PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 6, June 2018, Pages 2283–2298
http://dx.doi.org/10.1090/proc/13710
Article electronically published on March 9, 2018

ON CONJECTURES OF ANDREWS AND CURTIS

S. V. IVANOV

(Communicated by Pham Huu Tiep)

Abstract. It is shown that the original Andrews–Curtis conjecture on balanced presentations of the trivial group is equivalent to its "cyclic" version in which, in place of arbitrary conjugations, one can use only cyclic permutations. This, in particular, proves a satellite conjecture of Andrews and Curtis [Amer. Math. Monthly **73** (1966), 21–28]. We also consider a more restrictive "cancellative" version of the cyclic Andrews–Curtis conjecture with and without stabilizations and show that the restriction does not change the Andrews–Curtis conjecture when stabilizations are allowed. On the other hand, the restriction makes the conjecture false when stabilizations are not allowed.

1. Introduction

In 1965, Andrews and Curtis [1] put forward a conjecture on balanced presentations of the trivial group and indicated some interesting topological consequences of their conjecture related to the 4-dimensional and 3-dimensional Poincaré conjectures. Since then both the 4-dimensional and 3-dimensional Poincaré conjectures have been established; however, the Andrews–Curtis conjecture remains unsettled and has become one of the most notorious hypotheses in group theory and low-dimensional topology. In this paper, we show that the Andrews–Curtis conjecture is equivalent to its more restrictive "cyclic" version in which, in place of arbitrary conjugations, one can use only cyclic permutations. This, in particular, proves a satellite conjecture of Andrews and Curtis [2, Conjecture 3] made in 1966.

Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be an alphabet, $\mathcal{A}^{-1} := \{a_1^{-1}, \dots, a_m^{-1}\}$, where a_i^{-1} is the inverse of a letter $a_i \in \mathcal{A}$, $\mathcal{A}^{\pm 1} := \mathcal{A} \cup \mathcal{A}^{-1}$, and $\mathcal{F}(\mathcal{A})$ denotes the free group over $\mathcal{A}$ whose nontrivial elements are considered as reduced words over $\mathcal{A}^{\pm 1}$.

Let $\mathcal{W} = (W_1, \dots, W_n)$ be an n-tuple of elements of $\mathcal{F}(\mathcal{A})$. Recall that *Nielsen operations* over $\mathcal{W}$ have two types and are defined as follows.

(T1) For some i, W_i is replaced with its inverse W_i^{-1}.

(T2) For some pair of distinct indices i and j, W_i is replaced with W, where $W = W_i W_j$ in $\mathcal{F}(\mathcal{A})$.

Consider operations of a third type so that

(T3) For some i, W_i is replaced with a word W such that W_i and W are conjugate in $\mathcal{F}(\mathcal{A})$, i.e., $W = S W_i S^{-1}$ in $\mathcal{F}(\mathcal{A})$ for some $S \in \mathcal{F}(\mathcal{A})$.

Similarly to [1], [2], operations (T1)–(T3) are called extended Nielsen operations, or briefly *EN-operations*.

Received by the editors August 30, 2015 and, in revised form, June 22, 2016.
2010 *Mathematics Subject Classification.* Primary 20F05, 20F06, 57M20.
The author was supported in part by the National Science Foundation, grant DMS 09-01782.

The Andrews–Curtis conjecture [1], [2], abbreviated as the *AC-conjecture* (see also [4], [6], [12], [18]) states that, *for every balanced group presentation*

$$\mathcal{P} = \langle\, a_1, \dots, a_m \,\|\, R_1, \dots, R_m \,\rangle \tag{1.1}$$

that defines the trivial group, the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ of defining words $R_1, \dots, R_m$ can be brought to the letter tuple $(a_1, \dots, a_m)$ by a finite sequence of operations (T1)–(T3).

Extending the terminology by dropping the quantifier, we will say that the AC-conjecture holds for a balanced group presentation (1.1) if the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ can be brought to the letter tuple $(a_1, \dots, a_m)$ by a finite sequence of operations (T1)–(T3).

Consider a different, "cyclic", version of operation of type (T3) so that

(T3C) For some i, W_i is replaced with a cyclic permutation $\bar W_i$ of W_i.

A satellite hypothesis, made by Andrews and Curtis [2, Conjecture 3] regarding their main conjecture, claims for $m = 2$ that if a pair $\mathcal{R} = (R_1, R_2)$ of words can be transformed into (a_1, a_2) by a finite sequence of operations (T1)–(T3), then this can also be done by a finite sequence of operations (T1), (T2), (T3C). More informally, one could say that arbitrary conjugations in the AC-conjecture could be replaced with cyclic permutations. In this paper, we confirm this hypothesis by proving a more general result for all $m \ge 2$ that is the equivalence of the AC-conjecture to what we call a cyclic version of the AC-conjecture.

Let $\mathcal{W} = (W_1, \dots, W_n)$ be a tuple of words over $\mathcal{A}^{\pm 1}$ such that every W_i in $\mathcal{W}$ is either cyclically reduced, i.e., every cyclic permutation of W_i is reduced, or empty. We call such a tuple $\mathcal{W}$ *cyclically reduced*. Consider the following operations over cyclically reduced tuples.

(CT1) For some i, W_i is replaced with W_i^{-1}.
(CT2) For some pair of distinct indices i and j, W_i is replaced with W, where W is a cyclically reduced or empty word obtained from the product W_iW_j by making cancellations and cyclic cancellations.
(CT3) For some i, W_i is replaced with a cyclic permutation $\bar W_i$ of W_i.

Such redefined operations of type (CT1)–(CT3) are called *cyclically extended Nielsen operations*, or, briefly, *CEN-operations*. Thus, in place of arbitrary conjugations, we can use only cyclic permutations and we deal with cyclically reduced tuples only.

Now the cyclic version of the Andrews–Curtis conjecture, abbreviated as *CAC-conjecture*, claims that *for every presentation* (1.1) *such that* (1.1) *defines the trivial group and $\mathcal{R} = (R_1, \dots, R_m)$ is cyclically reduced, the m-tuple $\mathcal{R}$ can be brought to the letter tuple $(a_1, \dots, a_m)$ by a finite sequence of operations* (CT1)–(CT3).

As above, we will say that the CAC-conjecture holds for a balanced group presentation (1.1) if the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ is cyclically reduced and can be brought to the letter tuple $(a_1, \dots, a_m)$ by a finite sequence of operations (CT1)–(CT3).

The main technical result of this paper is the following.

Theorem 1.1. *Suppose that a balanced presentation* (1.1) *defines the trivial group, the m-tuple $\mathcal{R} = (R_1, \dots, R_m)$ is cyclically reduced and the original Andrews–Curtis conjecture holds true for all balanced presentations in ranks $< m$. If the original*

Andrews–Curtis conjecture holds true for the balanced presentation (1.1)*, then the cyclic version of the Andrews–Curtis conjecture also holds for* (1.1)*.*

As easy consequences of Theorem 1.1 we will obtain the following three corollaries.

Corollary 1.2. *Let $r \ge 2$ be an integer. The original Andrews–Curtis conjecture is true for all balanced presentations in ranks $\le r$ if and only if the cyclic version of the Andrews–Curtis conjecture is true for all balanced presentations in ranks $\le r$.*

Corollary 1.3. *The original Andrews–Curtis conjecture holds true if and only if the cyclic version of the Andrews–Curtis conjecture holds true.*

Corollary 1.4. *A satellite hypothesis of Andrews and Curtis* [2, Conjecture 3] *holds true. This hypothesis claims for $m = 2$ that if $\mathcal{R} = (R_1, R_2)$ can be brought to (a_1, a_2) by a finite sequence of operations* (T1)–(T3)*, then this result can also be achieved by operations* (T1), (T2), (T3C)*.*

More generally, if $r \ge 2$ is an integer and every m-tuple $\mathcal{R} = (R_1, \dots, R_m)$, where $2 \le m \le r$, that defines the trivial group by (1.1) *can be transformed to $(a_1, \dots, a_m)$ by a finite sequence of operations* (T1)–(T3)*, then such transformation can also be done by operations* (T1), (T2), (T3C)*.*

Recall that there is another, more general, version of the AC-conjecture, called the *AC-conjecture with stabilizations* (see [4], [6], [18]), in which a fourth type of operations, called *stabilizations*, is allowed.

