|f_I(a'_k) - f(a_I)|}{|f(a_k) - f(a_I)|} = 1 = \frac{|f_I(b'_k) - f(b_I)|}{|f(b_k) - f(b_I)|}.$$

On each interval $[a'_{k+1}, a'_k]$ or $[b'_k, b'_{k+1}]$ we extend f_I linearly. It follows from the choice of δ_0 that f_I is a homeomorphism.

Suppose that $J_1, J_2 \subset I$ are neighbor intervals. Then, there exists constant $C > 1$ depending only on η and c such that

$$C^{-1} \le |J_1|/|J_2| < C \quad \text{and} \quad C^{-1} \le \operatorname{diam} f_I(J_1)/\operatorname{diam} f_I(J_2) < C. \tag{3.2}$$

Thus, by Lemma 2.3, Remark 3.5 and the linearity of f_I on each J_i, the following remark can be easily verified.

Remark 3.9. Suppose that $J_1, J_2, J_3 \subset I$ are consecutive neighbor intervals. Then, there exists η_1 depending only on η and c such that $f_I|(J_1 \cup J_2 \cup J_3)$ is η_1-quasisymmetric.

Note that $f_I|\{a'_k\}_{k\geq 0}$ is η_2-quasisymmetric for some η_2 depending only on η and c. We show in the next lemma that f_I is quasisymmetric.

Lemma 3.10. *Let I be a component of $\mathbb{R}\setminus E$. There exists η' depending only on η and c such that f_I is η'-quasisymmetric.*

Proof. By Remark 3.9, $f_I|[a'_1, b'_1]$ is quasisymmetric. We show that $f_I|[a_I, a'_0]$ is quasisymmetric and similar arguments apply for $f_I|[b'_0, b_I]$. Then, by Lemma 2.3 and Remark 3.5, f_I is η'-quasisymmetric with η' depending only on η and c. Recall that $f_I|\{a'_k\}_{k\geq 0}$ is η_2-quasisymmetric with η_2 depending only on η and c.

To show that $f_I|[a_I, a'_0]$ is quasisymmetric, we apply Lemma 2.3. Let x, y, z be in $[a_I, a'_0]$, with x being between y and z, and $|x-y| \leq |x-z|$. Suppose $x \in [a'_k, a'_{k-1}]$.

Assume first that $y < x < z$. If $z \geq a'_{k-2}$, then $|f_I(x) - f_I(y)| \leq |f_I(a'_{k-1}) - f_I(a_I)| \leq \eta_2(2)|f_I(a'_{k-1}) - f_I(a'_{k-2})| \leq \eta_2(2)|f_I(x) - f_I(z)|$. If $z \leq a'_{k-2}$ and $y \geq a'_{k+1}$, then the quasisymmetry follows from Remark 3.9. If $z \leq a'_{k-2}$ and $y \leq a'_{k+1}$, then $|x-z| \geq |x-y| \geq C^{-1}|a'_{k-1} - a'_k|$, and by Remark 3.9,

$$\begin{aligned}|f_I(x) - f_I(y)| &\leq |f_I(a'_{k-1}) - f_I(a_I)| \leq \eta_2(2)|f_I(a'_k) - f_I(a'_{k-1})|\\ &\leq \eta_2(2)(|f_I(x) - f_I(a'_k)| + |f_I(x) - f_I(a'_{k-1})|)\\ &\leq 2\eta_2(2)\eta_1(C)|f_I(x) - f_I(z)|.\end{aligned}$$

Assume now that $z < x < y$. Then, there exists $m_0 \in \mathbb{N}$ depending only on c and η such that $y \leq a'_{k-m}$ for some $0 \leq m \leq m_0$. If $z \geq a'_{k+1}$, then we obtain quasisymmetry by applying Lemma 2.3 at most m_0 times. If $z \leq a'_{k+1}$, then $|f_I(x) - f_I(y)| \leq |f_I(a'_k) - f_I(a'_{k-m})| \leq \eta_2(m_0 C^{m_0})|f_I(a'_k) - f_I(a'_{k+1})| \leq \eta_2(m_0 C^{m_0})|f_I(x) - f_I(z)|$ where C is as in (3.2). $\square$

4. Proof of main results

We prove Theorem 1.1 in this section. The proof of Theorem 1.2 is given in Section 4.3 and is a minor modification of that of Theorem 1.1.

