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# LOCAL HOMOLOGICAL PROPERTIES AND CYCLICITY OF HOMOGENEOUS ANR-COMPACTA

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ABSTRACT. In accordance with the Bing-Borsuk conjecture, we show that if $X$ is an $n$-dimensional homogeneous metric $ANR$-compactum and $x \in X$, then there is a local basis at $x$ consisting of connected open sets $U$ such that the homological properties of $\overline{U}$ and $bd\,\overline{U}$ are similar to the properties of the closed ball $\mathbb{B}^n \subset \mathbb{R}^n$ and its boundary $\mathbb{S}^{n-1}$. We discuss also the following questions raised by Bing-Borsuk [Ann. of Math. (2) **81** (1965), 100–111], where $X$ is a homogeneous $ANR$-compactum with $\dim X = n$:

- Is it true that $X$ is cyclic in dimension $n$?
- Is it true that no non-empty closed subset of $X$, acyclic in dimension $n-1$, separates $X$?

It is shown that both questions simultaneously have positive or negative answers, and a positive solution to each one of them implies a solution to another question of Bing-Borsuk (whether every finite-dimensional homogenous metric $AR$-compactum is a point).

## 1. Introduction

There are few open problems concerning homogeneous compacta; see [2]. The most important one is the well-known Bing-Borsuk conjecture stating that every $n$-dimensional homogeneous metric $ANR$-compactum $X$ is an $n$-manifold. Another one is whether any such $X$ has the following properties: (i) $X$ is cyclic in dimension $n$; (ii) no closed non-empty subset of $X$, acyclic in dimension $n-1$, separates $X$. It is also unknown if there exists a non-trivial finite-dimensional metric homogeneous $AR$-compactum.

In this paper we address the above problems and investigate the homological structure of homogeneous metric $ANR$-compacta. In accordance with the Bing-Borsuk conjecture, we prove that any such compactum has local homological properties similar to the local structure of $\mathbb{R}^n$; see Theorem 1.1. It is also shown that the properties (i) and (ii) from the second of the above questions are equivalent, so each one of them implies that every finite-dimensional homogeneous metric $AR$ is a point.

Reduced Čech homology $H_n(X;G)$ and cohomology groups $H^n(X;G)$ with coefficients from $G$ are considered everywhere below, where $G$ is an abelian group. Suppose $(K,A)$ is a pair of closed subsets of a space $X$ with $A \subset K$. By $i^n_{A,K}$ we

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denote the homomorphism from $H_n(A;G)$ into $H_n(K;G)$ generated by the inclusion $A \hookrightarrow K$. Following [2], we say that $K$ is an *$n$-homology membrane spanned on $A$ for an element* $\gamma \in H_n(A;G)$ provided $\gamma$ is homologous to zero in $K$, but not homologous to zero in any proper closed subset of $K$ containing $A$. It is well known [2, property 5, p. 103] that for every compact metric space $X$ and a closed set $A \subset X$ the existence of a non-trivial element $\gamma \in H_n(A;G)$ with $i^n_{A,X}(\gamma) = 0$ yields the existence of a closed set $K \subset X$ containing $A$ such that $K$ is an $n$-homology membrane for $\gamma$ spanned on $A$. We also say that a space $K$ is a homological *$(n, G)$-bubble* if $H_n(K;G) \neq 0$, but $H_n(B;G) = 0$ for every closed proper subset $B \subset K$.

For any abelian group $G$, Alexandroff [1] introduced the dimension $d_G X$ of a space $X$ as the maximum integer $n$ such that there exist a closed set $F \subset X$ and a non-trivial element $\gamma \in H_{n-1}(F;G)$ with $\gamma$ being $G$-homologous to zero in $X$. According to [1, p. 207] we have the following inequalities for any metric finite-dimensional compactum $X$: $d_G X \leq \dim X$ and $\dim X = d_{\mathbb{Q}_1} X = d_{\mathbb{S}^1} X$, where $G$ is any abelian group, $\mathbb{S}^1$ is the circle group, and $\mathbb{Q}_1$ is the group of rational elements of $\mathbb{S}^1$.

Because the definition of $d_G X$ does not provide any information for the homology groups $H_{k-1}(F;G)$ when $F \subset X$ is closed and $k < d_G X - 1$, we consider the set $\mathcal{H}_{X,G}$ of all integers $k \geq 1$ such that there exist a closed set $F \subset X$ and a non-trivial element $\gamma \in H_{k-1}(F;G)$ with $i^{k-1}_{F,X}(\gamma) = 0$. Obviously, $d_G X = \max \mathcal{H}_{X,G}$.

Using the properties of the sets $\mathcal{H}_{X,G}$, we investigate in Section 2 the local homological properties of metric homogeneous $ANR$-compacta. The main result in that section is Theorem 1.1 below, which is a homological version of [11, Theorem 1.1].