(T4) Add (or remove) a new letter b, $b \notin \mathcal{A}^{\pm 1}$, both to the alphabet $\mathcal{A}$ and to the tuple $\mathcal{R}$ of defining words (when removing, b and b^{-1} may not occur in all other words of $\mathcal{R}$).

The AC-conjecture with stabilizations has a nice geometric interpretation due to Wright [27]: The AC-conjecture with stabilizations is equivalent to the conjecture that every finite contractible 2-complex can be 3-deformed into a point. Putting this result together with Perelman's proof [20], [21], [22] of the 3-dimensional Poincaré conjecture, one can see that the AC-conjecture with stabilizations is equivalent to the claim that every finite contractible 2-complex can be 3-deformed into a spine of a closed 3-manifold. Further generalizations of the AC-conjecture (with or without stabilizations), motivated by a problem of Magnus's on balanced presentations of the trivial group, can be found in [10]. Generalization of the original AC-conjecture in quite a different direction was investigated (and proved!) by Myasnikov [16] for solvable groups and by Borovik, Lubotzky and Myasnikov [3] for finite groups. In particular, it follows from results of [16], [3] that the natural idea to use a solvable or finite quotient of $F(\mathcal{A})$ to construct a counterexample to the AC-conjecture will necessarily fail.

It was earlier shown by the author [9] that the AC-conjecture with stabilizations holds for a presentation (1.1) if and only if the CAC-conjecture with stabilizations holds for (1.1) (the definition of the CAC-conjecture with stabilizations is analogous and uses operations (CT4) over cyclically reduced tuples in place of (T4)). The availability of stabilizations provides substantial aid in simulating required conjugations by compositions of cyclic permutations with operations (CT1)–(CT2). Such a simulation does not seem to be possible when stabilizations are not available. In particular, we are not able to prove the equivalence of the AC-conjecture to its cyclic version for a given presentation (which would be similar to the result of

[9] for the AC-conjecture with stabilizations). We are only able to prove a much weaker result, namely, that the absence of a counterexample to the AC-conjecture in rank $< m$ implies the absence of a counterexample to the CAC-conjecture in rank $\le m$, as stated in Theorem 1.1. As another illustration of subtlety of operations (CT1)–(CT3), we remark that operation (CT2) is not invertible in general and it is not clear whether operation (CT2) could be reversed with a composition of (CT1)–(CT3). Note that (T1)–(T3) are invertible.

In Section 2, we prove Theorem 1.1 and Corollaries 1.2–1.4. In Section 3, we discuss one more satellite conjecture made by Andrews and Curtis [2, Conjecture 4]. This conjecture turns out to be false, and we provide a counterexample based on results of Myasnikov [17].

We remark that in 1968, Rapaport [23], [24] gave a counterexample to [2, Conjecture 2]. Thus all satellite Conjectures 2–4 of [2] are now resolved with only [2, Conjecture 3] being true. Recall that [2, Conjecture 1] is the original Andrews–Curtis conjecture in rank $m = 2$.

In Section 4 we look at a more restrictive version of the CAC-conjecture with and without stabilizations, abbreviated as the CCAC-conjecture, in which the analogue of operation (CT2) requires a complete cancellation of one of the words W_i, W_j in the cyclic product W_iW_j. We will show that the CCAC-conjecture with stabilizations is still equivalent to the AC-conjecture with stabilizations, whereas the CCAC-conjecture without stabilizations is false.

2. Proofs of Theorem 1.1 and Corollaries 1.2–1.4

We start by proving Theorem 1.1.

Let (1.1) be a presentation of the trivial group, let the words $R_1, \dots, R_m$ be cyclically reduced and let the AC-conjecture be true for all presentations of rank $< m$. We need to show that if the AC-conjecture holds for the presentation (1.1), then the CAC-conjecture also holds for this presentation. To prove this, we argue by induction on $m \ge 1$, assuming that both AC- and CAC-conjectures hold true for all presentations of rank $< m$. Note that the basis step of this induction for $m = 1$ is obvious and we may assume that $m \ge 2$.

Suppose that $\sigma_1, \dots, \sigma_\ell$ are EN-operations that are applied to the m-tuple

$$\mathcal{R} = (R_1, \dots, R_m)$$

to obtain the letter tuple $(a_1, \dots, a_m)$. Denote $\mathcal{R}(0) := \mathcal{R}$ and $\mathcal{R}(k) := \sigma_k \dots \sigma_1(\mathcal{R})$, hence, $\mathcal{R}(\ell) = (a_1, \dots, a_m)$. Let $X \equiv Y$ denote the literal (or letter-by-letter) equality of words X, Y over $\mathcal{A}^{\pm 1}$. We also denote

$$\mathcal{R}(k) := (R_1(k), \dots, R_m(k)) \quad \text{and} \quad \bar{\mathcal{R}}(k) := (\bar{R}_1(k), \dots, \bar{R}_m(k)),$$

where $\bar{R}_1(k), \dots, \bar{R}_m(k)$ are cyclically reduced words such that, for every i, $\bar{R}_i(k)$ is obtained from $R_i(k)$ by cyclic cancellations, hence,

$$R_i(k) \equiv S_i(k)\bar{R}_i(k)S_i(k)^{-1},$$

with some words $S_i(k)$.

By induction on $k \ge 0$, we will be proving that $\bar{\mathcal{R}}(k)$ can be obtained from $\mathcal{R}$ by a sequence of CEN-operations, (CT1)–(CT3). Since $\mathcal{R}$ is cyclically reduced, we have $\bar{\mathcal{R}}(0) = \mathcal{R}$ and the basis step of the induction is true. We now address the induction step from k to $k + 1$.

If σ_{k+1} is of type (T1), then we can perform an analogous operation (CT1) over $\bar{\mathcal{R}}(k)$ and obtain $\bar{\mathcal{R}}(k+1)$. A reference to the induction hypothesis completes this case.

If σ_{k+1} has type (T3), then no change is needed, we can set $\bar{\mathcal{R}}(k+1) := \bar{\mathcal{R}}(k)$, and we can refer to the induction hypothesis.

From now on assume that σ_{k+1} has type (T2) and $R_s(k+1) = R_s(k)R_t(k)$ in $\mathcal{F}(\mathcal{A})$ with $t \neq s$. To simplify notation, rename $U_i := \bar{R}_i(k)$, $i = 1, \dots, m$, and $\mathcal{U} := \bar{\mathcal{R}}(k)$.

Suppose that $X, Y_1, \dots, Y_r$ are words over $\mathcal{A}^{\pm 1}$. We say that X *occurs* in words $Y_1, \dots, Y_r$ if there is an index i such that $Y_i \equiv Y_{i,1} X Y_{i,2}$ with some words $Y_{i,1}, Y_{i,2}$; i.e., X occurs in at least one of the words $Y_1, \dots, Y_r$. In this case, we may also say that X is a *subword* of $Y_1, \dots, Y_r$. The number of occurrences of X in $Y_1, \dots, Y_r$ is the sum of the number of occurrences of X in every Y_i.

Lemma 2.1. *For every letter $a_j \in \mathcal{A}$ and every word U_i in $\mathcal{U}$, one may assume that the number of occurrences of a_j in words U_i, U_i^{-1} is not one.*

Proof. Suppose, on the contrary, that the words U_i, U_i^{-1} contain a single occurrence of a letter $a_j \in \mathcal{A}$. Reindexing and using operations (CT1), (CT3) if necessary, we may assume that $i = j = 1$ and $U_1 \equiv a_1 U_{1,0}$, where $U_{1,0}$ has no occurrences of a_j and a_j^{-1}. Applying operations (CT1)–(CT3), we can turn $\mathcal{U}$ into $(U_1, V_2, \dots, V_m)$, where $V_2, \dots, V_m$ have no occurrences of letters a_1, a_1^{-1}. Hence, the presentation $\langle a_2, \dots, a_m \parallel V_2, \dots, V_m \rangle$ defines the trivial group and, by the induction hypothesis on m, the $(m-1)$-tuple $(V_2, \dots, V_m)$ can be transformed into $(a_2, \dots, a_m)$ by CEN-operations (CT1)–(CT3). Consequently, using CEN-operations (CT1)–(CT3), we can turn the m-tuple $\mathcal{U}$ into $(U_1, a_2, \dots, a_m)$ and, hence, into $(a_1, a_2, \dots, a_m)$. The proof is complete. $\square$

Reindexing if necessary, we may also assume that $s = 1$ and $t = 2$, hence $R_1(k+1) = R_1(k)R_2(k)$ in $\mathcal{F}(\mathcal{A})$. Let $|W|$ denote the length of a word W.