Let $N = n + n_0 + 1$ be as in Section 3.2. Define $F : \mathbb{R} \to \mathbb{R}^N$ with $F|E = f$ and $F|I = f_I$ whenever I is a component of $\mathbb{R}\setminus E$. We show in Section 4.2 that F satisfies (2.1), and then Lemma 2.2 concludes the proof of Theorem 1.1.

To limit the use of constants we write in the following $u \lesssim v$ (resp. $u \simeq v$) when the ratio u/v is bounded above (resp. bounded above and below) by a positive constant depending at most on η and c.

4.1. **A form of monotonicity.**

For the proof of the quasisymmetry of F we show first that F satisfies the following form of monotonicity.

Lemma 4.1. *Suppose that $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 < x_2 < x_3$. Then,*

$$|F(x_2) - F(x_1)| \vee |F(x_3) - F(x_2)| \lesssim |F(x_3) - F(x_1)|.$$

First we make an observation. Let $x, y \in \mathbb{R}$ with $x < y$ that are not on the same component of $\mathbb{R}\setminus E$. Denote by x', y' the minimum and maximum, respectively, of $E \cap [x, y]$. By Corollary 3.8 and the quasisymmetry of f,

$$|F(x) - F(y)| \simeq |F(x) - F(x')| + |F(x') - F(y')| + |F(y') - F(y)|. \tag{4.1}$$

Proof of Lemma 4.1. Let $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 < x_2 < x_3$. We only show that $|F(x_2)-F(x_1)| \lesssim |F(x_3)-F(x_1)|$; the inequality $|F(x_2)-F(x_1)| \lesssim |F(x_3)-F(x_1)|$ is similar.

If all three of them are in E or in the same component I of $\mathbb{R}\setminus E$, then the claim follows from the quasisymmetry of f and f_I. Therefore, we may assume that at least one of the x_1, x_2, x_3 is in $\mathbb{R}\setminus E$.

Case 1. Suppose that there exists a component I of $\mathbb{R}\setminus E$ that contains exactly two of the x_1, x_2, x_3. Assume, for instance, that $x_1, x_2 \in I$ and $x_3 \notin I$; the case $x_2, x_3 \in I$ is similar. Let x_2' and x_3' be the minimum and maximum, respectively, of $E \cap [x_2, x_3]$. By (4.1) and the quasisymmetry of F on I, $|F(x_3) - F(x_1)| \gtrsim |F(x_2') - F(x_1)| \gtrsim |F(x_2) - F(x_1)|$.

Case 2. Suppose that there is no component of $\mathbb{R}\setminus E$ containing two points from x_1, x_2, x_3. Let x_1' and x_2' be the minimum and maximum, respectively, of $E\cap[x_1, x_2]$ and x_2'', x_3' be the minimum and maximum, respectively, of $E \cap [x_2, x_3]$. Applying (4.1) on x_1, x_3 and quasisymmetry on x_1', x_2'', x_3', $|F(x_3)-F(x_1)| \gtrsim |F(x_2'')-F(x_2')|+|F(x_2')-F(x_1')|+|F(x_1')-F(x_1)|$. Applying quasisymmetry on x_2', x_2, x_2'' and then (4.1) on x_1, x_2, $|F(x_3) - F(x_1)| \gtrsim |F(x_2) - F(x_2')| + |F(x_2') - F(x_1')| + |F(x_1') - F(x_1)| \gtrsim |F(x_2) - F(x_1)|$. $\square$

4.2. **Proof of Theorem 1.1.** Let $x, y, z \in \mathbb{R}$ such that $|x - y| \le |x - z|$. By Lemma 4.1, we may assume that x is between y and z. Without loss of generality we assume that $y < x < z$.