**Theorem 1.1.** *Let $X$ be a finite-dimensional homogeneous metric $ANR$ with $\dim X \geq 2$. Then every point $x \in X$ has a basis $\mathcal{B}_x = \{U_k\}$ of open sets such that for any abelian group $G$ and $n \geq 2$ with $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$ almost all $U_k$ satisfy the following conditions:*

(1) $H_{n-1}(\mathrm{bd}\,\overline{U}_k;G) \neq 0$ *and* $\overline{U}_k$ *is an $(n-1)$-homology membrane spanned on* $\mathrm{bd}\,\overline{U}_k$ *for any non-zero* $\gamma \in H_{n-1}(\mathrm{bd}\,\overline{U}_k;G)$*;*
(2) $H_{n-1}(\overline{U}_k;G) = H_n(\overline{U}_k;G) = 0$ *and* $X \setminus \overline{U}_k$ *is connected;*
(3) $\mathrm{bd}\,\overline{U}_k$ *is a homological $(n-1, G)$-bubble.*

**Corollary 1.2.** *Let $X$ be as in Theorem* 1.1*. Then $X$ has the following property for any abelian group $G$ and $n \geq 2$ with $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$: If a closed subset $K \subset X$ is an $(n-1)$-homology membrane spanned on $B$ for some closed set $B \subset X$ and $\gamma \in H_{n-1}(B;G)$, then $(K \setminus B) \cap \overline{X \setminus K} = \varnothing$.*

In Section 3 we show that the following two statements are equivalent, where $\mathcal{H}(n)$ is the class of all homogeneous metric $ANR$-compacta $X$ with $\dim X = n$:

(1) For all $n \geq 1$ and $X \in \mathcal{H}(n)$ there is a group $G$ such that $H^n(X;G) \neq 0$ (resp., $H_n(X;G) \neq 0$).
(2) If $X \in \mathcal{H}(n)$ with $n \geq 1$ and $F \subset X$ is a closed separator of $X$ with $\dim F = n-1$, then there exists a group $G$ such that $H^{n-1}(F;G) \neq 0$ (resp.,$H_{n-1}(F;G) \neq 0$).

Therefore, we have the following result (see Corollary 3.3):

**Theorem 1.3.** *Suppose for all $n \geq 1$ and all $X \in \mathcal{H}(n)$ the following holds: For every closed separator $F$ of $X$ with $\dim F = n-1$ there exists a group $G$ such that either $H^{n-1}(F;G) \neq 0$ or $H_{n-1}(F;G) \neq 0$. Then there is no homogeneous metric $AR$-compactum $Y$ with $\dim Y < \infty$.*

## 2. Local homological properties of homogeneous $ANR$-compacta

We begin this section with the following analogue of Theorem 8.1 from [2].

**Proposition 2.1.** *Let $X$ be a locally compact and homogeneous separable metrizable $ANR$-space. Suppose there is a pair $F \subset K$ of compact proper subsets of $X$ such that $K$ is contractible in $X$ and $K$ is a homological membrane for some $\gamma \in H_{n-1}(F;G)$. If $(K \setminus F) \cap \overline{X \setminus K} \neq \varnothing$, then there exists a proper compact subset $P \subset X$ contractible in $X$ such that $H_n(P;G) \neq 0$.*

