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EXAMPLES OF NEW NONSTANDARD HULLS OF TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper, we construct new nonstandard hulls of topological vector spaces using convex subrings of ${}^*\mathbb{R}$ (or ${}^*\mathbb{C}$) and we show that such spaces are complete. Some examples of locally convex spaces are provided to illustrate our construction. Namely, we show that the new nonstandard hull of the space of polynomials is the algebra of Colombeau's entire holomorphic generalized functions. The proof is based on the existence of global representatives of entire generalized functions.

1. INTRODUCTION

The methods of nonstandard analysis have been applied to topology with illuminating and satisfying results; see Robinson [19] and Luxemburg[16, 17]. They provide an alternative to the classical description of a topological space by open sets. The notion of *monad* is a fundamental concept which encodes a topology and most of the subsequent development comes from their properties. Meanwhile, constructing nonstandard hulls turned out to be an effective method for obtaining new mathematical objects from those available. For metric spaces, this was carried out by Robinson. For normed spaces and uniform spaces, this was accomplished by Luxemburg and by Henson and Moore [7] for topological vector spaces. In the case of measure spaces, it is the Loeb spaces that play the role of nonstandard hulls [9]. Roughly speaking, the nonstandard hull is the quotient of the set of "bounded" elements by the equivalence relation of being infinitely close.

We construct new topologies on ${}^*E$, a nonstandard extension of a topological vector space E. Then, we define the set of $\mathbb{F}$-bounded points and we construct $\widehat{E}$, the $\mathbb{F}$-nonstandard hull of E. The space $\widehat{E}$ endowed with the quotient topology has the structure of a topological vector space over $\widehat{\mathbb{F}}$. We note that if $\mathbb{F} = {}^b\mathbb{R}$ or ${}^b\mathbb{C}$, then $\widehat{E}$ is the classical nonstandard hull of E constructed by Henson and Moore in [7]. Finally, we provide some examples of $\mathbb{F}$-nonstandard hulls of locally convex spaces.

The first is given by the $\mathbb{F}$-nonstandard hull of $\mathcal{E}(\Omega)$, the space of smooth functions over Ω, an open subset of $\mathbb{R}^n$. We obtain the nonstandard counterpart of the space of Colombeau's algebra of generalized functions. Such spaces are investigated by Todorov; see [21]. The second is the $\mathbb{F}$-nonstandard hull of $\mathbb{C}[T_1, \ldots, T_n]$,

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the space of polynomials in n-indeterminates. In this case, we get the nonstandard counterpart of the space Colombeau's holomorphic generalized functions. To achieve our goal, we prove that any entire generalized function has an entire generalized representative. Our proof is an adaptation to higher dimensions of [18], where the authors showed that any generalized holomorphic function has a global representative in any domain of $\mathbb{C}$. On polydiscs, we approximate internal holomorphic functions by internal polynomials. The degrees of such polynomials will be chosen in ${}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$, the monoid of $({}^b\mathbb{C}, \mathbb{F})$ naturals. One can easily check that an internal polynomial of *degree in* ${}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$ is $({}^b\mathbb{C}, \mathbb{F})$-bounded (resp. $({}^b\mathbb{C}, \mathbb{F})$-infinitesimal) if and only if its coefficients belongs to $\mathbb{F}$ (resp. ${}^i\mathbb{F}$). More generally, the relationship between boundedness of entire functions and their coefficients is established in [13].

Next, we provide a simple proof of the identity theorem for generalized entire functions using its Taylor expansion as a generalized power series over ${}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$.

These results constitute a decisive step toward the development of a theory of holomorphic generalized functions using nonstandard analysis. The need to go in such a theory to convex subrings $\mathbb{F}$ than only the ring M_ρ has been apparent in many applications and has inspired other authors to use subrings defined by asymptotic scales; see [3]. Characteristically, nonstandard analysis allows for a general framework at a minimal technical cost. The introduction of the sets of hypernatural numbers ${}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$ is a natural language to describe the behaviour of holomorphic generalized functions in such a general framework.

Finally, we investigate the $\mathbb{F}$-nonstandard hull of $\mathbb{Z}$, the ring of integers, equipped with the p-adic norm.

2. Preliminaries

This section of preliminary notions provides a background necessary for the comprehension of the paper.

2.1. **Nonstandard analysis.** The approach to nonstandard analysis that we use in the present paper follows that of Stroyan and Luxemburg [20]. One starts with a superstructure $V(S) = \bigcup V_n(S)$ over set S, which is often not specified explicitly but chosen large enough to contain all objects under the consideration, real numbers, necessary vector spaces, etc. We suppose that for the enlargement ${}^*S$ of the set S of basic elements, the natural embedding ${}^* : V(S) \to V({}^*S)$ satisfies the following principles:

The extension principle. ${}^*s = s$ for all $s \in S$.

The transfer principle. Let $\Phi(x_1, x_2, \ldots, x_n)$ be a bounded formula of the superstructure $V(S)$ and let $A_1, A_2, \ldots, A_n$ be elements of the superstructure $V(S)$.

Then the assertion $\Phi(A_1, A_2, \ldots, A_n)$ about elements of $V(S)$ holds true if and only if the assertion $\Phi({}^*A_1, {}^*A_2, \ldots, {}^*A_n)$ about elements of $V({}^*S)$ does.

Let $V({}^*S)$ be a nonstandard enlargement of a superstructure $V(S)$. An element $x \in V({}^*S)$ is called *standard* if $x = {}^*X$ for some $X \in V(S)$; *internal* if $x \in {}^*X$ for some $X \in V(S)$; *external* if x is not internal.

It is well known that a nonstandard enlargement $V({}^*S)$ of $V(S)$ can be chosen so that the following principle is satisfied; see for instance Goldblatt [6].

The general saturation principle. If a family $\{A_\gamma\}_\gamma \in \Gamma$ of internal sets possesses the finite intersection property and $\mathrm{card}(\Gamma) < \mathrm{card}(\mathrm{V(S)})$, then $\bigcap_{\gamma\in\Gamma} A_\gamma \neq \phi$.

In what follows, we always deal with nonstandard enlargements satisfying the general saturation principle (they are also called *polysaturated*).

2.2. **Convex subrings of** $^*\mathbb{R}$. Let $^*\mathbb{R}$ be a nonstandard extension of the field of real numbers $\mathbb{R}$ and $^i\mathbb{R}$, $^b\mathbb{R}$ and $^\infty\mathbb{R}$ stand for the sets of infinitesimals, bounded (or finite) numbers and infinitely large numbers in $^*\mathbb{R}$, respectively. For a comprehensive introduction to nonstandard analysis, the reader is referred to [6, 9, 14, 20].

First, we recall the definition and some properties of convex subrings of $^*\mathbb{R}$.