Since $F|E$ is already quasisymmetric, we may assume that at least one of the x, y, z is in $\mathbb{R}\setminus E$. The proof is divided into four cases.

For the first case, we use the following lemma, which can easily be verified.

Lemma 4.2. *Let $I = (a, b)$ be a component of $\mathbb{R}\setminus E$, $x_1 \in I$ and $x_2 \in E$.*

Suppose $x_1 < x_2$. If $|x_2-b| > (4c)^{-1}|x_1-b|$ set $x_1' = b$. If $|x_2-b| \le (4c)^{-1}|x_1-b|$ and $x_1 \le m_I$ set $x_1' = b_0$. If $|x_2 - b| \le (4c)^{-1}|x_1 - b|$ and $x_1 \in [b_{n+1}', b_n']$ set $x_1' = b_{n+1}$. In each case, $|x_2-x_1'| \simeq |x_2-x_1|$ and $|F(x_2)-F(x_1')| \simeq |F(x_2)-F(x_1)|$.

If $x_2 < x_1$ replace b, b_0, b_{n+1} by a, a_0, a_n, respectively, and define x_1' similarly. The claim of the lemma holds in this case as well.

Case 1. Suppose that exactly one of the x, y, z is in $\mathbb{R}\setminus E$.

Case 1.1. Assume that $y \in \mathbb{R}\setminus E$ and $x, z \in E$. Let y' be as in Lemma 4.2 for the pair $x_1 = y$, $x_2 = x$. Then, $|y' - x| \simeq |y - x| \lesssim |x - z|$ and

$$|F(y) - F(x)| \simeq |F(y') - F(x)| \lesssim |F(x) - F(z)|.$$

Case 1.2. Assume that $z \in \mathbb{R}\setminus E$ and $x, y \in E$. We work as in Case 1.1.

Case 1.3. Assume that $x \in \mathbb{R}\setminus E$ and $y, z \in E$. Let x' be the point defined in Lemma 4.2 for the pair $x_1 = x$, $x_2 = z$. Then, $|y-x'| = |y-x|+|x-x'| \lesssim |x-z| \simeq |x'-z|$ and by Lemma 4.1,

$$|F(x) - F(y)| \lesssim |F(x') - F(y)| \lesssim |F(x') - F(z)| \simeq |F(x) - F(z)|.$$

Case 2. Suppose that exactly two of the x, y, z are in the same component of $\mathbb{R}\setminus E$ and the third point is in E.

Case 2.1. Assume that x, y are in a component (a, b) of $\mathbb{R}\setminus E$ and $z \in E$.

If $|x - b| > |b - z|$ set $z' = b$. Note that $|x - z| \simeq |x - z'|$ and, by quasisymmetry of $F|(a, b)$ and Lemma 4.1,

$$|F(x) - F(y)| \lesssim |F(x) - F(z')| \lesssim |F(x) - F(z)|.$$

If $|x-b| \leq |b-z|$, then set $x' = b$. Note that $|x-y| \leq |x'-y| \lesssim |x-z| \simeq |x'-z|$. By Lemma 4.1 and Case 1 for y, x', z,

$$|F(x) - F(y)| \lesssim |F(x') - F(y)| \lesssim |F(x') - F(z)| \lesssim |F(x) - F(z)|.$$

Case 2.2. Assume that x, z are in a component (a, b) of $\mathbb{R} \setminus E$ and $y \in E$. If $|y-a| \leq |x-a|$ set $y' = a$, and if $|y-a| > |x-a|$ set $x' = a$. In each case we work as in Case 2.1.

For the next two cases we use the following lemma.

Lemma 4.3. *Let (a_1, b_1), (a_2, b_2) be two components of $\mathbb{R} \setminus E$ with $b_1 < a_2$ and $x_1 \in (a_1, b_1)$, $x_2 \in (a_2, b_2)$.*

If $|a_1 - b_1| \leq |a_2 - b_2|$ set $x_1' = b_1$. Then, $|x_1 - x_2| \simeq |x_1' - x_2|$ and $|F(x_2) - F(x_1)| \simeq |F(x_2) - F(x_1')|$.

