ONE DIMENSIONAL ESTIMATES
FOR THE BERGMAN KERNEL
AND LOGARITHMIC CAPACITY

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(Communicated by Filippo Bracci)

ABSTRACT. Carleson showed that the Bergman space for a domain on the plane is trivial if and only if its complement is polar. Here we give a quantitative version of this result. One is the Suita conjecture, established by the first-named author in 2012, and the other is an upper bound for the Bergman kernel in terms of logarithmic capacity. We give some other estimates for those quantities as well. We also show that the volume of sublevel sets for the Green function is not convex for all regular non-simply connected domains, generalizing a recent example of Fornæss.

1. INTRODUCTION

For $w \in \Omega$, where Ω is a domain in $\mathbb{C}$, Carleson [7] (see also [8]) showed the Bergman space $A^2(\Omega)$ of square integrable holomorphic functions in Ω is trivial if and only if the complement $\mathbb{C} \setminus \Omega$ is polar. The estimate conjectured by Suita [12] and proved in [4]

$$(1) \qquad c_\Omega(w)^2 \le \pi K_\Omega(w), \quad w \in \Omega,$$

gives a quantitative version of one of the implications. Here

$$(2) \qquad c_\Omega(w) = \exp\left(\lim_{z \to w} \big(G_\Omega(z,w) - \log|z-w|\big)\right)$$

is the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to w,

$$G_\Omega(z,w) = \sup\{u(z) \colon u \in SH(\Omega),\ u < 0,\ \limsup_{\zeta \to w}\left(u(\zeta) - \log|\zeta - w|\right) < \infty\}$$

is the (negative) Green function,

$$K_\Omega(w) = \sup\{|f(w)|^2 \colon f \in A^2(\Omega),\ ||f|| \le 1\}$$

is the Bergman kernel on the diagonal, and $||f|| = ||f||_{L^2(\Omega)}$.

Our first result is the following upper bound for the Bergman kernel:

Theorem 1. *Let Ω be a domain in $\mathbb{C}$ and $w \in \Omega$. Assume that $0 < r \le \delta_\Omega(w) :=$ dist $(w, \partial\Omega)$. Then*

$$K_\Omega(w) \le \frac{1}{-2\pi r^2 \max\limits_{z \in \overline{\Delta}(w,r)} G_\Omega(z,w)}.$$

As a consequence we will obtain the following quantitative version of the other implication in the Carleson characterization:

Theorem 2. *There exists a uniform constant $C > 0$ such that for $w \in \Omega$, where Ω is a domain in $\mathbb{C}$, we have*

$$K_\Omega(w) \le \frac{C}{\delta_\Omega(w)^2 \log\left(1/(\delta_\Omega(w)c_\Omega(w))\right)}.$$

We will also consider the following counterparts of the Bergman kernel for higher derivatives for $j = 0, 1, \ldots$:

$$K_\Omega^{(j)}(w) := \sup\{|f^{(j)}(w)|^2 : f \in A^2(\Omega), \ \|f\| \le 1,$$
$$f(0) = f'(0) = \cdots = f^{(j-1)}(0) = 0\}.$$

We will prove the following generalization of (1):

Theorem 3. *For $w \in \Omega \subset \mathbb{C}$ and $j = 0, 1, 2, \ldots$ we have*

$$K_\Omega^{(j)}(w) \ge \frac{j!(j+1)!}{\pi}(c_\Omega(w))^{2j+2}.$$

The inequality is optimal; one can easily check that the equality holds for $\Omega = \Delta$, the unit disc, and $w = 0$.

It is clear that the dimension of $A^2(\Omega)$ is infinite if and only if, for a given w, there exists infinitely many j's such that $K_\Omega^{(j)}(w) > 0$. Therefore, Theorem 3 gives a quantitative version of a result of Wiegerinck [13] who showed that if $\mathbb{C} \setminus \Omega$ is not polar, then $A^2(\Omega)$ is infinitely dimensional.