Lemma 2.2. *Up to cyclic permutations of the words $\bar{R}_1(k+1)$, U_1, U_2, one may assume that the cyclically reduced word $\bar{R}_1(k+1)$, conjugate to $R_1(k+1)$ in $\mathcal{F}(\mathcal{A})$, has one of the following four forms* (F1)–(F4)*, depicted in Figures* 1(a)–(c).

- (F1) $\bar{R}_1(k+1) \equiv U_1 C U_2 C^{-1}$, *where* $|C| > 0$*; see Figure* 1(a).
- (F2) $\bar{R}_1(k+1) \equiv DE$, *where* $U_1 \equiv DP^{-1}$, $U_2 \equiv PE$, *and* $|D|, |E| > 0$*; see Figure* 1(b).
- (F3) $\bar{R}_1(k+1) \equiv D$ *and* $U_1 \equiv DCU_2^{-1}C^{-1}$, *where* $|C| \geq 0$*; see Figure* 1(c).
- (F4) $\bar{R}_1(k+1) \equiv D$ *and* $U_2 \equiv DCU_1^{-1}C^{-1}$, *where* $|C| \geq 0$*; see Figure* 1(c).

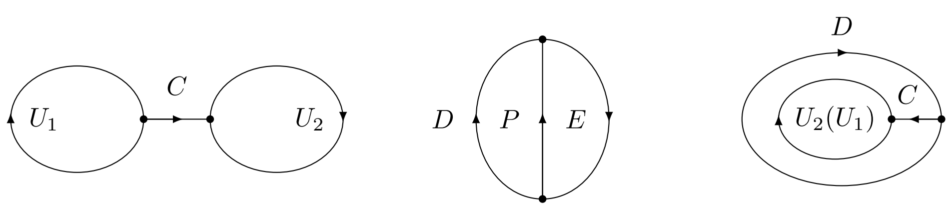

FIGURE 1(a) FIGURE 1(b) FIGURE 1(c)

Proof. It follows from the definitions that, letting $S_1 := S_1(k)$ and $S_2 := S_2(k)$, we have the following equalities:

$$R_1(k) \equiv S_1U_1S_1^{-1}, \quad R_2(k) \equiv S_2U_2S_2^{-1}, \quad R_1(k+1) \equiv S_1U_1S_1^{-1}S_2U_2S_2^{-1},$$
$$R_1(k+1) \equiv S_1(k+1)\bar{R}_1(k+1)S_1(k+1)^{-1}.$$

To analyze possible cancellations in the product $S_1U_1S_1^{-1}S_2U_2S_2^{-1}$, we consider a disk diagram Δ over the group presentation

$$\mathcal{P}_U = \langle\, a_1, \dots, a_m \,\|\, U_1, U_2 \,\rangle. \tag{2.1}$$

Recall that a disk diagram over a group presentation is a finite connected and simply connected 2-complex with a labeling function used for geometric interpretation of consequences of defining relations; details can be found in [7], [13], [19]. We define a disk diagram Δ over (2.1) so that Δ contains two faces Π_1, Π_2 whose clockwise oriented boundaries $\partial\Pi_1, \partial\Pi_2$ are labeled by words U_1, U_2, resp., and Δ contains a vertex o that is connected to $\partial\Pi_1$, $\partial\Pi_2$ by paths labeled by words S_1, S_2, resp.; see Figure 2. Then the clockwise oriented boundary $\partial|_o\Delta$ of Δ, starting at o, is labeled by the word $S_1U_1S_1^{-1}S_2U_2S_2^{-1}$, which we write in the form $\varphi(\partial|_o\Delta) \equiv S_1U_1S_1^{-1}S_2U_2S_2^{-1}$.

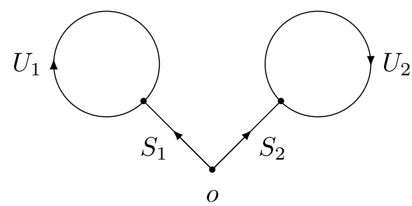

Figure 2

Cancellations in the cyclic word $S_1U_1S_1^{-1}S_2U_2S_2^{-1}$ can be interpreted as folding and pruning edges in Δ which, after all cancellations are done, will turn into a diagram Δ' that contains two faces Π_1', Π_2' and has $\varphi(\partial|_{o'}\Delta') \equiv \bar{R}_1(k+1)$ for a suitable vertex $o' \in \partial\Delta'$. It is not difficult to check that the following hold true. If the boundaries $\partial\Pi_1'$, $\partial\Pi_2'$ have no common vertex, then case (F1) holds. If the boundaries $\partial\Pi_1'$, $\partial\Pi_2'$ contain a common vertex and both $\partial\Pi_1'$, $\partial\Pi_2'$ have edges on $\partial\Delta'$, then case (F2) holds. If $(\partial\Pi_2')^{-1}$ is a subpath of $\partial\Pi_1'$, then case (F3) holds. Finally, if $(\partial\Pi_1')^{-1}$ is a subpath of $\partial\Pi_2'$, then case (F4) holds. ☐

In view of Lemma 2.2, we need to consider cases (F1)–(F4).

In case (F2), we apply (CT3) to U_1 to get DP^{-1} and apply (CT3) to U_2 to get PE. Then we use (CT2) to turn DP^{-1} into DE. Since DE is a cyclic permutation of $\bar{R}_1(k+1)$, the induction step is complete.

In case (F3), we apply (CT3) to U_1 to get $C^{-1}DCU_2^{-1}$ and use (CT2) to make the transformation $C^{-1}DCU_2^{-1} \to D$. Since D is a cyclic permutation of $\bar{R}_1(k+1)$, the induction step is complete.

Case (F4) is analogous to case (F3) with U_1 and U_2 switched.

It remains to study case (F1).

Let $\mathcal{W} = (W_1, \dots, W_m)$ be an m-tuple of nonempty cyclically reduced words over $\mathcal{A}^{\pm 1}$. Let $W_1 \equiv AB$, where $|A|, |B| > 0$, and C be a cyclic permutation of the word $W_{j'}^{\varepsilon}$, where $j' > 1$ and $\varepsilon = \pm 1$. If the word ACB is reduced, then the operation over $\mathcal{W}$ so that W_1 is replaced with ACB is called a *simple* 1*-insertion*.

Let $\mathcal{W}(i)$ denote an m-tuple obtained from a cyclically reduced tuple $\mathcal{W} = \mathcal{W}(0)$ by a sequence of i simple 1-insertions. It is easy to see that $\mathcal{W}(i)$ can be obtained from $\mathcal{W}(0)$ by a sequence of CEN-operations (CT1)–(CT3) and that $\mathcal{W}(0)$ can be obtained back from $\mathcal{W}(i)$ by a sequence of CEN-operations (CT1)–(CT3).

Consider the following properties of a reduced word W over the alphabet $\mathcal{A}^{\pm 1}$.

(P) For a given letter $a \in \mathcal{A}^{\pm 1}$, there are distinct letters $b, c \in \mathcal{A}^{\pm 1}$ such that ab and ac occur in the words W, W^{-1}.

(Q) For every letter $a \in \mathcal{A}^{\pm 1}$, there are distinct letters $b, c \in \mathcal{A}^{\pm 1}$, depending on a, such that ab and ac occur in the words W, W^{-1}.

Lemma 2.3. *Suppose the first and the last letters of the word U_1 are distinct. Then there exists a sequence of simple 1-insertions that transform the tuple*

$$\mathcal{U} = (U_1, \dots, U_m)$$

into $\mathcal{U}_V = (V, U_2, \dots, U_m)$ in which the cyclically reduced word V has property (Q).