If $|a_1 - b_1| > |a_2 - b_2|$ set $x_2' = a_2$. Then, $|x_1 - x_2| \simeq |x_1 - x_2'|$ and $|F(x_2) - F(x_1)| \simeq |F(x_2') - F(x_1)|$.

Proof. Assume that $|a_1 - b_1| \leq |a_2 - b_2|$; the case $|a_2 - b_2| \leq |a_1 - b_1|$ is similar. By Remark 2.6, $|x_1 - x_2| \simeq |x_1' - x_2|$. Moreover, by Lemma 4.1,

$$\begin{aligned}|F(x_2) - F(x_1')| \lesssim |F(x_2) - F(x_1)| &\leq |F(x_2) - F(x_1')| + |F(x_1') - F(a_1)| \\ &\lesssim |F(x_2) - F(x_1')| + |F(x_1') - F(a_2)| \lesssim |F(x_2) - F(x_1')|. \quad \square\end{aligned}$$

Case 3. Suppose that exactly two of the x, y, z are in $\mathbb{R} \setminus E$ but in different components.

Case 3.1. Assume that $y \in (a_1, b_1)$, $x \in (a_2, b_2)$ and $z \in E$ where for each $i = 1, 2$, (a_i, b_i) is a component of $\mathbb{R} \setminus E$ and $b_1 < a_2$.

If $|a_1 - b_1| \leq |a_2 - b_2|$, then, by Lemma 4.3, setting $y' = b_1$, we have $|x - y'| \simeq |x - y|$, $|F(x) - F(y')| \simeq |F(x) - F(y)|$. Now apply Case 1 for the points y', x, z.

If $|a_2 - b_2| < |a_1 - b_1|$, then, by Lemma 4.3, setting $x' = a_2$, we have $|x' - y| \simeq |x - y|$ and $|F(x') - F(y)| \simeq |F(x) - F(y)|$. Moreover, $|x - z| \leq |x' - z| = |x - z| + |x - x'| \leq |x - z| + |x - y| \leq 2|x - z|$. Thus, $|x - z| \simeq |x' - z|$, and applying Case 1 for the points x', x, z, we have $|F(z) - F(x)| \simeq |F(z) - F(x')|$. Now apply Case 1 for y, x', z.

Case 3.2. Assume that $x \in (a_1, b_1)$, $z \in (a_2, b_2)$ and $y \in E$ where for each $i = 1, 2$, (a_i, b_i) is a component of $\mathbb{R} \setminus E$ and $b_1 < a_2$.

If $|a_1 - b_1| \leq |a_2 - b_2|$, then $|x' - z| \simeq |x - z|$, $|F(x') - F(z)| \simeq |F(x) - F(z)|$, $|y - x| \lesssim |y - x'| \lesssim |x' - z|$, $|F(y) - F(x)| \lesssim |F(y) - F(x')| \lesssim |F(x') - F(z)|$, and we apply Case 1 for y, x', z.

If $|a_2 - b_2| < |a_1 - b_1|$, then set $z' = a_2$ and work as in Case 3.1.

Case 3.3. Assume that $y \in (a_1, b_1)$, $z \in (a_2, b_2)$ and $x \in E$ where for each $i = 1, 2$, (a_i, b_i) is a component of $\mathbb{R} \setminus E$ and $b_1 < a_2$.

If $|a_1 - b_1| \leq |a_2 - b_2|$, then set $y' = a_1$. Since $|x - z| \simeq |x - y| + |x - z| \gtrsim |b_1 - a_2|$ we have that $|x - y'| \simeq |x - y|$. Moreover, by Lemma 4.1, $|F(x) - F(y)| \lesssim |F(x) - F(y')|$, and we apply Case 1 for y', x, z.

If $|a_2 - b_2| < |a_1 - b_1|$, then set $z' = b_2$. As before, $|x - z| \simeq |x - z'|$. Furthermore, $|F(x) - F(z')| \simeq |F(x) - F(a_2)|$ when $|x - a_2| > |a_2 - z|$ and $|F(x) - F(z')| \simeq |F(b_2) - F(a_2)|$ when $|x - a_2| \leq |a_2 - z|$. In either case, $|F(x) - F(z)| \simeq |F(x) - F(z')|$, and we apply Case 1 for the points y, x, z'.