Definition 2.1. We say that $\mathbb{F}$ is a convex in $^*\mathbb{R}$ if

$$(\forall x \in {}^*\mathbb{R})(\forall \xi \in \mathbb{F})(|x| \leq |\xi| \Rightarrow x \in \mathbb{F}).$$

Remark 2.2. There is a one-to-one correspondence between convex subrings of $^*\mathbb{C}$ and those of $^*\mathbb{R}$: let $\mathbb{F}'$ be a convex subring of $^*\mathbb{C}$; then $\mathbb{F} = \mathbb{F}' \cap {}^*\mathbb{R}$ is a convex subring of $^*\mathbb{R}$. Conversely, let $\mathbb{F}$ be a convex subring of $^*\mathbb{R}$; then $\mathbb{F}' = \{a \in {}^*\mathbb{C} : |a| \in \mathbb{F}\}$ is a convex subring of $^*\mathbb{C}$.

Using the fact that any subring of $^*\mathbb{R}$ contains $\mathbb{Z}$, it is clear that if $\mathbb{F}$ is a convex subring of $^*\mathbb{R}$, then $\mathbb{F}$ contains $^b\mathbb{R}$. We prove that the converse remains true.

Proposition 2.3. *Let* $\mathbb{F}$ *be a subring of* $^*\mathbb{R}$. *Then* $\mathbb{F}$ *is convex if and only if* $\mathbb{F}$ *contains* $^b\mathbb{R}$.

Proof. Let $x \in {}^*\mathbb{R}$ and $\xi \in \mathbb{F} \setminus \{0\}$ such that $|x| \leq |\xi|$. Thus $x/\xi \in {}^b\mathbb{R}$, and we deduce that $x = (x/\xi).\xi \in {}^b\mathbb{R}.\mathbb{F} \subset \mathbb{F}$, that is, $\mathbb{F}$ is a convex subring of $^*\mathbb{R}$. ☐

Therefore, any convex subring $\mathbb{F}$ of $^*\mathbb{R}$ is a valuation ring. For the remainder of this paper we fix the following notation: $^i\mathbb{F}$ denotes the maximal ideal of $\mathbb{F}$, and $^a\mathbb{F} = \mathbb{F} \setminus {}^i\mathbb{F}$, the set of appreciable elements of $\mathbb{F}$ and $^\infty\mathbb{F} = {}^*\mathbb{R} \setminus \mathbb{F}$.

2.2.1. *Examples.*

(i) (Finite numbers). The ring of bounded nonstandard real numbers $^b\mathbb{R}$ is a convex subring of $^*\mathbb{R}$. Its maximal ideal is $^i\mathbb{R}$, the set of infinitesimals.

(ii) (Nonstandard real numbers). The field of nonstandard real numbers $^*\mathbb{R}$ is (trivially) a convex subring of $^*\mathbb{R}$. Its maximal ideal is $\{0\}$.

(iii) (Robinson rings). Let ρ be a positive infinitesimal in $^*\mathbb{R}$. The ring of the ρ-moderate nonstandard numbers is defined by

$$M_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N}\}.$$

M_ρ is a convex subring of $^*\mathbb{R}$. For its maximal ideal we have

$$N_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{n} \text{ for all } n \in \mathbb{N}\}.$$

(iv) Let ω be an infinite positive number in $^*\mathbb{R}$. Then

$$P_\omega = \{x \in {}^*\mathbb{R} : |x| \leq n^{\omega} \text{ for some } n \in \mathbb{N}\},$$

P_ω is a convex subring of $^*\mathbb{R}$, and its maximal ideal is given by

$$^iP_\omega = \{x \in {}^*\mathbb{R} : |x| \leq \frac{1}{n^\omega} \text{ for all } n \in \mathbb{N}\}.$$

One can easily check that $P_\omega = M_{\exp(-\omega)}$.

Definition 2.4. A sequence $(\lambda_n)_{n\in\mathbb{N}}$ of infinitesimal positive numbers (except possibly $n = 0$) is called an asymptotic scale if it satisfies the following conditions:

(i) for all $n \in \mathbb{N}$, $\frac{\lambda_{n+1}}{\lambda_n} \in {}^i\mathbb{R}$,
(ii) for every $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $\lambda_k \leq \lambda_n^2$.

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ extends to $(\lambda_n)_{n\in\mathbb{Z}}$ by putting

$$\lambda_{-n} = \frac{1}{\lambda_n} \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

Let F_n be the principal fractional ideal generated by λ_n, that is, for $n \in \mathbb{Z} \setminus \{0\}$ $F_n := \lambda_n{}^b\mathbb{R}$. One can easily check that if (λ_n) is an asymptotic scale, then

$$\mathbb{F} = \bigcup_{n\in\mathbb{Z}} F_n = \{x \in {}^*\mathbb{R} : x \in \lambda_{-n}{}^b\mathbb{R} \text{ for some } n \in \mathbb{N}\}$$

is a convex subring of ${}^*\mathbb{R}$ and its maximal ideal is given by

$${}^i\mathbb{F} = \bigcap_{n\in\mathbb{N}} F_n = \{x \in {}^*\mathbb{R} : x \in \lambda_n{}^b\mathbb{R} \text{ for all } n \in \mathbb{N}\}.$$

Using convex subrings of ${}^*\mathbb{R}$, a variety of fields $\widehat{\mathbb{F}}$ is constructed by Todorov [21]. These fields are called $\mathbb{F}$-asymptotic hulls and their elements $\mathbb{F}$-asymptotic numbers. This construction can be viewed as a generalization of A. Robinson's theory of asymptotic numbers; see Lightstone-Robinson [15].

Definition 2.5. Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{R}$. The $\mathbb{F}$-asymptotic hull is the factor ring $\widehat{\mathbb{F}} = \mathbb{F}/{}^i\mathbb{F}$.

Let $\widehat{\mathrm{st}} : \mathbb{F} \longrightarrow \widehat{\mathbb{F}}$ stand for the corresponding quotient mapping, called the *quasi-standard mapping.*

If $x \in \mathbb{F}$, we shall often write $\widehat{x}$ instead of $\widehat{\mathrm{st}}(x)$ for the quasi-standard part of x. We can define an order relation in $\widehat{\mathbb{F}}$, inherited from the order in ${}^*\mathbb{R}$, by

$$\widehat{x} \leq \widehat{y} \text{ if there are representatives } x, y \text{ with } x \leq y.$$

Using the convexity of $\mathbb{F}$, the following proposition is straightforward.

Proposition 2.6. $(\widehat{\mathbb{F}}, \leq)$ *is a completely ordered field.*

3. Nonstandard hulls of topological vector spaces

3.1. **Nonstandard topologies on ${}^*E$.** Let E be a $\mathbb{K}$-topological vector space, where $\mathbb{K}$ stands either for $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{N}_0$ the filter of neighborhoods of 0 in E. Let ${}^b\mathbb{K}$ be the set of bounded elements of ${}^*\mathbb{K}$ and let ${}^i\mathbb{K}$ be the set of infinitesimals of ${}^*\mathbb{K}$. Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{K}$, that is, ${}^b\mathbb{K} \subset \mathbb{F} \subset {}^*\mathbb{K}$. Let us recall that ${}^a\mathbb{F} = \mathbb{F} \setminus {}^i\mathbb{F}$ denotes the set of appreciable elements of $\mathbb{F}$.