Proof. Note that, by Lemma 2.1, $|U_i| > 1$ for every i. Next, we observe that if $U_1 \equiv AB$ with $|A|, |B| > 0$, then for every $j \geq 2$ at least one of the words AU_jB, $AU_j^{-1}B$ is cyclically reduced. We also note that if a letter $e \in \mathcal{A}$ does not occur in words $U_2, U_2^{-1}, \dots, U_m, U_m^{-1}$, then it follows from Lemma 2.1 and the triviality of the group given by presentation

$$\langle\, a_1, \dots, a_m \,\|\, U_1, \dots, U_m \,\rangle \tag{2.2}$$

that U_1 contains at least three occurrences of e, e^{-1} and, by the lemma's assumption, at most one of these occurrences is the first or the last letter of U_1. On the other hand, if a letter $e \in \mathcal{A}$ occurs in words U_i, U_i^{-1} for some $i > 1$, then, by Lemma 2.1, there are at least two such occurrences. Therefore, using the words $U_2, \dots, U_m$ and making at most $m-1$ simple 1-insertions, we can obtain a word V_{m-1} from $V_0 := U_1$ such that, for every $a \in \mathcal{A}^{\pm 1}$, the words V_{m-1}, V_{m-1}^{-1} contain at least two distinct occurrences of words ab, ac, where $b, c \in \mathcal{A}^{\pm 1}$. Below we will make more simple 1-insertions to guarantee that $b \neq c$, i.e., to guarantee that the resulting word would have property (P) relative to a.

Fix a letter $a \in \mathcal{A}^{\pm 1}$. As was shown above, the words V_{m-1}, V_{m-1}^{-1} contain two distinct occurrences of words ab, ac, where $b, c \in \mathcal{A}^{\pm 1}$. If $b \neq c$, then V_{m-1} has property (P) relative to a and we do not need to do anything. Suppose $b = c$.

First assume that the letter a does not occur in words $U_2, U_2^{-1}, \dots, U_m, U_m^{-1}$.

Let $V_{m-1}^{\delta} \equiv DE$ be the factorization of V_{m-1}^{δ}, $\delta = \pm 1$, defined by an occurrence of ab in V_{m-1}^{δ}, so that $D \equiv D_1 a$ and $E \equiv bE_1$. It follows from Lemma 2.1 and the triviality of the group given by presentation (2.2) that U_2, U_2^{-1} contain occurrences of at least two distinct letters of $\mathcal{A}$. This means that if b occurs in U_2, U_2^{-1}, then there is a cyclic permutation $\widetilde{U}_2^{\varepsilon}$ of U_2^{ε}, $\varepsilon = \pm 1$, such that $\widetilde{U}_2^{\varepsilon} \equiv b^{-1}Fd$, where $d \in \mathcal{A}^{\pm 1}$, $d \neq b^{-1}$. Then $V_{m-1} \to V_m \equiv (D\widetilde{U}_2^{\varepsilon}E)^{\delta}$ is a simple 1-insertion that creates a subword ab^{-1} in V_m, V_m^{-1}. Hence, both ab^{-1}, ab occur in V_m, V_m^{-1}, as desired. On the other hand, if b does not occur in U_2, U_2^{-1}, then $V_{m-1} \to V_m \equiv (DU_2E)^{\delta}$ is a simple 1-insertion that produces a subword ad in V_m^{δ} for some $d \in \mathcal{A}^{\pm 1}$, $d \neq b$. Hence, both ad, ab occur in V_m, V_m^{-1}, as required.

The case when a does not occur in words $U_2, U_2^{-1}, \dots, U_m, U_m^{-1}$ is complete.

Now assume that the letter a occurs in words $U_2, U_2^{-1}, \dots, U_m, U_m^{-1}$; say, a occurs in U_2, U_2^{-1}. As above, we let $V_{m-1}^\delta \equiv DE$ be the factorization of V_{m-1}^δ, $\delta = \pm 1$, defined by an occurrence of ab in V_{m-1}^δ, so that $D \equiv D_1 a$ and $E \equiv bE_1$.

Consider two cases: $b = a$ and $b \neq a$.

Assume that $b = a$. It follows from Lemma 2.1 and the triviality of the group given by presentation (2.2) that U_2 is not a power of a. Hence, there is a cyclic permutation $\widetilde{U}_2^\varepsilon$ of U_2^ε, $\varepsilon = \pm 1$, such that $\widetilde{U}_2^\varepsilon \equiv dFa$, $d \in \mathcal{A}^{\pm 1}$ and $d \neq a^{\pm 1}$. Then $V_{m-1} \to V_m \equiv (D\widetilde{U}_2^\varepsilon E)^\delta$ is a simple 1-insertion that creates a subword ad in V_m^δ with $d \neq b$. This means that ab and ad occur in V_m, V_m^{-1}, as required.

Suppose that $b \neq a$. Recall that a occurs in U_2, U_2^{-1}. If b does not occur in U_2, U_2^{-1}, then we consider a cyclic permutation $\widetilde{U}_2^\varepsilon$ of U_2^ε, $\varepsilon = \pm 1$, such that $\widetilde{U}_2^\varepsilon \equiv aF$. Then $V_{m-1} \to V_m \equiv (D\widetilde{U}_2^\varepsilon E)^\delta$ is a simple 1-insertion that creates a subword a^2 in V_m^δ. Hence, ab and aa occur in V_m, V_m^{-1}, as desired. On the other hand, if b occurs in U_2, U_2^{-1}, then there is a cyclic permutation $\widetilde{U}_2^\varepsilon$ of U_2^ε, $\varepsilon = \pm 1$, such that $U_2^\varepsilon \equiv b^{-1}Fd$, $d \in \mathcal{A}^{\pm 1}$ and $b^{-1} \neq d$. Then $V_{m-1} \to V_m \equiv (D\widetilde{U}_2^\varepsilon E)^\delta$ is a simple 1-insertion that creates a subword ab^{-1} in V_m^δ. This means that ab and ab^{-1} occur in V_m, V_m^{-1}, as required.

Let us overview our intermediate findings. When given a letter $a \in \mathcal{A}^{\pm 1}$ and two distinct occurrences of ab in V_{m-1}, V_{m-1}^{-1}, where $b \in \mathcal{A}^{\pm 1}$, we are always able to make a simple 1-insertion $V_{m-1} \to V_m \equiv (D\widetilde{U}_j^\varepsilon E)^\delta$ in such a way that $j > 1$, $\varepsilon = \pm 1$, $\delta = \pm 1$, and $V_{m-1}^\delta \equiv DE$ is the factorization of V_{m-1}^δ, defined by an occurrence of ab in V_{m-1}^δ, so that $D \equiv D_1 a$ and $E \equiv bE_1$. Moreover, if $\widetilde{U}_j^\varepsilon$ starts with a letter $d \in \mathcal{A}^{\pm 1}$, then $d \neq b$. In this situation, we say that the subword ab of $V_{m-1}^\delta \equiv D_1 abE_1$ was changed by the simple 1-insertion $V_{m-1} \to V_m$ into ad. Since $d \neq b$ and both ab, ad occur in V_m, V_m^{-1}, it follows that the word V_m has property (P) relative to a.

Clearly, for every $a' \in \mathcal{A}^{\pm 1}$, the words V_m, V_m^{-1}, similarly to V_{m-1}, V_{m-1}^{-1}, also contain at least two distinct occurrences of subwords of the form $a'b'$, $a'c'$, where $b', c' \in \mathcal{A}^{\pm 1}$. If V_m does not have property (P) relative to a', then $b' = c'$ and, arguing as above, we can make a simple 1-insertion that converts V_m into V_{m+1} and changes one of the subwords $a'b'$ into $a'd'$ with $d' \neq b'$. Note that this simple 1-insertion preserves property (P) of V_m relative to a because one of the occurrences of $a'b'$ is not affected by the performed 1-insertion. As a result, the word V_{m+1} has property (P) relative to a and relative to a'. Iterating our arguments for all $e \in \mathcal{A}^{\pm 1}$, we obtain property (P) for the word $V := V_{m'}$, where $m' \leq 3m - 1$, relative to every letter $e \in \mathcal{A}^{\pm 1}$, which means property (Q) for V. □

Lemma 2.4. *Suppose that the first word U_1 of the m-tuple $\mathcal{U} = (U_1, \dots, U_m)$ has property* (Q). *Furthermore, assume that the word $U_1' C U_2 C^{-1}$, where C is some word and U_1' is a cyclic permutation of $U_1^{\varepsilon_1}$, $\varepsilon_1 = \pm 1$, is cyclically reduced. Then the m-tuple $\mathcal{U}$ can be transformed into*

$$\mathcal{U}_C = (U_1' C U_2 C^{-1}, U_2, \dots, U_m)$$

by a finite sequence of CEN-operations (CT1)–(CT3).

