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ALTERNATING LINKS AND LEFT-ORDERABILITY

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ABSTRACT. Let $L \subset S^3$ denote an alternating link and $\Sigma(L)$ its branched double-cover. We give a short proof of the fact that the fundamental group of $\Sigma(L)$ admits a left-ordering iff L is an unlink. This result is originally due to Boyer-Gordon-Watson.

1. A group presentation

Consider a link $L \subset S^3$ presented by a connected planar diagram. Color its regions black and white in checkerboard fashion, and assign each crossing a sign as displayed in Figure 1. From this coloring we obtain the *white graph* $W = (V, E)$. This is the planar graph with one vertex for each white region, one signed edge for each crossing where two white regions touch, and one arbitrary distinguished vertex r (the *root*).

We form a group Γ as follows. It has one generator x_v and one relation $r_v = 1$ for each $v \in V$, as well as one additional relation $x_r = 1$ for the root. To describe the relation r_v, consider a small loop γ_v centered at v and oriented counter-clockwise. Starting at an arbitrary point along γ_v, the loop meets edges $(v, w_1), \ldots, (v, w_k)$ with respective signs $\epsilon_1, \ldots, \epsilon_k$ in order; then $r_v = \prod_{i=1}^{k} (x_{w_i}^{-1} x_v)^{\epsilon_i}$.

Let $\Sigma(L)$ denote the double-cover of S^3 branched along L.

Proposition 1.1. *The fundamental group of $\Sigma(L)$ is isomorphic to Γ.*

Proposition 1.1 is established in [5, §3.1], in which the presentation for Γ derives from a specific Heegaard diagram of the branched double-cover $\Sigma(L)$. We refer the reader there for a worked example, as well as to [5, §3.2] for another derivation of the relevant Heegaard diagram. Dylan Thurston points out that the standard derivation of the Wirtinger presentation of a knot group suggests an alternate route to Proposition 1.1.

2. Non-left-orderability

In this section we use Proposition 1.1 to establish the main result. Recall that a *left-ordering* of a group is a total ordering of its elements that is invariant under left-multiplication in the group.

Theorem 2.1 (Boyer-Gordon-Watson [1])**.** *If L is an alternating link, then $\pi_1(\Sigma(L))$ admits a left-ordering iff L is an unlink.*

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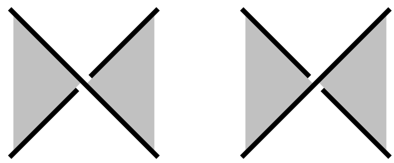

$+1$ -1

FIGURE 1. Crossings and signs.

Proof. First, suppose that $L = L_1 \cup L_2$ is a split link. In this case, L is a connect-sum of L_1, L_2, and the two-component unlink, so $\Sigma(L) \cong \Sigma(L_1)\#\Sigma(L_2)\#(S^1 \times S^2)$ and $\pi_1(\Sigma(L))$ decomposes as the free product $\pi_1(\Sigma(L_1)) * \pi_1(\Sigma(L_2)) * \mathbb{Z}$. Furthermore, a free product admits a left-ordering iff each of its factors do [12]. Therefore, to prove Theorem 2.1, it suffices to restrict attention to the case of a non-split alternating link L. With this assumption in place, Theorem 2.1 follows once we establish that $\pi_1(\Sigma(L))$ admits a left-ordering iff L is the unknot.

Present L by a connected, alternating diagram; color it, distinguish a root r, and let W denote the resulting white graph. It follows that every edge gets the same sign ϵ. Mirroring L if necessary (which leaves π_1 unchanged), we may assume that $\epsilon = 1$. Now suppose that $\Gamma \cong \pi_1(\Sigma(L))$ possesses a left-ordering $<$. Choose a vertex v for which $x_w \leq x_v$ for all $w \in V$. If $x_v = x_w$ for *all* $w \in V$, then from the relation $x_r = 1$ it follows that $1 = \Gamma \cong \pi_1(\Sigma(L))$, but then $1 = |H_1(\Sigma(L))| = \det(L)$, and since L is alternating, it follows that $L = U$.

Thus, we assume henceforth that $L \neq U$ and seek a contradiction. It follows that there exists some $w \in V$ for which $x_w < x_v$; from the connectivity of W, we may assume that $(v, w) \in E$. It follows that $1 < x_w^{-1} x_v$, while $1 \leq x_{w_i}^{-1} x_v$ for every other edge $(v, w_i) \in E$. Therefore, the product of all these terms in any order is greater than 1. In particular, $1 < \prod_{i=1}^{k} (x_{w_i}^{-1} x_v) = r_v = 1$, a contradiction. ☐

3. DISCUSSION

It remains an outstanding problem to relate $\pi_1(Y)$ to the Heegaard Floer homology of a 3-manifold Y. As of this writing, it remains a possibility that a rational homology sphere Y is an L-space iff $\pi_1(Y) \neq 1$ does not admit a left-ordering. Theorem 2.1 supports this conjecture, since $\Sigma(L)$ is a rational homology sphere L-space for a non-split alternating link L [10, Prop. 3.3]. Additional examples appear in [1–4, 11].

In this spirit, Peter Ozsváth raises an interesting question. Let (Y_0, Y_1, Y_2) denote a surgery triple of rational homology spheres. That is, there exists a manifold M with torus boundary and a triple of slopes $(\gamma_0, \gamma_1, \gamma_2)$ in ∂M such that Y_i results from filling M along slope γ_i and $\gamma_i \cdot \gamma_{i+1} = +1$, for all i (mod 3). Cyclically permuting the indices if necessary, assume that $|H_1(Y_0)| = |H_1(Y_1)| + |H_1(Y_2)|$.

Question 3.1. If $\pi_1(Y_0)$ admits a left-ordering, does it follow that one of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ must as well?

Note that if Y_1 and Y_2 are L-spaces, then so is Y_0 according to the surgery exact triangle in $\widehat{HF}$. This is the motivation behind Question 3.1. An affirmative answer would imply that Theorem 2.1 extends to quasi-alternating links.

Updates. Ito has applied the idea in this paper to a different presentation for $\pi_1(\Sigma(L))$ to recover yet another proof of Theorem 2.1 [7]. Levine and Lewallen proved that the fundamental group of any non-trivial *strong* L-space is not left-orderable [8, Theorem 1]. Their result generalizes Theorem 2.1 in the sense that $\Sigma(L)$ is a strong L-space whenever L is a non-split alternating link [5, Corollary 3.5], although no examples of strong L-spaces are known besides these [6, Question 1.2]. Li and Watson applied the presentation and technique used here in their study of genus one open books [9].

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