Case 4. Suppose that $y, x, z \in \mathbb{R} \setminus E$. By Lemma 3.10, we may assume that either y or z is not in the same component as x.

Case 4.1. Assume that $y \in (a_1, b_1)$ and $x \in (a_2, b_2)$ where (a_i, b_i) are components of $\mathbb{R} \setminus E$ and $b_1 < a_2$.

If $|b_1 - a_1| \le |b_2 - a_2|$, then set $y' = b_1$ and, by Lemma 4.3, $|x-y| \simeq |x-y'|$ and $|F(x) - F(y)| \simeq |F(x) - F(y')|$. Apply now Case 2 or Case 3 for the points y', x, z.

If $|b_2 - a_2| < |b_1 - a_1|$, then set $x' = a_2$ and, by Lemma 4.3, $|x-y| \simeq |x'-y|$ and $|F(x) - F(y)| \simeq |F(x') - F(y)|$. As in Case 3.1, $|x-z| \simeq |x'-z|$, and applying Case 2 or Case 3 for the points x', x, z we conclude that $|F(x) - F(x')| \lesssim |F(x) - F(z)|$, which implies $|F(x) - F(z)| \simeq |F(x') - F(z)|$. Now apply Case 2 or Case 3 on the points y, x', z.

Case 4.2. Assume that $x \in (a_1, b_1)$, $z \in (a_2, b_2)$ where (a_i, b_i) are components of $\mathbb{R} \setminus E$ and $b_1 < a_2$.

If $|b_2 - a_2| \le |b_1 - a_1|$, then set $z' = a_2$ and work as in Case 4.1.

If $|b_1 - a_1| < |b_2 - a_2|$, then set $x' = b_1$ and, by Lemma 4.1 and Lemma 4.3, $|x'-y| = |x-y| + |x-x'| \lesssim |x-z| \simeq |x'-z|$, $|F(x') - F(z)| \simeq |F(x) - F(z)|$ and $|F(x) - F(y)| \lesssim |F(x') - F(y)|$. Apply now Case 2 or Case 3 for the points y, x', z.

4.3. **Proof of Theorem 1.2.** By Corollary 2.5 we only need to show the sufficiency in Theorem 1.2. The proof is a mild modification of the proof of Theorem 1.1. We only outline the steps.

Let $E \subset \mathbb{R}$ be an M-relatively connected set and let $f : E \to \mathbb{R}$ be a monotone η-quasisymmetric mapping. As before, we may assume that E is a closed set that contains at least two points and f is power quasisymmetric. Moreover, we may assume that f is increasing.

Step 1. First, we reduce the proof to the case that E has no lower or upper bound, as in Section 2. This time, however, in Case 1 and Case 2 we define $\tilde{f}(-a_n) = -a_n$, where $\{a_n\} \subset E$ is as in Section 2. By Lemma 3.1, $\tilde{E}$ is a closed relatively connected set and $\tilde{f} : \tilde{E} \to \mathbb{R}$ is an increasing quasisymmetric embedding.

Step 2. We reduce the proof to the case that E has no isolated points. If E has isolated points, then define $\hat{E}$ and $\hat{f}$ as in Section 3.1. Since $f(E) \subset \mathbb{R}$, then $\hat{f} : E \to \mathbb{R}$ and $\hat{f}$ is increasing. By Lemma 3.3, $\hat{E}$ is a uniformly perfect closed set and $\hat{f}$ is quasisymmetric.

Step 3. Let $I = (a, b)$ be a component of $\mathbb{R} \setminus E$. The bridge $\mathcal{B}(f(a), f(b))$ in this case is simply the interval $[f(a), f(b)]$. The mapping f_I is defined as in Section 3.3. The rest of the proof is similar to that of Theorem 1.2.