Proof. First we will show that it suffices to prove that $\mathcal{U}$ can be turned into

$$\mathcal{U}_C' = (U_1, C U_2 C^{-1} U_1', U_3, \dots, U_m) \tag{2.3}$$

by operations (CT1)–(CT3). Indeed, starting with $\mathcal{U}'_C$ and using (CT1)–(CT2), we can switch U_1 and $CU_2C^{-1}U'_1$. Note that a switch is a composition of (CT1)–(CT2). Then we obtain U'_1 in place of U_1 by (CT3) and (CT1) if $\varepsilon_1 = -1$, and multiply U'_1 on the right by $(CU_2C^{-1}U'_1)^{-1}$ to get U_2^{-1} in place of U'_1. As a result, we obtain the m-tuple

$$(CU_2C^{-1}U'_1, U_2^{-1}, U_3, \dots, U_m)$$

which can be easily converted into $\mathcal{U}_C$ by (CT1), (CT3).

To prove that the tuple $\mathcal{U}'_C$, defined by (2.3), can be obtained from $\mathcal{U}$ by CEN-operations (CT1)–(CT3), we will argue by induction on the length $|C| \geq 0$ of the word C.

If $|C| = 0$, then our claim is obvious. Assume that $|C| > 0$ and let

$$C \equiv aC_1,$$

where $a \in \mathcal{A}^{\pm 1}$. By property (Q) for U_1, there are distinct letters $b, c \in \mathcal{A}^{\pm 1}$ such that ab, ac occur in words U_1, U_1^{-1}. Consequently, there is a cyclic permutation $U_{1,b}$ of U_1 or U_1^{-1} and there is a cyclic permutation $U_{1,c}$ of U_1 or U_1^{-1} such that

$$U_{1,b} \equiv a^{-1}D_b b^{-1}, \qquad U_{1,c} \equiv a^{-1}D_c c^{-1},$$

and at least one of the words $C_1U_2C_1^{-1}U_{1,b}$, $C_1U_2C_1^{-1}U_{1,c}$ is cyclically reduced. Let U_0 denote one of $U_{1,b}$, $U_{1,c}$ for which the product $C_1U_2C_1^{-1}U_0$ is cyclically reduced. We also denote $U_0 \equiv a^{-1}U_{0,1}$. By the induction hypothesis on $|C|$, $\mathcal{U}$ can be converted into

$$\mathcal{U}'_{C_1} := (U_1, C_1U_2C_1^{-1}U_0, U_3, \dots, U_m) \tag{2.4}$$

by a sequence of operations (CT1)–(CT3). Using more operations (CT1)–(CT3), we can cyclically permute $C_1U_2C_1^{-1}U_0$ to obtain the word $U_{0,1}C_1U_2C_1^{-1}a^{-1}$ and multiply it by U'_1 on the left (which is a composition of (CT1)–(CT2)) to obtain the word $U'_1U_{0,1}C_1U_2C_1^{-1}a^{-1}$. Consider the word

$$a^{-1}U'_1U_{0,1}C_1U_2C_1^{-1}. \tag{2.5}$$

Since $C_1U_2C_1^{-1}a^{-1}U_{0,1}$ is cyclically reduced, it follows that a first cancellation in the word (2.5), if it exists, occurs in the prefix subword $a^{-1}U'_1U_{0,1}$. We now discuss cancellations in this subword.

Let T denote an empty or reduced word obtained from $a^{-1}U'_1U_{0,1}$ by cancellations. Since $U'_1CU_2C^{-1}$ is cyclically reduced and $C \equiv aC_1$, it follows that U'_1 does not start with a. We also note that the words U'_1, $U_{0,1}$ are reduced, and $|U'_1| = |U_{0,1}| + 1$. Hence, we may conclude that $|T| > 0$. Observe that either of U'_1, $U_{0,1}a^{-1}$ is a cyclic permutation of U_1 or U_1^{-1}. Hence, the word $a^{-1}U'_1U_{0,1}$ represents the trivial element in the one-relator group given by presentation

$$\langle\, a_1, \dots, a_m \,\|\, U_1 \,\rangle. \tag{2.6}$$

Since the word U_1 has property (Q), it follows that every letter of $\mathcal{A}$ occurs in U_1, U_1^{-1} at least twice. By the classical Magnus's Freiheitssatz (see [13], [14]) for one-relator group presentation (2.6) applied to the word T, we obtain that every letter of $\mathcal{A}$ must occur in T, T^{-1}. Since either of U'_1, $U_{0,1}a^{-1}$ is a cyclic permutation of U_1 or U_1^{-1}, it follows that every letter of $\mathcal{A}$ occurs in $a^{-1}U'_1U_{0,1}$, $(a^{-1}U'_1U_{0,1})^{-1}$ even and positive number of times. Since cancellations of letters are done in pairs, $T = a^{-1}U'_1U_{0,1}$ in $\mathcal{F}(\mathcal{A})$, and every letter of $\mathcal{A}$ occurs in T, T^{-1}, we deduce that every letter of $\mathcal{A}$ occurs in T, T^{-1} at least twice and so $|T| \geq 2m \geq 4$. This implies that both the first and the last letters of $a^{-1}U_1'U_{0,1}$ remain uncanceled in the reduced word T. Therefore, we may conclude that the word $TC_1U_2C_1^{-1}$ is cyclically reduced.

Now we apply a sequence of CEN-operations (CT1)–(CT3) to the tuple $\mathcal{U}'_{C_1}$, defined by (2.4), so that the second component of $\mathcal{U}'_{C_1}$ would be changing as follows:

$$\begin{aligned} &C_1U_2C_1^{-1}U_0 \to a^{-1}U_{0,1}C_1U_2C_1^{-1} \to a^{-1}U_1'U_{0,1}C_1U_2C_1^{-1} \\ &\overset{\mathcal{F}(\mathcal{A})}{=} TC_1U_2C_1^{-1} \to TU_0^{-1}C_1U_2C_1^{-1} \overset{\mathcal{F}(\mathcal{A})}{=} a^{-1}U_1'U_{0,1}(a^{-1}U_{0,1})^{-1}C_1U_2C_1^{-1} \\ &\overset{\mathcal{F}(\mathcal{A})}{=} a^{-1}U_1'aC_1U_2C_1^{-1} \to U_1'aC_1U_2C_1^{-1}a^{-1} \equiv U_1'CU_2C^{-1} \to CU_2C^{-1}U_1'. \end{aligned}$$

The last word is a desired one and the induction step is complete. □

Recall that to prove Theorem 1.1 it remains to study case (F1), that is, to establish that the operation $U_1 \to U_1CU_2C^{-1}$, where $U_1CU_2C^{-1}$ is cyclically reduced, over the m-tuple $\mathcal{U}$ is a composition of CEN-operations (CT1)–(CT3). To do this, we first apply Lemma 2.3 and, using simple 1-insertions, turn U_1 into a word V with property (Q). Note that a simple 1-insertion, by the definition, does not change the first and the last letters of U_1 and so the word VCU_2C^{-1}, similarly to $U_1CU_2C^{-1}$, is cyclically reduced. Therefore, Lemma 2.4 applies and yields a sequence of CEN-operations (CT1)–(CT3) that transforms the m-tuple $\mathcal{U}_V = (V, U_2, \dots, U_m)$ into $(VCU_2C^{-1}, U_2, \dots, U_m)$. Now we can use CEN-operations that convert the subword V of VCU_2C^{-1} back into U_1 (these can be viewed as inverses of simple 1-insertions). As a result, we obtain the desired tuple $(U_1CU_2C^{-1}, U_2, \dots, U_m)$ and case (F1) is complete. Theorem 1.1 is proved. □

Proof of Corollary 1.2. Let $r \geq 2$ be an integer. Suppose that the AC-conjecture holds for every presentation (1.1) of rank $\leq r$. Then, by induction on m, where $1 \leq m \leq r$, it follows from Theorem 1.1 that the CAC-conjecture also holds for every presentation (1.1) of rank $\leq r$ for which the words $R_1, \dots, R_m$ are cyclically reduced.