We define a family of topologies on ${}^*E$ parametrized by convex subrings of ${}^*\mathbb{K}$ as follows: for each p in ${}^*E$, let

$$\mathcal{V}_p({}^*E, \mathbb{F}) = \{p + r\ {}^*U\ : U \in \mathcal{N}_0 \text{ and } r \in {}^a\mathbb{F}\}.$$

We will often write $\mathcal{V}({}^*E)$ in place of $\mathcal{V}_0({}^*E, \mathbb{F})$.

Proposition 3.1. $\mathcal{V}({}^*E)$ *is a neighborhood basis of zero in the group* $({}^*E, +)$*.*

Proof. First, we have to show that $\mathcal{V}(^*E)$ is a filter base on $^*E$.

(i) $0 \in r^*U$ for any $U \in \mathcal{N}_0$ and $r \in {}^a\mathbb{F}$.

(ii) For any $U, V \in \mathcal{N}_0$ and $r, s \in {}^a\mathbb{F}$. Let U_0, V_0 be two balanced neighborhoods of 0, such that $U_0 \subset U$ and $V_0 \subset V$. For $W = U_0 \cap V_0$ and $t = \min(|r|, |s|)$, we have $t \in {}^a\mathbb{F}$ and $t^*W \subset r^*U \cap s^*V$.

(iii) For any $U \in \mathcal{N}_0$, there exists $V \in \mathcal{N}_0$ such that $V - V \subset U$. Thus for any $r \in {}^a\mathbb{F}$, we get $r^*V - r^*V \subset r^*U$. □

If (E, τ) is a topological vector space, we denote by $(^*E, \tau_{\mathbb{F}})$ the topology on $^*E$ generated by $\mathcal{V}_p(^*E, \mathbb{F})$.

We notice that for $\mathbb{F} = {}^b\mathbb{K}$, the topology on $^*E$ generated by $\mathcal{V}(^*E)$ coincides with the topology generated by the zero neighborhood basis $\{^*U : U \in \mathcal{N}_0\}$. The latter topology was defined by Henson and Moore in [7].

3.2. $\mathbb{F}-$bounded elements of $^*E$.

Definition 3.2.

(i) A point p of $^*E$ is $\mathbb{F}$-bounded if, for each neighborhood U of 0, there exists $r \in {}^a\mathbb{F}$ which satisfies $p \in r\,^*U$.
    The set of $\mathbb{F}$-bounded elements of $^*E$ will be denoted by $\mathbb{F}(^*E)$.

(ii) We define the $\mathbb{F}$-halo of 0 by
$$\mu_{\mathbb{F}}(0) = \bigcap_{U \in \mathcal{N}_0,\, r \in {}^a\mathbb{F}} r\,^*U = \bigcap_{r \in {}^a\mathbb{F}} r\,\mu(0),$$
where $\mu(0) = \bigcap_{U \in \mathcal{N}_0} {}^*U$ stands for the classical halo of 0 in $^*E$.

(iii) For any point $p \in {}^*E$, the $\mathbb{F}$-halo of p,
$$\mu_{\mathbb{F}}(p) = p + \mu_{\mathbb{F}}(0).$$

Remark 3.3.

(i) The $\mathbb{F}$-halo of p is exactly the closure of p with respect to the topology generated by $\mathcal{V}_p(^*E, \mathbb{F})$.

(ii) $\mu_{\mathbb{F}}(0)$ is closed under addition and under multiplication by elements of $\mathbb{F}$.

(iii) The set of $\mathbb{F}$-bounded elements of $^*\mathbb{K}$ is $\mathbb{F}$, i.e., $\mathbb{F}(^*\mathbb{K}) = \mathbb{F}$.

(iv) The topology generated by $\mathcal{V}_p(^*\mathbb{K}, \mathbb{F})$ coincides with the QS-topology on $^*\mathbb{K}$; see [12].

Theorem 3.4. *An element p of $^*E$ is $\mathbb{F}$-bounded if and only if $\lambda p \in \mu_{\mathbb{F}}(0)$ whenever $\lambda \in {}^i\mathbb{F}$.*

In particular, this shows that
$${}^i\mathbb{F}.\,\mathbb{F}(^*E) \subset \mu_{\mathbb{F}}(0).$$

Proof. Suppose that p is $\mathbb{F}$-bounded. Let U be a balanced neighborhood of 0. Then $p \in r_0{}^*U$ for some $r_0 \in {}^a\mathbb{F}$. Therefore, $p \in \omega^*U$ for every $\omega \in {}^\infty\mathbb{F}$. Given $r \in {}^a\mathbb{F}$ and $\lambda \in {}^i\mathbb{F}$, $\lambda \neq 0$. Let $\omega_0 = r/\lambda$. Clearly, $\omega_0 \in {}^\infty\mathbb{F}$ and $\lambda p \in \lambda\omega_0{}^*U \subset r^*U$. It follows that λp is in $\mu_{\mathbb{F}}(0)$ whenever λ is in ${}^i\mathbb{F}$.

Conversely, if $\lambda p \in \mu_{\mathbb{F}}(0)$ for every λ in ${}^i\mathbb{F}$ and if U is a neighborhood of 0, then the internal set $\mathcal{A} = \{\omega \in {}^*\mathbb{R} : p \in \omega^*U\}$ contains ${}^\infty\mathbb{F}$. Thus by the underflow principle, $\mathcal{A}$ must contain $r \in {}^a\mathbb{F}$. Therefore, the condition implies that p is $\mathbb{F}$-bounded. □

The following is an immediate consequence of Theorem 3.4 and Remark 3.3 (ii).

Corollary 3.5.

(i) *If* $p \in \mathbb{F}({}^*E)$*, then* $\mu_{\mathbb{F}}(p) \subset \mathbb{F}({}^*E)$*.*
(ii) $\mathbb{F}({}^*E)$ *is an* $\mathbb{F}$*-module.*

Theorem 3.6.

(i) $\mathbb{F}({}^*E)$ *is a topological* $\mathbb{F}$*-module, that is, the addition* $\mathbb{F}({}^*E) \times \mathbb{F}({}^*E) \to \mathbb{F}({}^*E)$ *and the scalar multiplication* $\mathbb{F} \times \mathbb{F}({}^*E) \to \mathbb{F}({}^*E)$ *are continuous.*
(ii) $\mathbb{F}({}^*E)$ *is closed in* ${}^*E$*.*

Proof. (i) We have to check that the scalar multiplication $(\lambda, x) \mapsto \lambda x$ satisfies the following conditions; see Warner [23, page 86]:
(TM1) $(\lambda, x) \mapsto \lambda x$ is continuous at $(0, 0)$,
(TM2) for each $c \in \mathbb{F}({}^*E)$, $\lambda \mapsto \lambda c$ is continuous at 0,
(TM3) for each $\alpha \in \mathbb{F}$, $x \mapsto \alpha x$ is continuous at 0.