5. The quasisymmetric extension property in higher dimensions

This paper was motivated by the following question: given a uniformly perfect Cantor set $\mathcal{C}$ in $\mathbb{R}^n$ and a quasisymmetric mapping $f : \mathcal{C} \to \mathbb{R}^n$ that admits a homeomorphic extension on $\mathbb{R}^n$, is it always possible to extend f quasisymmetrically in $\mathbb{R}^n$? While Theorem 1.2 shows that the answer is yes when $n = 1$, this is not the case when $n \ge 2$. In fact we show a slightly stronger statement.

Theorem 5.1. *For any $n \ge 2$, there exist a compact, countable, relatively connected set $E \subset \mathbb{R}^n$ and a bi-Lipschitz mapping $f : E \to \mathbb{R}^n$ that admits a homeomorphic but no quasisymmetric extension on $\mathbb{R}^n$.*

Before describing the construction we recall a definition. A domain $\Omega \subset \mathbb{R}^n$ is a *C-John domain* if there exists $C \geq 1$ such that for any two points $x, y \in \Omega$, there is a path $\gamma \subset \Omega$ joining x, y such that $\operatorname{dist}(z, \partial\Omega) \leq C^{-1}\min\{|x-z|, |y-z|\}$ for all $z \in \gamma$. In this case, the arc γ is called a *C-John arc*. It is a simple consequence of quasisymmetry that quasisymmetric images of John arcs are John arcs quantitatively.

Now fix an integer $n \geq 2$ and define $h : \mathbb{R}^{n-1} \times \mathbb{R}$ with $h(v,t) = (v, 2-t)$. Set $Q_0 = Q'_0 = [-1,1]^{n-1} \times [-1,1]$ and for each $k \in \mathbb{N}$ set

$$Q_k = [-4^{-k}, 4^{-k}]^{n-1} \times [2^{-k}, 2^{1-k}],$$

$h_k = h|Q_k$ and $Q'_k = h(Q_k)$. For $k = 0$ we set $h_0 = \mathrm{Id}$. Define

$$U = \operatorname{int}(Q_0 \setminus \bigcup_{k\in\mathbb{N}} Q_k)\ ,\ U' = \operatorname{int}(Q'_0 \cup \bigcup_{k\in\mathbb{N}} Q'_k)$$

and $X = \partial U$, $X' = \partial U'$. Note that U is a C-John domain for some $C \geq 1$.

For each integer $m \geq 0$ let $\zeta_m : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity that maps $[-2,2]^n$ onto $[\frac{1}{2}4^{-m}, 4^{-m}] \times [-4^{-m-1}, 4^{-m-1}]^{n-1}$. For each $m, k \geq 0$ let $Q_{m,k}$, $Q'_{m,k}$, U_m, U'_m, X_m and X'_m be the images of Q_k, Q'_k, U, U', X and X', respectively, under ζ_m. Note that each U_m is a C-John domain.

For each $m, k \geq 0$ let $E_{m,k}$ be a finite set on $\partial Q_{m,k} \cap X_m$ such that

$$\operatorname{dist}(x, E_{m,k}) < 8^{-k-m} \text{ for all } x \in \partial Q_{m,k} \cap X_m. \tag{5.1}$$

Let $P_m = \zeta_m(0, \dots, 0, 0)$, $P^*_m = \zeta_m(0, \dots, 0, -1/2)$ and $P = (0, \dots, 0)$. Set

$$E = \{P\} \cup \{P_m, P^*_m\}_{m\geq 0} \cup \bigcup_{m,k\geq 0} E_{m,k}.$$

Clearly, E is compact and countable. Moreover, by choosing the sets $E_{m,k}$ to be relatively connected, we may assume that E is relatively connected.

Define $f : E \to \mathbb{R}^n$ with $f(P) = P$, $f(P^*_m) = P^*_m$, $f(P_m) = \zeta_m(0, \dots, 0, 2)$ and

$$f|E_{m,k} = \zeta_m \circ h_k \circ \zeta_m^{-1}|E_{m,k}.$$

Denote $E'_{m,k} = f(E_{m,k})$ and $E' = f(E)$. It is easy to show that f is bi-Lipschitz and can be extended to a homeomorphism of $\mathbb{R}^n$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be such an extension of f. We briefly describe why F cannot be quasisymmetric; the details are left to the reader.