Since the proof of Theorem 2 also easily gives the upper bound

$$K_\Omega^{(j)}(w) \le \frac{C_j}{\delta_\Omega(w)^{2+j} \log\left(1/(\delta_\Omega(w)c_\Omega(w))\right)}, \quad w \in \Omega,$$

and by Proposition 6 below we have the following characterization of domains in dimension one:

Theorem 4. *For $w \in \Omega \subset \mathbb{C}$ and $j = 0, 1, 2, \ldots$ the following are equivalent:*
  i) $\mathbb{C} \setminus \Omega$ *is not polar;*
  ii) $K_\Omega^{(j)}(w) > 0$;
  iii) $\log K_\Omega^{(j)}$ *is smooth and strongly subharmonic;*
  iv) $A^2(\Omega) \ne \{0\}$;
  v) $\dim A^2(\Omega) = \infty$.

A different proof of the Suita conjecture (1) was given in [5]. It follows from the lower bound

$$(3) \qquad\qquad K_\Omega(w) \ge \frac{1}{e^{-2t}\lambda(\{G_\Omega(\cdot, w) < t\})}$$

for $t < 0$. This inequality was proved in [5] using the tensor power trick which requires a corresponding inequality for pseudoconvex domains in $\mathbb{C}^n$ for arbitrary n. As noticed by Lempert (see also [3]), (3) can also be proved using the variational formula for the Bergman kernel in $\mathbb{C}^2$ of Maitani-Yamaguchi [11] (generalized by Berndtsson [2] to higher dimensions). Both proofs therefore make crucial use of several complex variables. It would be interesting to find a purely one dimensional proof of (3).

It was shown in [6] that the right-hand side of (3) is non-increasing in t (it is an open problem in higher dimensions). Also, a more general conjecture was given, namely that the function

$$(4) \qquad (-\infty, 0) \ni t \longmapsto \log \lambda(\{G_\Omega(\cdot, w) < t\})$$

is convex. A counterexample was found by Fornæss [10]. It was also shown numerically in [1] that the conjecture does not hold in an annulus. Here we will generalize and simplify both results proving the following:

Theorem 5. *Assume that $w \in \Omega$, where Ω is a domain in $\mathbb{C}$, are such that $\nabla G(z_0) = 0$ for some $z_0 \in \Omega \setminus \{w\}$, where $G = G_\Omega(\cdot, w)$. Then the function (4) is not convex near $t_0 = G(z_0)$.*

Note that for example any regular domain Ω which is not simply connected satisfies the assumption of Theorem 5 for any w: it is enough to take maximal t_0 such that $\{G < t_0\}$ is simply connected. Then there exists z_0 such that $\nabla G(z_0) = 0$.

2. Upper bounds for the Bergman kernel

In this section we will prove Theorems 1 and 2.

Proof of Theorem 1. We may assume that Ω is bounded and smooth, $w = 0$, and $r < \delta_\Omega(0)$. Take $f \in A^2(\Omega)$. Without loss of generality we may take such an f which is defined in a neighborhood of $\overline{\Omega}$. Let $u \in C^\infty(\overline{\Omega} \setminus \Delta_r)$, where $\Delta_r := \Delta(0, r)$, be harmonic in $\Omega \setminus \overline{\Delta}_r$ and such that $u = 1$ on $\partial\Omega$ and $u = 0$ on $\partial\Delta_r$. Then

$$f(0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z} \, dz = \frac{1}{2\pi i} \int_{\partial(\Omega \setminus \overline{\Delta}_r)} \frac{fu}{z} \, dz = \frac{1}{\pi} \int_{\Omega \setminus \overline{\Delta}_r} \frac{fu_{\bar{z}}}{z} \, d\lambda.$$