*Proof.* We follow the proof of [6, Lemma 1] (let us note that the proof of Proposition 2.1 can also be obtained following the arguments of [2, Theorem 8.1]). Let $a \in (K \setminus F) \cap \overline{X \setminus K}$. Then $a$ is a boundary point for $K$. Because $K$ is contractible in $X$, there is a homotopy $g : K \times [0,1] \to X$ such that $g(x,0) = x$ and $g(x,1) = c \in X$ for all $x \in K$. Then we can find an open set $U \subset X$ containing $K$ and a homotopy $\overline{g} : \overline{U} \times [0,1] \to X$ extending $g$ and connecting the identity on $\overline{U}$ and the constant map $\overline{U} \to c$ (this can be done since $X$ is an $ANR$). So, $\overline{U}$ is also contractible in $X$. Moreover, we can assume that $\overline{U}$ is compact. Fix a metric $d$ on $X$ generating its topology in the following way: consider $X$ as a subspace of its one-point compactification $\alpha X$ and take $d$ to be the restriction to $X$ of some admissible metric on $\alpha X$. Let $2\epsilon = d(a,F)$ and take an open cover $\omega$ of $U$ such that for any two $\omega$-close maps $f_1, f_2 : K \to U$ (i.e., for all $x \in K$ the points $f_1(x), f_2(x)$ are contained in some element of $\omega$) there is an $\epsilon$-homotopy $\Phi : K \times [0,1] \to U$ between $f_1$ and $f_2$ (i.e., each set $M_\Phi(x) = \{\Phi(x,t) : t \in [0,1]\}$, $x \in K$, is of diameter $< \epsilon$). This can be done because $U$ is an $ANR$. Now, we fix an open set $V \subset X$ containing $K$ with $\overline{V} \subset U$ and let $\delta$ be the Lebesgue number of the open cover $\{\Gamma \cap \overline{V} : \Gamma \in \omega\}$ of $\overline{V}$. According to Effros' theorem [4], there is a positive number $\eta$ such that if $x, y \in X$ are two points with $d(x,y) < \eta$, then $f(x) = y$ for some homeomorphism $h\colon X \to X$, which is $\min\{\delta, d(K, X \setminus V\}$-close to the identity on $X$ (Effros' theorem can be applied because of the special choice of the metric $d$). Since $a$ is a boundary point for $K$, we can choose a point $b \in V \setminus K$ with $d(a,b) < \eta$. Then, there exists a homeomorphism $h_1' : X \to X$ such that $h_1'(a) = b$ and $d(x, h_1'(x)) < \min\{\delta, d(K, X \setminus V\}$, $x \in X$. Let $h_1$ be the restriction $h_1'|K$. Obviously, $h_1 : K \to h_1(K)$ is a homeomorphism with $h_1(K) \subset V$ and $h_1$ is $\delta$-close to the identity on $K$. Then, according to the choice of $\delta$, there is homotopy $h : K \times [0,1] \to U$ such that $h(x,0) = x$, $h(x,1) = h_1(x)$, and $d(x, h(x,t)) < \epsilon$ for all $x \in K$ and $t \in [0,1]$.

Let $K_1 = K \cup h(F \times \mathbb{I})$, $K_2 = h_1(K)$, and $K_0 = K_1 \cap K_2$, where $\mathbb{I} = [0,1]$. Since $2\epsilon = d(a,F)$ and $h$ is an $\epsilon$-small homotopy, $b \in K_2 \setminus K_1$. So, $K_0$ is a proper subset of $K_2$ containing $h_1(F)$. Hence, $D = h_1^{-1}(K_0)$ is a proper subset of $K$ containing $F$, which implies $\gamma_1 = i^{n-1}_{F,D}(\gamma) \neq 0$. Because $h_1$ is a homeomorphism, $(\varphi_1)_* : H_{n-1}(D;G) \to H_{n-1}(K_0;G)$ is an isomorphism, where $\varphi_1 = h_1|D$. Thus, $\hat{\gamma} = (\varphi_1)_*(\gamma_1) \neq 0$.

*Claim* 1. We have $i^{n-1}_{K_0,K_1}(\hat{\gamma}) = 0$ and $i^{n-1}_{K_0,K_2}(\hat{\gamma}) = 0$.

Let $\lambda = i^{n-1}_{F,h(F\times\mathbb{I})}(\gamma)$. Since $h|(F\times\mathbb{I})$ is a homotopy between the identity on $F$ and the map $\varphi_2 = h_1|F$, $\lambda = i^{n-1}_{h_1(F),h(F\times\mathbb{I})}((\varphi_2)_*(\gamma))$. We consider the following commutative diagram:

$$\begin{tikzcd}
\gamma \in H_{n-1}(F;G) \arrow[r, "i^{n-1}_{F,D}"] \arrow[d, "(\varphi_2)_*"'] & H_{n-1}(D;G) \ni \gamma_1 \arrow[d, "(\varphi_1)_*"'] \\
(\varphi_2)_*(\gamma) \in H_{n-1}(h_1(F);G) \arrow[r, "i^{n-1}_{h_1(F),K_0}"] \arrow[d, "i^{n-1}_{h_1(F),h(F\times\mathbb{I})}"'] & H_{n-1}(K_0;G) \ni \hat{\gamma} \arrow[ddl, "i^{n-1}_{K_0,K_1}"'] \\
\lambda \in H_{n-1}(h(F\times\mathbb{I});G) \arrow[d, "i^{n-1}_{h(F\times\mathbb{I}),K_1}"'] & \\
0 \in H_{n-1}(K_1;G) &
\end{tikzcd}$$

Obviously, $i^{n-1}_{F,K_1}(\gamma) = i^{n-1}_{h(F\times\mathbb{I}),K_1}(\lambda) = i^{n-1}_{K_0,K_1}(\hat{\gamma})$. On the other hand, $i^{n-1}_{F,K_1}(\gamma) = i^{n-1}_{K,K_1}(i^{n-1}_{F,K}(\gamma)) = 0$ because $i^{n-1}_{F,K}(\gamma) = 0$. Hence, $i^{n-1}_{K_0,K_1}(\hat{\gamma}) = 0$.