Assume that F is η-quasisymmetric. Fix $m \in \mathbb{N}$ to be chosen later. Let $x \in U_m$ with $\operatorname{dist}(x, X_m) = \operatorname{dist}(x, E_{m,k}) = 4^{-m}4^{-k}$. By quasisymmetry, (5.1) and the fact that $F|E_{m,k}$ is an isometry, its image $x' = F(x)$ satisfies $c_1 4^{-m}4^{-k} \leq \operatorname{dist}(x', E'_{m,k}) \leq c_2 4^{-m}4^{-k}$ for some $0 < c_1 < c_2$ depending on η. We claim that if m is chosen big enough, $x' \in U'_m$. Indeed, let γ be a C-John arc connecting x and P^*_m in $\mathbb{R}^n \setminus E$. If $x' \in \mathbb{R}^n \setminus \overline{U'_m}$, then there would be a point $z \in F(\gamma) \cap X'_m$. If $z \in \partial Q'_{m,l}$, then $\operatorname{dist}(z, E'_{m,l}) \leq 8^{-m-l} < 2^{-m}\min\{|z-x'|, |z-P^*_m|\}$, which contradicts the quasisymmetry of F if m is sufficiently big.

Now let m be chosen as above. Let $x, y \in U_m$ with

$$\operatorname{dist}(x, X_m) = \operatorname{dist}(x, E_{m,k}) = \operatorname{dist}(y, X_m) = \operatorname{dist}(y, E_{m,k}) = 4^{-m}4^{-k}$$

and with $|x-y| = 4^{-m}2^{-k-1}$ where k is chosen later. Let a, b be the points in $E_{m,k}$ closest to x, y respectively. By quasisymmetry of F, (5.1) and the fact that $F|E_{m,k}$ is an isometry, there exist constants $C_1, C_2 > 0$ depending only on η such that the images x', y' of x, y satisfy $\operatorname{dist}(x', E'_{m,k}), \operatorname{dist}(y', E'_{m,k}) \leq C_1 4^{-m}4^{-k}$ and

$|x'-y'| \geq C_2 4^{-m} 2^{-k}$. Let σ be a C-John arc joining x and y in $\mathbb{R}^n \setminus E$. As before, we can show that σ is contained in U_m and its image σ' is contained in U'_m. Let $z \in \sigma' \cap Q'_{m,k}$ such that $|z-x'| = |z-y'|$. Then, $\min\{|z-x'|, |z-y'|\} \geq \frac{1}{2} C_2 2^{-k} 4^{-m}$ while $\operatorname{dist}(z, E'_{m,k}) \leq \frac{1}{2} 4^{-k} 4^{-m}$ and the John condition for σ' fails if k is sufficiently big. The latter contradicts the quasisymmetry of F.

Remark 5.2. Let $\mathscr{C}$ be the standard ternary Cantor set in $[-\frac{1}{2}, \frac{1}{2}]$. If in the above construction we replace the finite sets $E_{m,k}$ by uniformly perfect Cantor sets $\mathcal{C}_{m,k}$ satisfying (5.1) and the points P^*_m by sets $\mathcal{C}_m = \zeta_m(\mathscr{C} \times \{(0, \ldots, 0, \frac{1}{2})\})$, then we obtain a Cantor set

$$\mathcal{C} = \{P\} \cup \{P_m\}_{m \geq 0} \cup \bigcup_{m \geq 0} \mathcal{C}_m \cup \bigcup_{m,k \geq 0} \mathcal{C}_{m,k},$$

for which the mapping f defined as above is bi-Lipschitz and admits a homeomorphic extension on $\mathbb{R}^n$ but no quasisymmetric extension on $\mathbb{R}^n$.

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Jyväskylä, Finland

Current address: Department of Mathematics, University of Connecticut, 341 Mansfield Rd, Storrs, CT 06269, USA

E-mail address: vyron.vellis@uconn.edu