Therefore,

$$|f(0)|^2 \leq \frac{||f||^2}{\pi^2 r^2} \int_{\Omega \setminus \overline{\Delta}_r} |u_{\bar{z}}|^2 d\lambda = \frac{||f||^2}{4\pi^2 r^2} \int_{\partial\Omega} u_n \, d\sigma,$$

where u_n denotes the outer normal derivative of u at $\partial\Omega$. Denoting $G = G_\Omega(\cdot, 0)$, we have

$$\frac{G}{-\max\limits_{\overline{\Delta}_r} G} + 1 \leq u,$$

and therefore on $\partial\Omega$

$$u_n \leq \frac{G_n}{-\max\limits_{\overline{\Delta}_r} G}.$$

The required estimate now follows from the fact that

$$\int_{\partial\Omega} G_n d\sigma = 2\pi.$$

$\square$

Proof of Theorem 2. Denote $G = G_\Omega(\cdot, w)$, $R = \delta_\Omega(w)$ and assume that $0 < r < R$. Then by the Poisson formula

$$\max_{\overline{\Delta}(w,r)} G \leq \frac{R-r}{2\pi(R+r)} \int_0^{2\pi} G(w + Re^{it}) dt = \frac{R-r}{R+r} \log(Rc_\Omega(w)).$$

By Theorem 1

$$K_\Omega(w) \leq \frac{R+r}{2\pi(R-r)r^2 \log(1/(Rc_\Omega(w)))}.$$

We can now take for example $r = R/2$ and the estimate follows. $\qquad\square$

The smallest constant the above proof gives will be obtained for $r = (\sqrt{5}-1)R/2$; then

$$C = \frac{11 + 5\sqrt{5}}{4\pi}.$$

3. PROOF OF THE LOWER BOUND FOR $K_\Omega^{(j)}$

In this section we prove Theorem 3. We follow the method from [4]. We could have also used another method from [5] but this would require us to go to several complex variables and we prefer to have a purely one dimensional argument.

Proof of Theorem 3. We assume that $w = 0$, Ω is bounded and smooth, and denote $G = G_\Omega(\cdot, 0)$. Set

$$\alpha := \frac{\partial}{\partial \bar{z}} \left(z^j \chi(|z|) \right) = \frac{z^{j+1} \chi'(|z|)}{2|z|}$$

and

$$\varphi := (2j+2)G + \eta \circ G, \quad \psi := \gamma \circ G,$$

where $\chi \in C^{0,1}((0,\infty))$, $\eta \in C^{1,1}((-\infty,0))$, $\gamma \in C^{0,1}((-\infty,0))$ will be defined later. We assume that η is convex and non-decreasing (so that φ is subharmonic), $(\gamma')^2 < \eta''$, and that $(\gamma' \circ G)^2 \leq \delta \eta'' \circ G$ on the support of α for some constant δ with $0 < \delta < 1$. Then by Theorem 2 in [4] one can find $u \in L^2_{loc}(\Omega)$ such that $F := z^j \chi(|z|) - u$ is holomorphic and

$$(5) \qquad \int_\Omega |u|^2 \, \Gamma \circ G \, d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega \frac{|\alpha|^2}{\eta'' \circ G \, |G_z|^2} \, e^{2\psi - \varphi} \, d\lambda,$$

where

$$\Gamma := \left(1 - \frac{(\gamma')^2}{\eta''} \right) e^{2\gamma - \eta - (2j+2)t}.$$

Take $\varepsilon > 0$ and assume that $\chi(|z|) \subset \{|z| \leq \varepsilon\}$. We choose $T = T(\varepsilon) < 0$ such that $\{|z| \leq \varepsilon\} \subset \{G \leq T\}$. Since $|G - \log|z|| \leq C_0$ near 0, we may take $T := \log \varepsilon + C_0$.