Given $U \in \mathcal{N}_0$, there exists U_0 a balanced neighborhood of 0 such that $U_0 \subset U$. Let $r \in {}^a\mathbb{F}_+$.
(TM1): $(|\lambda| \leq 1)(r\,{}^*U_0) \subset r\,{}^*U_0 \subset r\,{}^*U$.
(TM2): Let $c \in \mathbb{F}({}^*E)$; then there exists $r_0 \in {}^a\mathbb{F}$, such that $c \in r_0{}^*U_0$. We have $r/r_0 \in {}^a\mathbb{F}_+$ and $(|\lambda| \leq |r|/|r_0|)\, c \subset r\,(|\lambda| \leq 1){}^*U_0 \subset r\,{}^*U$.
(TM3): Let $\alpha \in \mathbb{F}$. We have $\frac{r}{|\alpha|+1} \in {}^a\mathbb{F}_+$ and $\alpha\,\frac{r}{|\alpha|+1}\,{}^*U_0 \subset r\,{}^*U$.

(ii) To see that $\mathbb{F}({}^*E)$ is closed, let $x \in {}^*E$ with $x \notin \mathbb{F}({}^*E)$. Then there exists U a neighborhood of 0 in E such that $x \notin r\,{}^*U$ for any $r \in {}^a\mathbb{F}_+$. Let O be a balanced neighborhood of 0 such that $O - O \subset U$. Let $V := x + {}^*O$. It follows that V is a neighborhood of x in ${}^*E$ satisfying $V \cap \mathbb{F}({}^*E) = \emptyset$.

Indeed, assume that there exists $y \in V \cap \mathbb{F}({}^*E)$. Then we find $r_0 \in {}^a\mathbb{F}$ with $y \in r_0\,{}^*O$. Since ${}^*O$ is balanced, this implies $x \in r_0{}^*O - {}^*O \subset (|r_0|+1)({}^*O - {}^*O) \subset (|r_0| + 1){}^*U$, a contradiction. □

For each $U \in \mathcal{N}_0$ and $r \in {}^a\mathbb{F}$ define

$$V_{r,U} = \{(x, y) \in {}^*E \times {}^*E \;:\; x - y \in r\,{}^*U\}.$$

Let $\mathcal{U}_{\mathbb{F}}$ be the filter on ${}^*E \times {}^*E$ generated by the filter base $\{V_{r,U} : U \in \mathcal{N}_0,\, r \in {}^a\mathbb{F}\}$. Then $\mathcal{U}_{\mathbb{F}}$ is a translation-invariant uniformity on ${}^*E$ which determines the topology on ${}^*E$ generated by $\mathcal{V}_p({}^*E, \mathbb{F})$.

Theorem 3.7. *If* $\mathbb{F}$ *is generated by an asymptotic scale, then* $({}^*E, \mathcal{U}_{\mathbb{F}})$ *is complete.*

Proof. Assume that $\mathbb{F}$ is generated by the asymptotic scale λ_n. Let $\mathcal{G}$ be a Cauchy filter on ${}^*E$. Then for each $n \in \mathbb{N}$ and each $U \in \mathcal{N}_0$, there exists $F_{n,U} \in \mathcal{G}$ such that

$$F_{n,U} - F_{n,U} \subset \lambda_n\,{}^*U.$$

Choose some $x_{n,U} \in F_{n,U}$ and consider the system of internal sets $\mathcal{A}_{n,U} := x_{n,U} + \lambda_n\,{}^*U$. As $F_{n,U} \subset \mathcal{A}_{n,U}$, then $\mathcal{A}_{n,U}$ has the finite intersection property. Hence, by the saturation property, we conclude that $\cap \mathcal{A}_{n,U}$ contains some element $x \in {}^*E$.

We claim that the filter $\mathcal{G}$ converges to x, that is, any neighborhood of x belongs to $\mathcal{G}$.

Let W be any neighborhood of x; then there exists $n \in \mathbb{N}$ and $U \in \mathcal{N}_0$ such that $x + \lambda_n{}^*U \subset W$. Let V be a neighborhood of 0 such that $V - V \subset U$. We have

$$F_{V,n} \subset x_{n,V} + \lambda_n{}^*V \subset (x_{n,V} - x) + (x + \lambda_n{}^*V) \subset -\lambda_n{}^*V + (x + \lambda_n{}^*V) \subset x + \lambda_n{}^*U.$$

Hence $x + \lambda_n{}^*U \in \mathcal{G}$ and so $W \in \mathcal{G}$, as claimed. □

Using Theorem 3.6 (ii), we deduce the following

Corollary 3.8. *If* $\mathbb{F}$ *is generated by an asymptotic scale, then* $\mathbb{F}({}^*E)$ *is complete.*

Theorem 3.9. *Let* (G,τ) *and* (H,τ') *be* $\mathbb{K}$*-topological vector spaces and let* $f : G \to H$ *be a linear mapping. Consider the following:*

(i) f *is continuous at* 0.
(ii) ${}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0)$.
(iii) ${}^*f\left(\mathbb{F}({}^*G)\right) \subset \mathbb{F}({}^*H)$.

Then (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii).

Furthermore, if $\mathbb{F}$ *is generated by an asymptotic scale, then* (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Before giving the proof, we need the following lemmas.

Lemma 3.10. *Let* E *be* $\mathbb{K}$*-topological vector space. Then there exists* W*, a* $*$*-open neighborhood of* 0*, such that* $W \subset \mu_{\mathbb{F}}(0)$.

Proof. According to the saturation principle, there exists V, a $*$- open neighborhood of 0, such that $V \subset \mu(0)$; see [9]. Let α be a nonzero element in ${}^i\mathbb{F}$. We claim that $W := \alpha V$ is a $*$-open neighborhood of 0 satisfying $W \subset \mu_{\mathbb{F}}(0)$. Indeed, the transfer principle shows that W is a *-open neighborhood of 0. Furthermore, for any $r \in {}^a\mathbb{F}$ and for any U, a balanced neighborhood 0, we have $\alpha V \subset \alpha\, {}^*U \subset r\, {}^*U$, thus, $W \subset \mu_{\mathbb{F}}(0)$. □

We prove the converse of Theorem 3.4 under some additional assumptions on $\mathbb{F}$.