For the second part of the claim, observe that $i^{n-1}_{D,K}(\gamma_1) = i^{n-1}_{F,K}(\gamma) = 0$. Then, the equality $i^{n-1}_{K_0,K_2}(\hat{\gamma}) = 0$ follows from the diagram

$$\begin{tikzcd}
\gamma_1 \in H_{n-1}(D;G) \arrow[r, "i^{n-1}_{D,K}"] \arrow[d, "(\varphi_1)_*"'] & H_{n-1}(K;G) \ni 0 \arrow[d, "(h_1)_*"'] \\
\hat{\gamma} \in H_{n-1}(K_0;G) \arrow[r, "i^{n-1}_{K_0,K_2}"] & H_{n-1}(K_2;G) \ni 0
\end{tikzcd}$$

We are in a position now to complete the proof of Proposition 2.1. Let $P = K_1 \cup K_2$. Since $h(K\times\mathbb{I}) \subset U$, $P \subset U$. Therefore, $P$ is contractible in $X$ (recall that $\overline{U}$ is contractible in $X$). Finally, by Claim 1 and the Phragmen-Brouwer theorem (see [2]), there exists a non-trivial $\alpha \in H_n(P;G)$. $\square$

For simplicity, we say that a closed set $F \subset X$ is *strongly contractible in* $X$ if $F$ is contractible in a closed set $A \subset X$ and $A$ is contractible in $X$.

**Corollary 2.2.** *Let $X$ be a homogeneous compact metrizable ANR-space such that $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$. Then for every closed set $F \subset X$ we have:*

1. $H_n(F;G) = 0$ *provided $F$ is contractible in $X$;*
2. *$F$ separates $X$ provided $H_{n-1}(F;G) \neq 0$ and $F$ is strongly contractible in $X$;*
3. *if $K$ is a homological membrane for some non-trivial element of $H_{n-1}(F;G)$ and $K$ is contractible in $X$, then $(K\setminus F)\cap\overline{X\setminus K} = \varnothing$.*

*Proof.* Since $F$ is contractible in $X$, every $\gamma \in H_n(F;G)$ is homologous to zero in $X$. So, the existence of a non-trivial element of $H_n(F;G)$ would imply $n+1 \in \mathcal{H}_{X,G}$, a contradiction.

To prove the second item, suppose $H_{n-1}(F;G) \neq 0$ and $F$ is strongly contractible in $X$. So, there exists a closed set $A \subset X$ such that $F$ is contractible in $A$ and

$A$ is contractible in $X$. Then, by [2, property 5, p. 103], we can find a closed set $K \subset A$ containing $F$ which is a homological membrane for some non-trivial $\gamma \in H_{n-1}(F;G)$. Because $K$ (as a subset of $A$) is contractible in $X$, the assumption $(K \setminus F) \cap \overline{X \setminus K} \neq \varnothing$ would yield the existence of a proper closed set $P \subset X$ contractible in $X$ with $H_n(P;G) \neq 0$ (see Proposition 2.1). Consequently, there would be a non-trivial $\alpha \in H_n(P;G)$ homologous to zero in $X$. Hence, $n+1 \in \mathcal{H}_{X,G}$, a contradiction. Therefore, $(K \setminus F) \cap \overline{X \setminus K} = \varnothing$. This means that $X \setminus F = (K \setminus F) \cup (X \setminus K)$ with both $K \setminus F$ and $X \setminus K$ being non-empty open disjoint subsets of $X$.