Similarly as in [4] for $s < 0$ we define

$$\eta_0(s) := -\log(-s + e^s - 1),$$
$$\gamma_0(s) := -\log(-s + e^s - 1) + \log(1 - e^s),$$

so that

$$\left(1 - \frac{(\gamma_0')^2}{\eta_0''} \right) e^{2\gamma_0 - \eta_0 - s} = 1$$

and

$$(6) \qquad \lim_{s \to -\infty} (2\gamma_0(s) - \eta_0(s) - \log \eta_0'(s)) = 0.$$

We now set

$$\eta(t) := \begin{cases} \eta_0\big((2j+2)t\big), & t \geq T, \\ -\delta \log(T - t + a) + b, & t < T, \end{cases}$$

and

$$\gamma(t) := \begin{cases} \gamma_0\big((2j+2)t\big), & t \geq T, \\ -\delta \log(T - t + a) + \widetilde{b}, & t < T, \end{cases}$$

where $\delta = \delta(\varepsilon) > 0$ will be determined later and $a, b, \widetilde{b}$ are uniquely determined by the conditions $\eta \in C^{1,1}$, $\gamma \in C^{0,1}$:

$$a = a(\varepsilon) = \frac{\delta}{(2j+2)\eta_0'\big((2j+2)T\big)},$$

$$b = b(\varepsilon) = \eta_0\big((2j+2)T\big) + \delta \log a,$$

$$\widetilde{b} = \widetilde{b}(\varepsilon) = \gamma_0\big((2j+2)T\big) + \delta \log a.$$

We see that if we choose $\delta = \sqrt{-T}$, then $\delta(\varepsilon) \to 0$ and $a(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

On $(-\infty, T)$ we have $(\gamma')^2 = \delta \eta''$ and

$$\Gamma = (1 - \delta)e^{2\widetilde{b} - b}\frac{e^{-(2j+2)t}}{(T - t + a)^\delta},$$

so that $|z|^{2j}\Gamma \circ G$ is not locally integrable near 0. By (5) it implies that $F(0) = F'(0) = \cdots = F^{(j-1)}(0) = 0$ and $F^{(j)}(0) = j! \chi(0)$. One can also check that $\Gamma \geq 1$ on $(-\infty, T)$.

Since $|2G_z - 1/z| \leq C_1$ near 0, we have $2|z||G_z| \geq 1 - C_1\varepsilon$ on $\{|z| \leq \varepsilon\}$. There we also have

$$\frac{e^{2\psi - \varphi}}{\eta'' \circ G} = \frac{e^{2\widetilde{b} - b}}{\delta}(T - G + a)^{2-\delta}e^{-(2j+2)G}$$

$$\leq \frac{e^{2\widetilde{b} - b}}{\delta}(\log \varepsilon - \log |z| + 2C_0 + a)^{2-\delta}e^{-(2j+2)G}.$$

Therefore, the right-hand side of (5) can be bounded from above by

$$(7) \quad \frac{(1 + \sqrt{\delta})e^{2\widetilde{b} - b}\mathcal{A}(\varepsilon)}{\delta(1 - \sqrt{\delta})(c_\Omega(0))^{2j+2}} \int_{\{|z| \leq \varepsilon\}} (\chi'(|z|))^2 (\log \varepsilon - \log |z| + 2C_0 + a)^{2-\delta}d\lambda,$$

where $\mathcal{A}(\varepsilon) \to 1$ as $\varepsilon \to 0$. The optimal choice for χ is

$$\chi(r) = (2C_0 + a)^{\delta-1} - (\log \varepsilon - \log r + 2C_0 + a)^{\delta-1};$$

then (7) takes the form

$$\frac{2\pi(1 + \sqrt{\delta})^2 e^{2\widetilde{b} - b}(2C_0 + a)^{\delta-1}\mathcal{A}(\varepsilon)}{\delta(c_\Omega(0))^{2j+2}}.$$

Note that

$$\frac{e^{2\widetilde{b} - b}(2C_0 + a)^{1-\delta}}{\delta} = \left(\frac{2C_0 + a}{a}\right)^{1-\delta}\frac{e^{2\gamma_0(S) - \eta_0(S) - \log \eta_0'(S)}}{2j+2},$$

where $S = (2j+2)T$. Combining this with (6) and the fact that $\chi(0) = (2C_0+a)^{\delta-1}$

we will obtain

$$\liminf_{\varepsilon \to 0} \frac{|F^{(j)}(0)|^2}{||F||^2} = \frac{j!(j+1)!}{\pi}(c_\Omega(w))^{2j+2}.$$

$\square$

Using standard methods we will also prove the following formula for the Laplacian of $\log K_\Omega^{(j)}$. It is of course well known for $j = 0$ and the proof is essentially the same in general.