Lemma 3.11. *Assume that* $\mathbb{F}$ *is generated by an asymptotic scale. Then, for each* $p \in \mu_{\mathbb{F}}(0)$ *there exists* $\omega \in {}^\infty\mathbb{F}$ *such that* $\omega p \in \mu_{\mathbb{F}}(0)$*. Hence*

$$ {}^i\mathbb{F}\,.\,\mu_{\mathbb{F}}(0) = \mu_{\mathbb{F}}(0). $$

Proof. Let $p \in \mu_{\mathbb{F}}(0)$. For each $n \in \mathbb{N}$ and $U \in \mathcal{N}_0$ define the internal set $A(n,U)$ by

$$ A(n,U) = \{x \in {}^*\mathbb{R} : x \geq \lambda_{-n} \text{ and } xp \in \lambda_n {}^*U\}. $$

Since $\mu_{\mathbb{F}}(0)$ is closed by multiplication by elements of $\mathbb{F}$, each set $A(n,U)$ is nonempty. It follows that the sets $A(n,U)$ is a collection of internal subsets of ${}^*\mathbb{R}$ which has the finite intersection property. Hence, by the saturation principle, there is ω in the intersection of the collection. That is, $\omega \in {}^\infty\mathbb{F}$ and satisfies $\omega p \in \lambda_n {}^*U$ for each $n \in \mathbb{N}$ and each $U \in \mathcal{N}_0$. It follows that $\omega p \in \mu_{\mathbb{F}}(0)$, which completes the proof. □

Remark 3.12. We remark that if $(E, |.|)$ is a normed space, then ${}^i\mathbb{F}\,.\,\mu_{\mathbb{F}}(0) = \mu_{\mathbb{F}}(0)$ holds for any $\mathbb{F}$ a convex subring of ${}^*\mathbb{R}$. Indeed, let p be a nonzero element in $\mu_{\mathbb{F}}(0)$, i.e., $|p| \in {}^i\mathbb{F}$. Let $\omega = 1/\sqrt{|p|}$. Clearly, $\omega \in {}^\infty\mathbb{F}$ and $\omega p \in \mu_{\mathbb{F}}(0)$.

Proof (Theorem 3.9).

(i) $\Longrightarrow$ (ii) f is continuous at 0. So for any V neighborhood of 0 in H there exists U, a neighborhood of 0 in G, such that $f(U) \subset V$. By the transfer principle, we get ${}^*f(r\,{}^*U) \subset r\,{}^*V$, for any $r \in {}^a\mathbb{F}$. Hence ${}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0)$.

(ii) $\Longrightarrow$ (i) Conversely, assume that ${}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0)$. Let V be an arbitrary neighborhood of 0 in H. Using Lemma 3.10, we obtain :

$$ \text{There exists } W, \text{a *-open nieghborhood of } 0, \text{such that } {}^*f(W) \subset {}^*V. $$

The transfer principle shows that f is continuous.

(ii) $\Longrightarrow$ (iii) Let $p \in \mathbb{F}({}^*G)$. According to Theorem 3.4, the condition ${}^*f(p) \in \mathbb{F}({}^*H)$ is equivalent to ${}^i\mathbb{F}.{}^*f(p) \in \mu_{\mathbb{F}}^{\tau'}(0)$. Indeed, let $\lambda \in {}^i\mathbb{F}$,

$$\lambda^* f(p) \ = {}^*f(\lambda p) \in {}^*f(\mu_{\mathbb{F}}^{\tau}(0)) \subset \mu_{\mathbb{F}}^{\tau'}(0),$$

which completes the proof.

(iii) $\Longrightarrow$ (ii) Assume that $\mathbb{F}$ is generated by an asymptotic scale. By Lemma 3.10, we have

$${}^*f(\mu_{\mathbb{F}}^{\tau}(0)) = {}^*f({}^i\mathbb{F}.\,\mathbb{F}({}^*G)) = {}^i\mathbb{F}.\,{}^*f(\mathbb{F}({}^*G)) \subset {}^i\mathbb{F}.\,\mathbb{F}({}^*H) \subset \mu_{\mathbb{F}}^{\tau'}(0).$$

□

Corollary 3.13. *If τ and τ' are two vector topologies on E and $\mathbb{F}$ is a convex subring of ${}^*\mathbb{R}$ generated by an asymptotic scale, then*

(i) $\tau \subset \tau' \Longleftrightarrow \mathbb{F}_\tau({}^*E) \supset \mathbb{F}_{\tau'}({}^*E)$.
(ii) $\tau = \tau' \Longleftrightarrow \mathbb{F}_\tau({}^*E) = \mathbb{F}_{\tau'}({}^*E)$.

Definition 3.14. Let (E, τ) be a $\mathbb{K}$-topological vector space. The $\mathbb{F}$-nonstandard hull of E is the vector space $\widehat{E}$ defined by

$$\widehat{E} = \widehat{E}_{\mathbb{F}} = \mathbb{F}({}^*E)/\mu_{\mathbb{F}}(0)$$

equipped with $\widehat{\tau_{\mathbb{F}}}$, the quotient topology of $\tau_{\mathbb{F}}$ on $\mathbb{F}({}^*E)$.

The canonical mapping of $\mathbb{F}({}^*E)$ on $\widehat{E}$ will be denoted by π, thus $\pi(p) = p + \mu_{\mathbb{F}}(0)$ for all $p \in \mathbb{F}({}^*E)$.

We remark that the quotient topology on $\widehat{\mathbb{F}}$ coincides with the (product of) order topology, that is,

$$\widehat{B}(0, r) = \{\alpha \in \widehat{\mathbb{F}} \ : \ |\alpha| < r\}, \quad r \in \widehat{\mathbb{F}}_+,$$

is a neighborhood basis of 0 is $\widehat{\mathbb{F}}$.

Proposition 3.15. *The quotient mapping $\pi : \mathbb{F}({}^*E) \to \widehat{E}$ is continuous and open.*

By Theorem 3.6 and the universal property of the quotient topology, we have

Theorem 3.16. *$\widehat{E}$ is a Hausdorff topological $\widehat{\mathbb{F}}$-vector space. Then*

$$\begin{array}{ccc} \mathbb{F}({}^*E) \times \mathbb{F}({}^*E) & \xrightarrow{\ +\ } & \mathbb{F}({}^*E) \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ \widehat{E} \times \widehat{E} & \xrightarrow{\ +\ } & \widehat{E} \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \mathbb{F}({}^*E) & \xrightarrow{\ \cdot\ } & \mathbb{F}({}^*E) \\ \downarrow{\scriptstyle \widehat{\mathrm{st}}\times\pi} & & \downarrow{\scriptstyle \pi} \\ \widehat{\mathbb{F}} \times \widehat{E} & \xrightarrow{\ \cdot\ } & \widehat{E} \end{array}$$

Theorem 3.17. *If $\mathbb{F}$ is generated by an asymptotic scale, then $\widehat{E}$ is complete.*

Proof. Let $\widehat{\mathcal{G}}$ be a Cauchy filter on $\widehat{E}$ and let $\mathcal{G}$ be the filter on $\mathbb{F}({}^*E)$ generated by $\pi^{-1}(\widehat{\mathcal{G}})$. One can easily check that $\pi^{-1}(\widehat{\mathcal{G}})$ is a Cauchy filter on $\mathbb{F}({}^*E)$, hence by Corollary 3.8, it converges to some $x \in \mathbb{F}({}^*E)$. The continuity of the mapping π implies that $\widehat{\mathcal{G}} = \pi(\mathcal{G})$ converges to $\pi(x)$. □

Proposition 3.18. *If E is a normed space, then topology of $\widehat{\tau_{\mathbb{F}}}$ induces on E the discrete topology.*

Proof. If E is Hausdorff, then E is a subspace of $\widehat{E}$. Indeed, let $x \in E$ such that $\pi(x) = 0$. Hence $x \in \mu_{\mathbb{F}}(0) \cap E \subset \mu(0) \cap E = \overline{\{0\}} = \{0\}$. □