The above arguments provide also the proof of the third item. $\square$

*Proof of Theorem* 1.1. Suppose $X$ satisfies the hypotheses of Theorem 1.1. By [11, Theorem 1.1], every $x \in X$ has a basis $\mathcal{B}_x = \{U_k\}_{k \geq 1}$ of open sets satisfying the following conditions: $\mathrm{bd}U_k = \mathrm{bd}\,\overline{U}_k$; the sets $U_k$, $\mathrm{bd}\,\overline{U}_k$, and $X \setminus \overline{U}_k$ are connected; $H^{\dim X - 1}(A;\mathbb{Z}) = 0$ for all proper closed sets $A \subset \mathrm{bd}\,\overline{U}_k$. We may also suppose that each $\overline{U}_{k+1}$ is contractible in $U_k$ and all $\overline{U}_k$ are strongly contractible in $X$. Let $G$ be an abelian group and let $n \geq 2$ with $n+1 \notin \mathcal{H}_{X,G}$ and $n \in \mathcal{H}_{X,G}$. So, there exist a closed set $B \subset X$ and a non-trivial element $\gamma \in H_{n-1}(B;G)$ with $i^{n-1}_{B,X}(\gamma) = 0$. Then, by [2, property 5, p. 103], there is a closed set $K \subset X$ containing $B$ which is a homological membrane for $\gamma$. We fix a point $\widetilde{x} \in K \setminus B$ and its open in $K$ neighborhood $W$ with $\overline{W} \cap B = \varnothing$. According to [2, property 6, p. 103], $\overline{W}$ is an $(n-1)$-homological membrane for some non-trivial element of $H_{n-1}(\mathrm{bd}_K\overline{W};G)$. We can choose $W$ so small that $\overline{W}$ is contractible in $X$. Then Corollary 2.2 yields $(\overline{W} \setminus \mathrm{bd}_K\overline{W}) \cap \overline{X \setminus \overline{W}} = \varnothing$. So, $\overline{W} \setminus \mathrm{bd}_K\overline{W}$ is open in $X$ and contains $\widetilde{x}$. Hence, there exists $k_0$ such that $U_k \subset \overline{W} \setminus \mathrm{bd}_K\overline{W}$ for all $U_k \in \mathcal{B}_{\widetilde{x}}$ with $k \geq k_0$. Below we consider only the elements $U_k$ with $k \geq k_0$. Applying again [2, property 6, p. 103], we conclude that every $\overline{U}_k$ is a homological membrane for some non-trivial element of $H_{n-1}(\mathrm{bd}U_k;G)$. By Corollary 2.2(1), $H_n(\overline{U}_k;G) = 0$. Suppose $\gamma \in H_{n-1}(\mathrm{bd}U_k;G)$ is non-trivial. Since $X \setminus \overline{U}_k$ is connected, Corollary 2.2(2) implies that $H_{n-1}(\overline{U}_k;G) = 0$. Consequently, $\gamma$ is homologous to zero in $\overline{U}_k$. So, by [2, property 5, p. 103], $\overline{U}_k$ contains a closed set $P$ such that $P$ is a homological membrane for $\gamma$. Then Corollary 2.2(3) implies $(P \setminus \mathrm{bd}U_k) \cap \overline{X \setminus P} = \varnothing$. Hence, $X \setminus \mathrm{bd}U_k$ is the union of the disjoint open sets $P \setminus \mathrm{bd}U_k$ and $X \setminus P$. Because $U_k$ is connected and $U_k \cap P \neq \varnothing$, $U_k \subset P \setminus \mathrm{bd}U_k$. Therefore, $P = \overline{U}_k$. This provides the proof of the first two conditions of Theorem 1.1.

To prove the last item of Theorem 1.1, assume that $H_{n-1}(F;G) \neq 0$ for some closed proper subset $F$ of $\mathrm{bd}U_{k+1}$, where $k \geq k_0$. Because $F$ (as a subset of $\overline{U}_{k+1}$) is strongly contractible in $X$, according to Corollary 2.2(2), $F$ separates $X$. So, $X \setminus F$ is the union of two disjoint non-empty open in $X$ sets $V_1$ and $V_2$ with $\overline{V}_1 \cap \overline{V}_2 \subset F$. Let us show that $F$ separates $\overline{U}_k$. Indeed, otherwise $\overline{U}_k \setminus F$ would be connected. Then $\overline{U}_k \setminus F$ should be contained in one of the sets $V_1, V_2$, say $V_1$. Since $X \setminus \overline{U}_k$ is also connected and $V_2 \neq \varnothing$, $X \setminus \overline{U}_k \subset V_2$. Hence, $\overline{\overline{U}_k \setminus F} \cap \overline{X \setminus \overline{U}_k} \subset F$. On the other hand, because $\overline{U}_k \setminus F$ is dense in $\overline{U}_k$ (recall that $F$ does not contain interior points), $\overline{\overline{U}_k \setminus F} \cap \overline{X \setminus \overline{U}_k} = \mathrm{bd}U_k$. So, $F \supset bdU_k$, a contradiction. Therefore, $F$ separates $\overline{U}_k$.

The proof of Theorem 1.1(3) will be done if we show that $F$ cannot separate $\overline{U}_k$. According to [11, Theorem 1.1], $\overline{U}_k$ is an $(m-1)$-cohomology membrane spanned on

$\mathrm{bd}U_k$ for some non-trivial $\alpha \in H^{m-1}(\mathrm{bd}U_k;\mathbb{Z})$, where $m = \dim X$. This means that $\alpha$ (considered as a map from $\mathrm{bd}U_k$ to the Eilenberg-MacLane complex $K(\mathbb{Z}, m-1)$) is not extendable over $\overline{U}_k$, but it is extendable over any proper closed subset of $\overline{U}_k$. Hence, by [9, Proposition 2.10], the couple $(\overline{U}_k, \mathrm{bd}U_k)$ is a strong $K^m_{\mathbb{Z}}$-manifold (see [9] for the definition of a strong $K^m_{\mathbb{Z}}$-manifold). So, according to [9, Theorem 3.3], $H^{m-1}(F;\mathbb{Z}) \neq 0$ because $F$ separates $\overline{U}_k$ and $F \cap \mathrm{bd}U_k = \varnothing$. Finally, we obtained a contradiction because $H^{m-1}(A;\mathbb{Z}) = 0$ for every proper closed subset $A$ of $\mathrm{bd}U_{k+1}$. Therefore, all $U_k$, $k \geq k_0 + 1$, satisfy conditions (1) – (3) from Theorem 1.1. □