Conversely, suppose that the CAC-conjecture holds for every presentation (1.1) for which the words $R_1, \dots, R_m$ are cyclically reduced and $m \leq r$. Consider an arbitrary presentation

$$\langle\, a_1, \dots, a_m \,\|\, W_1, \dots, W_m \,\rangle \tag{2.7}$$

of the trivial group, where $W_1, \dots, W_m$ are reduced words over $\mathcal{A}^{\pm 1}$. Let $\bar W_1, \dots, \bar W_m$ be cyclically reduced words obtained from $W_1, \dots, W_m$, resp., by cyclic cancellations. Since the CAC-conjecture holds for the presentation

$$\langle\, a_1, \dots, a_m \,\|\, \bar W_1, \dots, \bar W_m \,\rangle,$$

it follows that there is a finite sequence of operations (CT1)–(CT3) that changes the tuple $(\bar W_1, \dots, \bar W_m)$ into $(a_1, \dots, a_m)$. Note that every operation of type (CT1)–(CT3) over a cyclically reduced tuple can be presented as a composition of operations (T1)–(T3). Therefore, the tuple $(\bar W_1, \dots, \bar W_m)$ can also be converted into $(a_1, \dots, a_m)$ by a sequence of operations (T1)–(T3). Since $(W_1, \dots, W_m)$ can be turned into $(\bar W_1, \dots, \bar W_m)$ by operations (T3), the AC-conjecture is also true for presentation (2.7). □

Proof of Corollary 1.3. This is straightforward from Corollary 1.2. □

Proof of Corollary 1.4. Let $r \geq 2$ be an integer and assume that every m-tuple $\mathcal{R} = (R_1, \dots, R_m)$, where $2 \leq m \leq r$, that defines the trivial group by (1.1) can be transformed to $(a_1, \dots, a_m)$ by a finite sequence of operations (T1)–(T3). Let $\bar{R}_1, \dots, \bar{R}_m$ be cyclically reduced words obtained from $R_1, \dots, R_m$, resp., by cyclic cancellations. It follows from Corollary 1.3 that the m-tuple $\bar{\mathcal{R}} = (\bar{R}_1, \dots, \bar{R}_m)$ can be turned into $(a_1, \dots, a_m)$ by operations (CT1)–(CT3). Note that every operation of type (CT1)–(CT3) over a cyclically reduced tuple can be presented as a composition of operations (T1), (T2), (T3C). Hence, the tuple $(\bar{R}_1, \dots, \bar{R}_m)$ can also be converted into $(a_1, \dots, a_m)$ by a sequence of operations (T1), (T2), (T3C). It remains to observe that the original m-tuple $(R_1, \dots, R_m)$ can be turned into $(\bar{R}_1, \dots, \bar{R}_m)$ by operations (T3C). □

3. One more conjecture of Andrews and Curtis

Here we discuss one more satellite hypothesis of Andrews and Curtis [2, Conjecture 4] concerning nonminimal pairs of words. According to [2], a pair (W_1, W_2) of reduced words W_1, W_2 over the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ is called *minimal* if no sequence of operations (T1)–(T3) can decrease the total length $|W_1| + |W_2|$ of (W_1, W_2).

In [2, Conjecture 4], Andrews and Curtis speculate that if (W_1, W_2) is not a minimal pair, then W_1 and W_2, considered as cyclic words, contain a common subword V such that

$$|W_1| + |W_2| - 2|V| < \max(|W_1|, |W_2|). \tag{3.1}$$

In other words, there are cyclic permutations $\bar{W}_1$, $\bar{W}_2$ of cyclically reduced words $W_1^{\varepsilon_1}$, $W_2^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, such that $\bar{W}_1 \equiv V_1 V$, $\bar{W}_2 \equiv V^{-1} V_2$ and the product $\bar{W}_1 \bar{W}_2 \equiv V_1 V_2$ is cyclically reduced and shorter than the longer of W_1, W_2. This conjecture would provide a strong Nielsen-type reduction for nonminimal pairs. However, the conjecture is false and, as a counterexample, one could use the pair $(a^2 b^{-3}, abab^{-1}a^{-1}b^{-1})$. For this pair, if V is a common subword of cyclic permutations of words $W_1^{\varepsilon_1}$ and $W_2^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, then it is easy to see that $|V| \leq 2$. Hence, the inequality (3.1) could not be satisfied. On the other hand, as was found out by Myasnikov [17] (see also [5], [18]), the AC-conjecture holds for the pair $(a^2 b^{-3}, abab^{-1}a^{-1}b^{-1})$. Therefore, the pair $(a^2 b^{-3}, abab^{-1}a^{-1}b^{-1})$ is not minimal and gives a counterexample to [2, Conjecture 4].

4. Cancellative cyclic version of the Andrews–Curtis conjecture

The significance and power of stabilizations does not look clear even in the special case of presentations coming from spines of the 3-sphere and is totally obscure for arbitrary presentations. For this reason, it seems worthwhile to consider a more restrictive version of the CAC-conjecture with and without stabilizations, called the cancellative cyclic version of the Andrews–Curtis conjecture and abbreviated as the CCAC-conjecture. In this new version, in the analogue of operation (CT2) we require complete cancellation of one of the words W_i, W_j in the cyclic product $W_i W_j$. This CCAC-conjecture enables us to give the first evidence of importance of stabilizations in the context of the AC-conjecture. We will show in Theorem 4.1 that the CCAC-conjecture with stabilizations is still equivalent to the AC-conjecture with stabilizations, whereas the CCAC-conjecture *without* stabilizations is false.

As before, let $\mathcal{W} = (W_1, \dots, W_n)$ be a cyclically reduced n-tuple of words over $\mathcal{A}^{\pm 1}$. Consider the following transformation over $\mathcal{W}$.

(CCT2) For some pair of distinct indices i and j, W_i is replaced with a word W, where W is a cyclically reduced or empty word obtained from the product W_iW_j by making cancellations and cyclic cancellations and W is such that $|W| \le \max(|W_i|, |W_j|) - \min(|W_i|, |W_j|)$.

It is easy to see that the latter inequality is equivalent to the condition that one of the words W_i, W_j cancels out completely in the cyclic product W_iW_j. In particular, this means that the analogue of operation (T4) over cyclically reduced tuples would be meaningless when combined with operations (CT1), (CCT2), (CT3). Indeed, if one of the words W_i, W_j is a letter $b \notin \mathcal{A}^{\pm 1}$ and the other one is a word over $\mathcal{A}^{\pm 1}$, then (CCT2) would not be applicable to the pair W_i, W_j. For this reason, a suitable analogue of operation (T4) over pairs $\mathcal{A}, \mathcal{R}$, where $\mathcal{R}$ is a cyclically reduced tuple of words over $\mathcal{A}^{\pm 1}$, is defined as follows.

(CCT4) Let $b \notin \mathcal{A}^{\pm 1}$ be a letter and let U be a word over $\mathcal{A}^{\pm 1}$. Add b to the alphabet $\mathcal{A}$ and append the word bU to the tuple $\mathcal{R}$. Conversely, if bU is a word of $\mathcal{R}$, where $b \in \mathcal{A}$, and b, b^{-1} have no occurrences in U and in all words of $\mathcal{R}$ other than bU, then delete b from $\mathcal{A}$ and delete bU from $\mathcal{R}$.

The cancellative cyclic version of the Andrews–Curtis conjecture, abbreviated as *CCAC-conjecture*, states that, *for every balanced group presentation*

$$\mathcal{P} = \langle\, a_1, \dots, a_m \,\|\, R_1, \dots, R_m \,\rangle \tag{4.1}$$

such that (4.1) *defines the trivial group and* $\mathcal{R} = (R_1, \dots, R_m)$ *is cyclically reduced, the* m*-tuple* $\mathcal{R}$ *can be brought to the letter tuple* $(a_1, \dots, a_m)$ *by a finite sequence of operations* (CT1), (CCT2), (CT3).

Similarly, the cancellative cyclic version of the Andrews–Curtis conjecture with stabilizations, briefly *CCAC-conjecture with stabilizations*, claims that *for every balanced group presentation* (4.1) *such that* (4.1) *defines the trivial group and* $\mathcal{R} = (R_1, \dots, R_m)$ *is cyclically reduced, the* m*-tuple* $\mathcal{R}$ *can be brought to the letter tuple* $(a_1, \dots, a_m)$ *by a finite sequence of operations* (CT1), (CCT2), (CT3), (CCT4).