Using Theorem 3.9, we deduce the following

Theorem 3.19. *Let G and H be two $\mathbb{K}$-topological vector spaces and let $f : G \to H$ be a continuous linear mapping. Then f gives rise to a continuous $\widehat{\mathbb{F}}$-linear mapping $\widehat{f} : \widehat{G} \to \widehat{H}$ defined by*

$$\widehat{f}(\widehat{x}) = \widehat{f(x)} \quad \text{for all } \widehat{x} \in \widehat{G}.$$

3.3. **Nonstandard hulls of locally convex spaces.** Let E be a locally convex topological vector space topologized through a family of seminorms $(p_j)_{j\in J}$. Then $\mathbb{F}(^*E)$, the set of $\mathbb{F}$-bounded points of $^*E$ defined in Section 3.2, is given by

$$\mathbb{F}(^*E) = \{x \in {}^*E : p_j(x) \in \mathbb{F} \text{ for all } j \in J\},$$

and

$$\mu_{\mathbb{F}}(0) = \{x \in {}^*E : p_j(x) \in {}^i\mathbb{F} \text{ for all } j \in J\}.$$

The topology $(^*E, \tau_{\mathbb{F}})$ is generated by $\{p_j^{-1}(0, r) : r \in {}^a\mathbb{F}_+\}$ as a subbase. In other words, $\tau_{\mathbb{F}}$ coincides with the initial topology on $^*E$ making all $^*p_j : {}^*E \longrightarrow {}^*\mathbb{R}_+$ continuous, where $^*\mathbb{R}$ is equipped with the QS-topology generated by $\mathbb{F}$.

The family of seminorms p_j induces on $\widehat{E}$

$$\widehat{p_j} : \widehat{E} \to \widehat{\mathbb{F}}_+.$$

The quotient topology $\widehat{\tau_{\mathbb{F}}}$ on $\widehat{E}$ coincides with the initial topology making all the seminorms $\widehat{p_j} : \widehat{E} \to \widehat{\mathbb{F}}_+$ continuous, where $\widehat{\mathbb{F}}$ is equipped with the order topology.

4. Examples

4.1. $E = \mathcal{E}(\Omega)$ **the space of smooth functions.** Let Ω be an open subset of $\mathbb{R}^n$ and let E be the space of smooth functions over Ω. E is topologized through the family of seminorms $p_{K_i,j}(f) = \sup_{x\in K_i, |\alpha|\le j} |\partial^\alpha f(x)|$, where $(K_i)_{i\in\mathbb{N}}$ is an exhausting sequence of compact subsets of Ω,

$$\mathbb{F}(^*\mathcal{E}(\Omega)) = \mathcal{M}(^*\mathcal{E}(\Omega)) = \{f \in {}^*\mathcal{E}(\Omega) : \partial^\alpha f(\text{ns}(^*\Omega)) \subset \mathbb{F} \text{ for all } \alpha \in \mathbb{N}^n\},$$

$$\mu_{\mathbb{F}}(0) = \mathcal{N}(^*\mathcal{E}(\Omega)) = \{f \in {}^*\mathcal{E}(\Omega) : \partial^\alpha f(\text{ns}(^*\Omega)) \subset {}^i\mathbb{F} \text{ for all } \alpha \in \mathbb{N}^n\},$$

where $\text{ns}(^*\Omega)$ stands for the nearstandard points of $^*\Omega$.

The space $\widehat{\mathcal{E}_{\mathbb{F}}(\Omega)} = \mathbb{F}(^*\mathcal{E}(\Omega))/\mu_{\mathbb{F}}(0)$ was studied in detail in [21] as the nonstandard counterpart of Colombeau algebras.

4.2. $E = \mathbb{C}[T_1, \dots, T_n]$ **the space of polynomials.** Let $E = \mathbb{C}[T_1, \dots, T_n]$ be the space of polynomials in n-indeterminates over $\mathbb{C}$ equipped with the topology of compact convergence.

$$\mathbb{F}(^*\mathbb{C}[T_1, \dots, T_n]) = \{f \in {}^*(\mathbb{C}[T_1, \dots, T_n]) : f(^b\mathbb{C}^n) \subset \mathbb{F}\},$$

$$\mu_{\mathbb{F}}(0) = \{f \in {}^*(\mathbb{C}[T_1, \dots, T_n]) : f(^b\mathbb{C}^n) \subset {}^i\mathbb{F}\},$$

where $^b\mathbb{C}^n$ stands for the nearstandrad points of $^*\mathbb{C}^n$.

$\mathbb{F}(^*\mathbb{C}[T_1, \dots, T_n])$ is the ring of $(^b\mathbb{C}, \mathbb{F})$-bounded polynomials and will be denoted by ${}^{\mathbb{F}}_{^b\mathbb{C}}\mathbb{C}[T_1, \dots, T_n]$ and $\mu_{\mathbb{F}}(0)$ is the ideal of $(^b\mathbb{C}, \mathbb{F})$-infinitesimal polynomials. In [13], we provided a characterization of $(^b\mathbb{C}, \mathbb{F})$-bounded polynomials in terms of their coefficients using ${}^{\mathbb{F}}_{^b\mathbb{C}}\mathbb{N}$ the set of naturals in $(^b\mathbb{C}, \mathbb{F})$.

For the sake of completeness, we recall the definition of ${}^{\mathbb{F}}_{^b\mathbb{C}}\mathbb{N}$, the set of naturals in $(^b\mathbb{C}, \mathbb{F})$, and their main properties.

Definition 4.1 ([13])**.** Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{C}$. Define

$$ {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N} := \{\nu \in {}^*\mathbb{N} \,:\, \forall R \in {}^b\mathbb{C}_+,\ R^\nu \in \mathbb{F}\} $$

the set of $({}^b\mathbb{C}, \mathbb{F})$ naturals, where ${}^b\mathbb{C}_+ = {}^b\mathbb{C} \cap {}^*\mathbb{R}_+$.

Moreover, by convexity of $\mathbb{F}$, we have

$$ \nu \in {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N} \text{ if and only if } \forall n \in \mathbb{N},\ n^\nu \in \mathbb{F}. $$

Example 4.2. If $\mathbb{F} = M_\rho$, then

$$ {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N} = \{\nu \in {}^*\mathbb{N} \,:\, \nu \leq \alpha|\ln\rho| \text{ for some } \alpha \in \mathbb{R}_+\} = {}^*\mathbb{N} \cap (|\ln\rho|\,{}^b\mathbb{R}_+). $$

Proposition 4.3 ([13])**.** *Let* $\mathbb{F}$ *be a convex subring of* ${}^*\mathbb{C}$ *. Then*

(i) ${}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$ *is a monoid.*
(ii) $\mathbb{N} \subset {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N} \subset \mathbb{F} \cap {}^*\mathbb{N}$.
(iii) *Let* $n, m \in {}^*\mathbb{N}$ *such that* $m \leq n$*. If* $n \in {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$*; then* $m \in {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$*.*
(iv) *If* ${}^b\mathbb{C} \subsetneq \mathbb{F}$*, then* $\mathbb{N} \subsetneq {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$*.*

Remark 4.4. Let $P \in {}^*(\mathbb{C}[T_1, \ldots, T_n])$ be an internal polynomial of degree in ${}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$,

$$ P = \sum_{|\nu| \leq d} a_\nu Z^\nu,\ \ a_\nu \in {}^*\mathbb{C},\ \ \ d \in {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}. $$

Then

(i) $P({}^b\mathbb{C}^n) \subset \mathbb{F}$ if and only if $a_\nu \in \mathbb{F}$ for all $|\nu| \leq d$.
(ii) $P({}^b\mathbb{C}^n) \subset {}^i\mathbb{F}$ if and only if $a_\nu \in {}^i\mathbb{F}$ for all $|\nu| \leq d$.