*Proof of Corollary* 1.2. Suppose there exists a point $a \in (K \setminus B) \cap \overline{X \setminus K}$ and take a set $U \in \mathcal{B}_a$ satisfying conditions (1) – (3) from Theorem 1.1 such that $U \cap B = \varnothing$. Then $F_U = \mathrm{bd}_K(U \cap K)$ is non-empty, and it follows from [2, property 6, p. 103] that $\overline{U \cap K}$ is a homology membrane for some non-zero $\alpha \in H_{n-1}(F_U;G)$. Because $F_U \subset \mathrm{bd}U$, by Theorem 1.1(3), $F_U = \mathrm{bd}U$. So, $\overline{U}$ is a homological membrane for $\alpha$; see Theorem 1.1(1). This implies that $i^{n-1}_{\mathrm{bd}U,\overline{U\cap K}}(\alpha) \neq 0$ provided $\overline{U \cap K}$ is a proper subset of $\overline{U}$. Therefore, $\overline{U \cap K} = \overline{U}$, which yields $U \subset K$. The last inclusion contradicts the fact that $a \in \overline{X \setminus K}$. Hence, $(K \setminus B) \cap \overline{X \setminus K} = \varnothing$. □

## 3. Cyclicity of homogeneous *ANR*'s

Let $\mathcal{H}(n)$ be the class of all homogeneous metric $ANR$-compacta $X$ with $\dim X = n$.

**Theorem 3.1.** *The following conditions are equivalent:*

(1) *If $n \geq 1$, then for every space $X \in \mathcal{H}(n)$ there exists a group $G$ with $H^n(X;G) \neq 0$.*
(2) *If $n \geq 1$ and $X \in \mathcal{H}(n)$, then for every closed set $F \subset X$ separating $X$ there exists a group $G$ with $H^{n-1}(F;G) \neq 0$.*
(3) *If $n \geq 1$ and $X \in \mathcal{H}(n)$, then for every $(n-1)$-dimensional closed set $F \subset X$ separating $X$ there exists a group $G$ with $H^{n-1}(F;G) \neq 0$.*

*Proof.* (1) ⇒ (2) Suppose $n \geq 1$ and $X \in \mathcal{H}(n)$. Then $H^n(X;G) \neq 0$ for some group $G$, and by [10, Corollary 1.2], $H^{n-1}(F;G) \neq 0$ for every non-empty closed set $F \subset X$ separating $X$.

(2) ⇒ (3) This implication is trivial.

(3) ⇒ (1) Suppose that condition (3) holds, but there exists $n \geq 1$ and $X \in \mathcal{H}(n)$ such that $H^n(X;G) = 0$ for all groups $G$. Consider the two-dimensional sphere $\mathbb{S}^2$ and a circle $\mathbb{S}^1$ separating $\mathbb{S}^2$. Then $X \times \mathbb{S}^2 \in \mathcal{H}(n+2)$ and $X \times \mathbb{S}^1$ is a closed separator of $X \times \mathbb{S}^2$ of dimension $n+1$. So, there is a group $G'$ such that $H^{n+1}(X \times \mathbb{S}^1;G') \neq 0$. On the other hand, according to the Künneth formula, we have the exact sequence

$$\sum_{i+j=n+1} H^i(X) \otimes H^j(\mathbb{S}^1) \to H^{n+1}(X \times \mathbb{S}^1) \to \sum_{i+j=n+2} H^i(X) * H^j(\mathbb{S}^1),$$

where the coefficient group $G'$ is suppressed. Because $\dim X = n$ and $\dim \mathbb{S}^1 = 1$, $H^{n+i}(X;G') = 0$ and $H^{1+i}(\mathbb{S}^1;G') = 0$ for all $i \geq 1$. Moreover, $H^n(X;G') = 0$. So,

$$\sum_{i+j=n+1} H^i(X;G') \otimes H^j(\mathbb{S}^1;G') = \sum_{i+j=n+2} H^i(X;G') * H^j(\mathbb{S}^1;G') = 0.$$

Hence, $H^{n+1}(X \times \mathbb{S}^1;G') = 0$, a contradiction. □

A homological version of Theorem 3.1 also holds.