Proposition 6. *For a domain Ω in $\mathbb{C}$ such that $\mathbb{C} \setminus \Omega$ is not polar and $j = 0, 1, \ldots$ we have*

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\log K_\Omega) = \frac{K_\Omega^{(j+1)}}{K_\Omega^{(j)}}.$$

Proof. Denote $H_0 = A^2(\Omega)$ and for a fixed $w \in \Omega$ and $k = 1, 2, \ldots$ set

$$H_k := \{f \in A^2(\Omega) : f(w) = f'(w) = \cdots = f^{(k-1)}(w) = 0\}.$$

Since H_k is of codimension at most 1 in H_{k-1}, we can find an orthonormal system $\varphi_0, \varphi_1, \ldots$ in $A^2(\Omega)$ such that $\varphi_k \in H_k$ for all k. This means that $\varphi_l^{(j)}(w) = 0$ for $l > j$. For $f \in H_j$ we have

$$f = \sum_{l \geq j} \langle f, \varphi_l \rangle \, \varphi_l.$$

Therefore,

$$K_\Omega^{(j)}(z) = \sum_{l \geq j} |\varphi_l^{(j)}(z)|^2$$

and

$$K_\Omega^{(j)}(w) = |\varphi_j^{(j)}(w)|^2.$$

Since

$$(\log K)_{z\bar{z}} = \frac{KK_{z\bar{z}} - |K_z|^2}{K^2}$$

and

$$(K_\Omega^{(j)})_{z\bar{z}}(w) = |\varphi_j^{(j+1)}(w)|^2 + |\varphi_{j+1}^{(j+1)}(w)|^2,$$

$$(K_\Omega^{(j)})_z(w) = \varphi_j^{(j+1)}(w) \, \overline{\varphi_j^{(j)}(w)},$$

we will obtain

$$(\log K_\Omega^{(j)})_{z\bar{z}}(w) = \frac{|\varphi_{j+1}^{(j+1)}(w)|^2}{|\varphi_j^{(j)}(w)|^2}$$

and the proposition follows.

$\square$

4. PROOF OF THEOREM 5

Let $t_j \to t_0$ be a sequence of regular values for G. It will be enough to show that $\gamma'(t_j) \to \infty$, where $\gamma(t) = \lambda(\{G_\Omega(\cdot, w) < t\})$. By the co-area formula we have

$$\gamma(t) = \int_{-\infty}^{t} \int_{\{G=s\}} \frac{d\sigma}{|\nabla G|} \, ds,$$

and, therefore,

$$\gamma'(t_j) = \int_{\{G=t_j\}} \frac{d\sigma}{|\nabla G|}.$$

It is convenient to assume that $z_0 = 0$. Since G is harmonic in $\Omega \setminus \{w\}$, it follows that there exists a holomorphic h near 0 such that $h(0) \neq 0$ and for some $n \geq 2$ we have

$$G(z) = t_0 + \operatorname{Re}(z^n h(z)).$$

It follows that near 0 we have

$$|\nabla G(z)| \leq C_1 |z|^{n-1}.$$

We can also find a biholomorphic F near 0 such that $G(F(\zeta)) = t_0 + \operatorname{Re}(\zeta^n)$. We then have

$$|\nabla G(F(\zeta))| \leq C_2 |\zeta|^{n-1}$$

and for some $r > 0$

$$\int_{\{G=t_j\}} \frac{d\sigma}{|\nabla G|} \geq \frac{1}{C_3} \int_{\{\zeta \in \Delta(0,r) \colon \operatorname{Re}(\zeta^n) = t_j - t_0\}} \frac{d\sigma}{|\zeta|^{n-1}} \longrightarrow \infty$$

as $j \to \infty$. $\qquad\square$

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