More generally, for internal entire functions, a characterization of $({}^b\mathbb{C}, \mathbb{F})$-boundedness in terms of their coefficients is proved in [13].

Our main result is Theorem 4.10 showing that

$$ \widehat{\mathbb{C}[T_1, \ldots, T_n]}_{\mathbb{F}} = \mathbb{F}({}^*\mathbb{C}[T_1, \ldots, T_n])/\mu_{\mathbb{F}}(0), $$

the $\mathbb{F}$-nonstandard hull of $\mathbb{C}[T_1, \ldots, T_n]$, is $\widehat{\mathcal{O}_{\mathbb{F}}}(\mathbb{C}^n)$, the nonstandard counterpart of Colombeau's generalized holomorphic functions over $\mathbb{C}^n$.

We should mention that for $\mathbb{F} = {}^b\mathbb{C}$, the nonstandard hull of $\mathbb{C}[T_1, \ldots, T_n]$ is $\mathcal{O}(\mathbb{C}^n)$ the algebra of entire functions on $\mathbb{C}^n$, see [11], and for $n = 1$, the $\mathbb{F}$-nonstandard hull of $\mathbb{C}[T]$ is the nonstandard counterpart of Colombeau's generalized holomorphic functions over $\mathbb{C}$; see [13]. The proof is based on the existence of global holomorphic representatives of Colombeau's generalized holomorphic functions. For $n = 1$, our argument is that the $\overline{\partial}$-operator has a right inverse; see also [18]. Some substantial modifications are needed to adapt the proof in [18] to higher dimensions spaces. It appears that the set of naturals is an essential tool in our proofs. Indeed, the Runge approximation theorem is explicit on polydiscs since we approximate internal analytic functions (on polydiscs) by internal polynomials given by their Taylor expansion. The degrees of the polynomials will be carefully chosen to satisfy some crucial estimates in the proof.

4.2.1. *Existence of global holomorphic representaives.* Let Ω be an open subset of $\mathbb{C}^n$. We construct variants of nonstandard counterparts of the space of holomorphic generalized functions introduced by Colombeau, see [1, 2],

$$\widehat{\mathcal{O}_{\mathbb{F}}}(\Omega) := \{f \in \widehat{\mathcal{E}_{\mathbb{F}}(\Omega)} \,:\, \overline{\partial}_j f = 0 \ \forall j = 1, \dots, n\}.$$

In [13], we used bounded polynomials to provide natural counterexamples showing the disparity with the classical theory of analytic functions. Indeed, any polynomial $P \in {}^*(\mathbb{C}[T_1, \dots, T_n])$ which is $({}^b\mathbb{C}, \mathbb{F})$-bounded generates $\widehat{P} \in \widehat{\mathcal{O}_{\mathbb{F}}}(\mathbb{C}^n)$ a holomorphic generalized function.

Now, we prove the converse that any holomorphic generalized function over $\mathbb{C}^n$ is given by a $({}^b\mathbb{C}, \mathbb{F})$-bounded polynomial of degree in ${}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$. First, we show

Theorem 4.5. *Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{C}$. Then any holomorphic generalized function $F \in \widehat{\mathcal{O}_{\mathbb{F}}}(\mathbb{C}^n)$ admits an internal holomorphic function representative $f \in {}^*\mathcal{O}(\mathbb{C}^n)$.*

In [4], the authors proved that any generalized holomorphic function has a *local* holomorphic representative. One can easily translate the proof from the setting of Colombeau generalized functions to our setting since the proof is based on the Cauchy-Pompeiu formula; see [8].

Lemma 4.6 ([4])**.** *Let Ω be an open subset of $\mathbb{C}^n$, $a = (a_1, \dots, a_n) \in \Omega$ and let $r_1, \dots, r_n > 0$ be such that the closure of the polydisc $P(a, r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i - a_i| < r_i, \ \forall\, i = 1, \dots, n\}$ is contained in Ω. Then every $F \in \widehat{\mathcal{O}_{\mathbb{F}}}(\Omega)$ has an internal holomorphic representative f in the polydisc $P(a, r)$.*

Proof. For $j \in \mathbb{N}_{\geq 1}$, we set V_j the polydisc in $\mathbb{C}^n$ of center the origin and radius j and ${}^*V_j$ its nonstandard extension defined for $j \in {}^*\mathbb{N}_{\geq 1}$. By Theorem 4.5, for every $j \in \mathbb{N}_{\geq 1}$, there exists f_j a holomorphic representative of F on V_j. The sequence $(f_j)_{j \in \mathbb{N}_{\geq 1}}$ extends to $(f_j)_{j \in {}^*\mathbb{N}_{\geq 1}}$ an internal sequence of internal analytic functions on ${}^*V_j$.

Let

$$u_j(z) = f_{j+1}(z) - f_j(z), \quad z \in {}^*V_j, \ \ j \in {}^*\mathbb{N}, j \geq 1.$$

u_j is an internal holomorphic function on ${}^*V_j$, therefore it has an internal power series expansion

$$u_j(z) = \sum_{l \in {}^*\mathbb{N}^n} a_j^l z^l, \ \ z \in {}^*V_j, \ \ j \in {}^*\mathbb{N}_{\geq 1}$$

and for j *standard*, $u_j \in \mathcal{N}({}^*\mathcal{E}(V_j)) \cap {}^*\mathcal{O}(V_j)$.

Claim. For all $j \in {}^*\mathbb{N}_{\geq 2}$, there exists $\alpha_j \in {}^i\mathbb{F}_+$ and $v_j \in {}^*(\mathbb{C}[T_1, \dots, T_n])$, an internal polynomial, such that $\sup_{z \in {}^*V_{j-1}} |u_j(z) - v_j(z)| \leq 2^{-j}\alpha_j$ and v_j is a $({}^b\mathbb{C}, \mathbb{F})$-infinitesimal polynomial for j *standard.*

For $j \geq 2$ *standard*, let us choose $N_j \in {}^{\mathbb{F}}_{{}^b\mathbb{C}}\mathbb{N}$ such that if

$$v_j(z) = \sum_{|l| \leq N_j} a_j^l z^l$$

we have

$$\sup_{z\in {}^*V_{j-1}} |u_j(z) - v_j(z)| \leq 2^{-j}\alpha_j$$

for some $\alpha_j \in {}^i\mathbb{F}$ and v_j is a $({}^b\mathbb{C}, \mathbb{F})$-infinitesimal polynomial.