**Theorem 3.2.** *The following conditions are equivalent:*

(1) *If $n \geq 1$, then for every space $X \in \mathcal{H}(n)$ there exists a group $G$ with $H_n(X;G) \neq 0$.*
(2) *If $n \geq 1$ and $X \in \mathcal{H}(n)$, then for every closed set $F \subset X$ separating $X$ there exists a group $G$ with $H_{n-1}(F;G) \neq 0$.*
(3) *If $n \geq 1$ and $X \in \mathcal{H}(n)$, then for every $(n-1)$-dimensional closed set $F \subset X$ separating $X$ there exists a group $G$ with $H_{n-1}(F;G) \neq 0$.*

*Proof.* Everywhere below, $\widehat{H}_*$ denotes the exact homology (see [5], [7]), which for locally compact metric spaces is equivalent to Steenrod's homology [8]. For every compact metric space $X$ and every $k$ there exists a surjective homomorphism $T^k_X : \widehat{H}_k(X;G) \to H_k(X;G)$. According to [7, Theorem 4], $T^k_X$ is an isomorphism in each of the following cases: $G$ is a vector space over a field, both $\widehat{H}_k(X;G)$ and $G$ are countable modules, $\dim X = k$, $H^{k+1}(X;\mathbb{Z})$ is finitely generated.

(1) $\Rightarrow$ (2) Suppose $n \geq 1$ and $X \in \mathcal{H}(n)$. Then $H_n(X;G) \neq 0$ for some group $G$. By [7, Theorem 3], we have the exact sequence

$$(*) \qquad \operatorname{Ext}(H^{n+1}(X;\mathbb{Z}),G) \to \widehat{H}_n(X;G) \to \operatorname{Hom}(H^n(X;\mathbb{Z}),G) \to 0.$$

Since $\dim X = n$, $H^{n+1}(X;\mathbb{Z}) = 0$. Moreover $\widehat{H}_n(X;G)$ is non-trivial because so is $H_n(X;G)$ and $T^n_X$ is a surjective homomorphism. Hence, $H^n(X;\mathbb{Z}) \neq 0$ and there exists a non-trivial homomorphism $\varphi\colon H^n(X;\mathbb{Z}) \to G$. Now, let $F \subset X$ be a closed separator of $X$ and $X \setminus F = X_1 \cup X_2$, where $X_1, X_2 \subset X$ are closed proper subsets with $X_1 \cap X_2 = F$. Since $H^n(P;\mathbb{Z}) = 0$ for every closed proper subset $P \subset X$ (see [10]), $H^n(F;\mathbb{Z}) = H^n(X_1;\mathbb{Z}) = H^n(X_2;\mathbb{Z}) = 0$. Then it follows from the Mayer-Vietoris sequence

$$H^{n-1}(F;\mathbb{Z}) \xrightarrow{\ \partial\ } H^n(X;\mathbb{Z}) \xrightarrow{\ \psi\ } H^n(X_1;\mathbb{Z}) \oplus H^n(X_1;\mathbb{Z})$$

that $H^{n-1}(F;\mathbb{Z}) \neq 0$ and $\partial$ is a surjective homomorphism. Consequently, $\varphi \circ \partial : H^{n-1}(F;\mathbb{Z}) \to G$ is also a non-trivial surjective homomorphism. Hence, $\operatorname{Hom}(H^{n-1}(F;\mathbb{Z}),G) \neq 0$, and the exact sequence

$$0 \to \operatorname{Ext}(H^n(F;\mathbb{Z}),G) \to \widehat{H}_{n-1}(F;G) \to \operatorname{Hom}(H^{n-1}(F;\mathbb{Z}),G) \to 0$$

yields $\widehat{H}_{n-1}(F;G) \neq 0$. Finally, since $H^n(F;\mathbb{Z}) = 0$, $\widehat{H}_{n-1}(F;G)$ is isomorphic to $H_{n-1}(F;G)$.

(2) $\Rightarrow$ (3) This implication is obvious.

(3) $\Rightarrow$ (1) As in the proof of Theorem 3.1, (3) $\Rightarrow$ (1), suppose there exists $n \geq 1$ and $X \in \mathcal{H}(n)$ such that $H_n(X;G) = 0$ for all groups $G$. Since $\widehat{H}_n(X;G)$ is isomorphic to $H_n(X;G)$ and $H^{n+1}(X;\mathbb{Z}) = 0$ (recall that $\dim X = n$), it follows from the exact sequence $(*)$ that $\operatorname{Hom}(H^n(X;\mathbb{Z}),G) = 0$ for all groups $G$. This implies that $H^n(X;\mathbb{Z}) = 0$. As above, the product $X \times \mathbb{S}^1$ is a closed separator of $X \times \mathbb{S}^2$, and according to our assumption, $H_{n+1}(X \times \mathbb{S}^1;G') \neq 0$ for some group $G'$. Because $\dim X \times \mathbb{S}^1 = n+1$, $H_{n+1}(X \times \mathbb{S}^1;G') \cong \widehat{H}_{n+1}(X \times \mathbb{S}^1;G')$ and $H^{n+2}(\dim X \times \mathbb{S}^1,\mathbb{Z}) = 0$. Therefore, the exact sequence