Theorem 4.1.

(a) *Suppose that a balanced presentation* (4.1) *defines the trivial group and the tuple* $\mathcal{R} = (R_1, \dots, R_m)$ *is cyclically reduced. Then the AC-conjecture with stabilizations holds true for* (4.1) *if and only if the CCAC-conjecture with stabilizations holds for* (4.1).

(b) *The CCAC-conjecture* without *stabilizations is false.*

Proof. (a) First we will show that if the AC-conjecture with stabilizations holds for (4.1), then the CCAC-conjecture with stabilizations also holds for (4.1).

Assume that the AC-conjecture with stabilizations holds for (4.1) and $\mathcal{R}$ can be converted into $(a_1, \dots, a_m)$ by a sequence of EN-operations (T1)–(T3) and $2s$ stabilizations (T4). It is clear that in this process of turning $\mathcal{R}$ into $(a_1, \dots, a_m)$, one can do all s positive stabilizations (that increase $|\mathcal{A}|$) in the very beginning and all s negative stabilizations (or destabilizations that decrease $|\mathcal{A}|$) in the very end. Therefore, one can avoid stabilizations altogether and assume that the $(m+s)$-tuple

$$(\mathcal{R}, \mathcal{B}) := (R_1, \dots, R_m, b_1, \dots, b_s),$$

where $b_1, \dots, b_s$ are all new letters that were introduced by s positive stabilizations, can be converted into $(m+s)$-tuple $(\mathcal{A}, \mathcal{B}) := (a_1, \dots, a_m, b_1, \dots, b_s)$ by a sequence of EN-operations of type (T1)–(T3).

Similarly to the proof of Theorem 1.1, let $\sigma_1, \dots, \sigma_\ell$ be operations of type (T1)–(T3) that are applied to the $(m+s)$-tuple $(\mathcal{R}, \mathcal{B})$ to obtain $(\mathcal{A}, \mathcal{B})$. Denote

$$\mathcal{W}(0) := (\mathcal{R}, \mathcal{B}) \qquad \text{and} \qquad \mathcal{W}(k) := \sigma_k(\mathcal{W}(k-1))$$

for $k = 1, \dots, \ell$, hence, $\mathcal{W}(k) = \sigma_k \dots \sigma_1(\mathcal{W}(0))$ and $\mathcal{W}(\ell) = (\mathcal{A}, \mathcal{B})$. Also, we denote

$$\mathcal{W}(k) := (W_1(k), \dots, W_{m+s}(k)) \qquad \text{and} \qquad \bar{\mathcal{W}}(k) := (\bar{W}_1(k), \dots, \bar{W}_{m+s}(k)),$$

where $\bar{W}_1(k), \dots, \bar{W}_{m+s}(k)$ are cyclically reduced words such that, for every i, $\bar{W}_i(k)$ is obtained from $W_i(k)$ by cyclic cancellations, so $W_i(k) \equiv S_i(k)\bar{W}_i(k)S_i(k)^{-1}$ in the free group $\mathcal{F}(\mathcal{A} \cup \mathcal{B})$ with some word $S_i(k)$ for $k = 0, \dots, \ell$.

By induction on $k \geq 0$, we will be proving that $\bar{\mathcal{W}}(k)$ can be obtained from $\mathcal{W}(0)$ by a sequence of operations (CT1), (CCT2), (CT3), (CCT4). Since the basis step of this induction is obvious, we only need to make the induction step from k to $k+1$.

If σ_{k+1} is of type (T1), then we can perform an analogous operation (CT1) over $\bar{\mathcal{W}}(k)$ and obtain $\bar{\mathcal{W}}(k+1)$. A reference to the induction hypothesis completes this case.

If σ_{k+1} has type (T3), then no change is needed, we can set $\bar{\mathcal{W}}(k+1) := \bar{\mathcal{W}}(k)$, and we can refer to the induction hypothesis.

Therefore, we may assume that σ_{k+1} has type (T2) and $W_t(k+1) = W_t(k)W_r(k)$ in $\mathcal{F}(\mathcal{A} \cup \mathcal{B})$ with $t \neq r$. To simplify notation, rename $U_i := \bar{W}_i(k)$, $i = 1, \dots, m+s$, and $\mathcal{U} := \bar{\mathcal{W}}(k)$. Reindexing if necessary, we may also suppose that $t = 1$, $r = 2$, hence, $W_1(k+1) = U_1U_2$ in $\mathcal{F}(\mathcal{A} \cup \mathcal{B})$.

It follows from the analogue of Lemma 2.2 in which the word $\bar{R}_1(k+1)$ is replaced with $\bar{W}_1(k+1)$ that we need to consider cases (F1)–(F4).

In case (F3), we apply (CT3) to U_1 to get $C^{-1}DCU_2^{-1}$ and use (CCT2) to convert $C^{-1}DCU_2^{-1}$ to D. Since D is a cyclic permutation of $\bar{U}_1(k+1)$, a reference to the induction hypothesis completes the induction step in case (F3).

Case (F4) is analogous to case (F3) with U_1 and U_2 switched.

It remains to study cases (F1)–(F2).

Suppose that case (F1) holds. Hence, up to cyclic permutations of the words $\bar{W}_1(k+1)$, U_1, U_2, we have $\bar{W}_1(k+1) \equiv U_1CU_2C^{-1}$, where $|C| > 0$; see Figure 1(a). Applying operations (CT3) to U_1, U_2, $\bar{W}_1(k+1)$ if necessary, we may assume that $\bar{W}_1(k+1) \equiv U_1CU_2C^{-1}$.

Now we apply a sequence of operations (CT1), (CCT2), (CT3), (CCT4) to the tuple $\mathcal{U}$ so that the first two components of $\mathcal{U}$ would be changing as indicated below. Note that the addition and deletion of the third component is done by (CCT4) and x is a letter, $x \notin (\mathcal{A} \cup \mathcal{B})^{\pm 1}$:

$$\begin{aligned}(U_1, U_2) &\to (xCU_2C^{-1}U_1, U_1, U_2) \to (U_1^{-1}CU_2^{-1}C^{-1}x^{-1}, U_1, U_2) \\ &\to (U_1^{-1}CU_2^{-1}C^{-1}x^{-1}, CU_2^{-1}C^{-1}x^{-1}, U_2) \to (xCU_2C^{-1}U_1, xCU_2C^{-1}, U_2^{-1}) \\ &\to (xCU_2C^{-1}U_1, C^{-1}xCU_2, U_2^{-1}) \to (xCU_2C^{-1}U_1, x, U_2^{-1}) \\ &\to (CU_2C^{-1}U_1x, x^{-1}, U_2^{-1}) \to (CU_2C^{-1}U_1, x^{-1}, U_2^{-1}) \to (U_1CU_2C^{-1}, U_2).\end{aligned}$$

Thus it is shown that $\mathcal{U} = \bar{W}(k)$ can be changed into $\bar{W}(k+1)$ by operations (CT1), (CCT2), (CT3), (CCT4). A reference to the induction hypothesis completes the induction step in case (F1).

Assume that case (F2) holds. Hence, up to cyclic permutations of the words $\bar{W}_1(k+1)$, U_1, U_2, we have $\bar{W}_1(k+1) \equiv DE$, where $U_1 \equiv DP^{-1}$, $U_2 \equiv PE$, and $|D|, |E| > 0$; see Figure 1(b). Applying operations (CT3) to U_1, U_2, $\bar{W}_1(k+1)$, if necessary, we may assume that $\bar{W}_1(k+1) \equiv DE$, where $U_1 \equiv DP^{-1}$, $U_2 \equiv PE$, and $|D|, |E| > 0$.

Let us apply a sequence of operations (CT1), (CCT2), (CT3), (CCT4) to the tuple $\mathcal{U}$ so that the first two components of $\mathcal{U}$ would be changing as indicated below. As above, $x \notin (\mathcal{A} \cup \mathcal{B})^{\pm 1}$ is a new letter.

$$\begin{gathered}(U_1, U_2) = (DP^{-1}, PE) \to (xPEDP^{-1}, DP^{-1}, PE) \\ \to (PD^{-1}E^{-1}P^{-1}x^{-1}, DP^{-1}, PE) \to (PD^{-1}E^{-1}P^{-1}x^{-1}, E^{-1}P^{-1}x^{-1}, PE) \\ \to (xPEDP^{-1}, xPE, E^{-1}P^{-1}) \to (xPEDP^{-1}, x, E^{-1}P^{-1}) \\ \to (PEDP^{-1}x, x^{-1}, PE) \to (ED, x, PE) \to (DE, PE) = (\bar{W}_1(k+1), U_2).\end{gathered}$$

Thus $\mathcal{U} = \bar{W}(k)$ can be changed into $\bar{W}(k+1)$ by operations (CT1), (CCT2), (CT3), (CCT4). A reference to the induction hypothesis completes the induction step in case (F2).