$$\operatorname{Ext}(H^{n+2}(X \times \mathbb{S}^1),G') \to \widehat{H}_{n+1}(X \times \mathbb{S}^1;G') \to \operatorname{Hom}(H^{n+1}(X \times \mathbb{S}^1),G'),$$

where the coefficient groups $\mathbb{Z}$ in $H^{n+2}(X \times \mathbb{S}^1)$ and $H^{n+1}(X \times \mathbb{S}^1)$ are suppressed, yields that $H^{n+1}(X \times \mathbb{S}^1;\mathbb{Z}) \neq 0$. On the other hand, the Künneth formula from the proof of Theorem 3.1 (with $\mathbb{Z}$ being the coefficient group in all cohomology groups) implies $H^{n+1}(X \times \mathbb{S}^1; \mathbb{Z}) = 0$, a contradiction. □

**Corollary 3.3.** *Suppose for all $n \geq 1$ and all $X \in \mathcal{H}(n)$ the following holds: For every closed separator $F$ of $X$ with $\dim F = n-1$ there exists a group $G$ such that either $H^{n-1}(F; G) \neq 0$ or $H_{n-1}(F; G) \neq 0$. Then there is no homogeneous metric $AR$-compactum $Y$ with $\dim Y < \infty$.*

If $\mathcal{H}(G, n)$ denotes the class of all homogeneous metric $ANR$-compacta $X$ with $\dim_G X = n$, the arguments from Theorem 3.1 provide the following result:

**Proposition 3.4.** *The following conditions are equivalent:*

(1) *$H^n(X; G) \neq 0$ for all $X \in \mathcal{H}(G, n)$ and all $n \geq 1$.*
(2) *If $X \in \mathcal{H}(G, n)$ and $n \geq 1$, then $H^{n-1}(F; G) \neq 0$ for every closed set $F \subset X$ separating $X$.*
(3) *If $X \in \mathcal{H}(G, n)$ and $n \geq 1$, then $H^{n-1}(F; G) \neq 0$ for every closed set $F \subset X$ separating $X$ with $\dim_G F = n-1$.*

The corresponding homological analogue of Proposition 3.4 also holds for some groups $G$.

**Proposition 3.5.** *The following conditions are equivalent, where $G$ is either a field or a torsion free group:*

(1) *$H_n(X; G) \neq 0$ for all $X \in \mathcal{H}(n)$ and all $n \geq 1$.*
(2) *If $X \in \mathcal{H}(n)$, $n \geq 1$, and $F \subset X$ is a closed set separating $X$, then $H_{n-1}(F; G) \neq 0$.*
(3) *If $X \in \mathcal{H}(n)$, $n \geq 1$, and $F \subset X$ is a closed set separating $X$ with $\dim F = n-1$, then $H_{n-1}(F; G) \neq 0$.*

*Proof.* All implications except (3) ⇒ (1) follow from the proof of Theorem 3.2. To prove (3) ⇒ (1), we suppose there exists a space $X \in \mathcal{H}(n)$ with $H_n(X; G) = 0$. Considering the $(n+1)$-dimensional separator $X \times \mathbb{S}^1$ of $X \times \mathbb{S}^2$, we conclude that $H_{n+1}(X \times \mathbb{S}^1; G) \neq 0$. Because $X$ and $X \times \mathbb{S}^1$ are $ANR$'s, their Čech homology groups are isomorphic to the singular homology groups. Thus, we can apply the Künneth formula

$$\sum_{i+j=n+1} H_i(X) \otimes H_j(\mathbb{S}^1) \to H_{n+1}(X \times \mathbb{S}^1) \to \sum_{i+j=n} H_i(X) * H_j(\mathbb{S}^1),$$

where $G$ is the coefficient group. Since $H_n(X; G) = H_{n+1}(X; G) = 0$ and $H_j(\mathbb{S}^1; G) = 0$ for all $j > 1$, $\sum_{i+j=n+1} H_i(X; G) \otimes H_j(\mathbb{S}^1; G) = 0$. If $G$ is a torsion free group, then the group $\sum_{i+j=n} H_i(X; G) * H_j(\mathbb{S}^1; G)$ is also trivial because $H_1(\mathbb{S}^1; G) = G$ yields $H_{n-1}(X; G) * H_1(\mathbb{S}^1; G) = 0$. Therefore, $H_{n+1}(X \times \mathbb{S}^1; G) = 0$, a contradiction.

When $G$ is a field, the group $H_{n+1}(X \times \mathbb{S}^1; G)$ is isomorphic to the trivial group $\sum_{i+j=n+1} H_i(X; G) \otimes H_j(\mathbb{S}^1; G)$. So, again we have a contradiction. □

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