The induction step is now complete in all cases (F1)–(F4), and it is shown that $\bar{\mathcal{W}}(k)$, for every $k \geq 0$, can be obtained from $\mathcal{W}(0) = (\mathcal{R}, \mathcal{B})$ by a sequence of operations (CT1), (CCT2), (CT3), (CCT4). Since $\bar{\mathcal{W}}(\ell) = \mathcal{W}(\ell) = (\mathcal{A}, \mathcal{B})$, it follows that, using operations (CCT4), one can transform the tuple $\mathcal{R}$ into $(\mathcal{R}, \mathcal{B})$. Then, applying operations (CT1), (CCT2), (CT3), (CCT4), one can get $(\mathcal{A}, \mathcal{B})$ from $(\mathcal{R}, \mathcal{B})$ and then, using (CCT4), obtain $(a_1, \dots, a_m)$ from $(\mathcal{A}, \mathcal{B})$. Thus the CCAC-conjecture with stabilizations holds for $\mathcal{R}$.

Conversely, assume that the CCAC-conjecture with stabilizations holds for $\mathcal{R}$. It is easy to see that every operation (CT1), (CCT2), (CT3), (CCT4) is a composition of (T1)–(T4). Hence, the AC-conjecture with stabilizations also holds for $\mathcal{R}$.

(b) As a counterexample to the CCAC-conjecture, we use the presentation

$$\langle\, a, b \,\|\, a^2b^{-3}, abab^{-1}a^{-1}b^{-1} \,\rangle,$$

where $(a^2b^{-3}, abab^{-1}a^{-1}b^{-1}) = (W_1, W_2)$ is the pair of Section 3. As was observed in Section 3, if V is a common subword of cyclic permutations of words $W_1^{\varepsilon_1}$ and $W_2^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, then $|V| \leq 2 < \min(|W_1|, |W_2|) = 5$. Therefore, no operation of type (CCT2) is applicable to any pair obtained from (W_1, W_2) by a sequence of operations (CT1), (CT3). Since operations (CT1), (CT3) do not change the length $|W_1| + |W_2|$, this proves that (W_1, W_2) cannot be turned into (a, b) by operations (CT1), (CCT2), (CT3). Theorem 4.1 is proved. □

In conclusion, we recall that the Andrews–Curtis conjecture with stabilizations is known to hold for presentations that come from spines of the 3-sphere, and it would be of interest to find out whether there is an upper bound on the number of operations (T1)–(T4) in this situation. Note that such a computable bound for spine presentations associated with 3-manifolds, together with the 3-dimensional Poincaré conjecture, would imply a purely algebraic algorithm to recognize the 3-sphere and to detect the triviality of spine presentations associated with 3-manifolds. It might be the case that available algorithms for recognition of the 3-sphere, together with analysis of their computational complexity (see [8], [11], [15], [25], [26]) would be useful towards this goal.

Acknowledgement

The author wishes to thank the referee for many meticulous remarks.

References

[1] J. J. Andrews and M. L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc. **16** (1965), 192–195, DOI 10.2307/2033843. MR0173241

[2] J. J. Andrews and M. L. Curtis, *Extended Nielsen operations in free groups*, Amer. Math. Monthly **73** (1966), 21–28, DOI 10.2307/2313917. MR0195928

[3] Alexandre V. Borovik, Alexander Lubotzky, and Alexei G. Myasnikov, *The finitary Andrews-Curtis conjecture*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 15–30, DOI 10.1007/3-7643-7447-0_2. MR2195451

[4] R. G. Burns and Olga Macedońska, *Balanced presentations of the trivial group*, Bull. London Math. Soc. **25** (1993), no. 6, 513–526, DOI 10.1112/blms/25.6.513. MR1245076

[5] George Havas and Colin Ramsay, *Breadth-first search and the Andrews-Curtis conjecture*, Internat. J. Algebra Comput. **13** (2003), no. 1, 61–68, DOI 10.1142/S0218196703001365. MR1970867

[6] Cynthia Hog-Angeloni and Wolfgang Metzler, *The Andrews-Curtis conjecture and its generalizations*, Two-dimensional homotopy and combinatorial group theory, London Math. Soc. Lecture Note Ser., vol. 197, Cambridge Univ. Press, Cambridge, 1993, pp. 365–380, DOI 10.1017/CBO9780511629358.014. MR1279186

[7] Sergei V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. **4** (1994), no. 1-2, ii+308, DOI 10.1142/S0218196794000026. MR1283947

[8] S. V. Ivanov, *Recognizing the 3-sphere*, Illinois J. Math. **45** (2001), no. 4, 1073–1117. MR1894888

[9] S. V. Ivanov, *On Rourke's extension of group presentations and a cyclic version of the Andrews-Curtis conjecture*, Proc. Amer. Math. Soc. **134** (2006), no. 6, 1561–1567, DOI 10.1090/S0002-9939-05-08450-9. MR2204265

[10] S. V. Ivanov, *On balanced presentations of the trivial group*, Invent. Math. **165** (2006), no. 3, 525–549, DOI 10.1007/s00222-005-0497-1. MR2242626

[11] S. V. Ivanov, *The computational complexity of basic decision problems in 3-dimensional topology*, Geom. Dedicata **131** (2008), 1–26, DOI 10.1007/s10711-007-9210-4. MR2369189

[12] Martin Lustig, *Nielsen equivalence and simple-homotopy type*, Proc. London Math. Soc. (3) **62** (1991), no. 3, 537–562, DOI 10.1112/plms/s3-62.3.537. MR1095232

[13] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin-New York, 1977. MR0577064

[14] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966. MR0207802

[15] S. V. Matveev, *Algorithms for the recognition of the three-dimensional sphere (after A. Thompson)* (Russian, with Russian summary), Mat. Sb. **186** (1995), no. 5, 69–84, DOI 10.1070/SM1995v186n05ABEH000037; English transl., Sb. Math. **186** (1995), no. 5, 695–710. MR1341085

[16] A. G. Myasnikov, *Extended Nielsen transformations and the trivial group* (Russian), Mat. Zametki **35** (1984), no. 4, 491–495. MR744511

[17] Alexei D. Miasnikov, *Genetic algorithms and the Andrews-Curtis conjecture*, Internat. J. Algebra Comput. **9** (1999), no. 6, 671–686, DOI 10.1142/S0218196799000370. MR1727164

[18] Alexei D. Myasnikov, Alexei G. Myasnikov, and Vladimir Shpilrain, *On the Andrews-Curtis equivalence*, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 183–198, DOI 10.1090/conm/296/05074. MR1921712

[19] A. Yu. Olʹshanskiĭ, *Geometry of defining relations in groups*, translated from the 1989 Russian original by Yu. A. Bakhturin, Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991. MR1191619

[20] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv.org, November 11, 2002.

[21] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv.org, March 10, 2003.
[22] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv.org, July 17, 2003.
[23] Elvira Strasser Rapaport, *Remarks on groups of order* 1, Amer. Math. Monthly **75** (1968), 714–720, DOI 10.2307/2315181. MR0236251
[24] Elvira Strasser Rapaport, *Groups of order* 1*: Some properties of presentations*, Acta Math. **121** (1968), 127–150, DOI 10.1007/BF02391911. MR0229704
[25] Joachim H. Rubinstein, *An algorithm to recognize the* 3*-sphere*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 601–611. MR1403961
[26] Abigail Thompson, *Thin position and the recognition problem for* S^3, Math. Res. Lett. **1** (1994), no. 5, 613–630, DOI 10.4310/MRL.1994.v1.n5.a9. MR1295555
[27] Perrin Wright, *Group presentations and formal deformations*, Trans. Amer. Math. Soc. **208** (1975), 161–169, DOI 10.2307/1997282. MR0380813

Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: ivanov@illinois.edu