Since $u_j \in \mathcal{N}({}^*\mathcal{E}(V_j)) \cap {}^*\mathcal{O}(V_j)$, then it follows from Cauchy's inequalities that there exits $\alpha_j \in {}^i\mathbb{F}$ such that

$$(\star) \qquad |a_j^l| \leq \frac{\alpha_j}{(j-1/2)^{|l|}}.$$

Therefore if $z \in {}^*V_{j-1}$,

$$|u_j(z) - v_j(z)| \leq \sum_{|l|\geq N_j+1} |a_j^l|(j-1)^{|l|} \quad \leq \alpha_j \sum_{|l|\geq N_j+1} r_j^{|l|} \leq \alpha_j\, r_j^{(Nj+1)n}(2j-1)^n,$$

where $r_j = \dfrac{j-1}{j-\frac{1}{2}}$.

Since r_j is a real number in $(0,1)$, we have $r_j^{(Nj+1)n} \in {}^i\mathbb{R}$ for any $N_j \in {}^\infty\mathbb{N} \cap {}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$, thus $(2j-1)^n r_j^{(Nj+1)n} < 2^{-j}$. Such N_j exists since ${}^\infty\mathbb{N} \cap {}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$ is nonempty by Proposition 4.3 (iv).

Hence, on ${}^*V_{j-1}$ we have

$$|u_j - v_j| \leq \alpha_j\, 2^{-j}.$$

Moreover, $\sum_{j\geq 1} \alpha_j 2^{-j} \in {}^i\mathbb{F}$.

For j *standard*, the polynomial v_j is a $({}^b\mathbb{C}, \mathbb{F})$-infinitesimal polynomial . Indeed, by the Cauchy's estimate $(\star)$, we obtain $a_j^l \in {}^i\mathbb{F}$ for all $|l| \leq N_j$. Thus as $N_j \in {}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$, we have $v_j({}^b\mathbb{C}^n) \subset {}^i\mathbb{F}$; see Remark 4.4.

Define g on ${}^*V_{j-1}$, $j \in {}^*\mathbb{N}$, $j \geq 2$, by

$$\begin{aligned} g = &\quad f_j + \sum_{k\geq j}(u_k - v_k) - v_2 - \ldots - v_{j-1} \\ = &\quad f_{j+1} + \sum_{k\geq j+1}(u_k - v_k) - v_2 - \ldots - v_j. \end{aligned}$$

The $*$-sheaf property shows that g defines an internal holomorphic function on ${}^*\mathbb{C}^n$ and g is a global representative of F. □

Remark 4.7. If $\mathbb{F}$ is generated by an asymptotic scale, then we strengthen the claim by showing that there exists a uniform $\alpha \in {}^i\mathbb{F}$ such that $\sup_{z\in {}^*V_{j-1}} |u_j(z) - v_j(z)| \leq 2^{-j}\alpha$; see Lemma A.3.

Lemma 4.8 ([13]). *Let $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ be a $({}^b\mathbb{C}, \mathbb{F})$-bounded internal holomorphic function. Then for each $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}$, the tail $(f - f_N)$ is $({}^b\mathbb{C}, \mathbb{F})$ infinitesimal, that is,*

$$(f - f_N)({}^b\mathbb{C}^n) \subset {}^i\mathbb{F}$$

and

$$\widehat{f}(z) = \lim_{N\in {}^{\mathbb{F}}_{b\mathbb{C}}\mathbb{N}, N\to\infty} \widehat{f_N}(z).$$

$\widehat{f}(z) = \lim_{N\in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}, N\to\infty} \widehat{f_N}(z)$ means that for any $\varepsilon \in \widehat{\mathbb{F}}$, $\varepsilon > 0$, there exists $N \in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}$, such that for any $n \in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}$, $n > N$, we have $|\widehat{f}(z) - \widehat{f_n}(z)| < \varepsilon$.

Combining Theorem 4.5 with the previous lemma proved in [13], we obtain

Corollary 4.9. *Every holomorphic generalized function $F \in \widehat{\mathcal{O}_\mathbb{F}}(\mathbb{C}^n)$ admits a $({}^b\mathbb{C}, \mathbb{F})$-bounded polynomial representative P of degree $N \in {}^*\mathbb{N}\backslash {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}$, that is, $\widehat{P} = F$.*

The $({}^b\mathbb{C}, \mathbb{F})$-bounded polynomial is obtained by truncation of the internal entire function up to any order in ${}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}$.

Theorem 4.10. *Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{C}$. Then the quasi-standard mapping*

$$\widehat{\mathrm{st}} : {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{C}[T_1, \ldots, T_n] \to \widehat{\mathcal{O}_\mathbb{F}}(\mathbb{C}^n)$$

is surjective and its kernel is given by $({}^b\mathbb{C}, \mathbb{F})$-infinitesimal polynomials and

$$\widehat{\mathrm{st}}(\sum_{\nu\in{}^*\mathbb{N}^n} a_\nu z^n) = \sum_{|\nu|\in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}} \widehat{a_\nu}\widehat{z}^n.$$

This shows that the $\mathbb{F}$-nonstandard hull of the space $\mathbb{C}[T_1, \ldots, T_n]$ is $\widehat{\mathcal{O}_\mathbb{F}}(\mathbb{C}^n)$, the algebra of holomorphic generalized functions on $\mathbb{C}^n$.

4.2.2. *Identity theorem.* The classical identity theorem for power series is very simple: given an entire function $f = \sum a_n z^n$. If f vanishes near the origin, then f vanishes everywhere as $a_n = f^{(n)}(0)/n! = 0$ for all n. We can apply similar arguments to the case of generalized holomorphic functions.

Proposition 4.11. *Let f be a generalized holomorphic function in $\widehat{\mathcal{O}_\mathbb{F}}(\mathbb{C}^n)$. If f vanishes on Ω, a nonempty open subset of $\mathbb{C}^n$, then f vanishes everywhere.*

Proof. Let F be a global generalized holomorphic representative of f. Let $a \in \Omega$ and $r > 0$ such that $\overline{P(a, r)} \subset \Omega$. Let us write the Taylor expansion of F

$$F(z) = \sum_{\nu\in{}^*\mathbb{N}^n} a_\nu (z - a)^\nu.$$

By hypothesis, $F({}^*\overline{P(a, r')}) \subset {}^i\mathbb{F}$, for any $0 < r' < r$. The Cauchy's estimate shows that $a_\nu \in {}^i\mathbb{F}$ for every $|\nu| \in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}$. Thus

$$\widehat{\mathrm{st}}(F) = \widehat{\mathrm{st}}\left(\sum_{\nu\in{}^*\mathbb{N}^n} a_\nu (z - a)^\nu\right) = \sum_{|\nu|\in {}^{\mathbb{F}}_{b_\mathbb{C}}\mathbb{N}} \widehat{a_\nu}(\widehat{z} - a)^\nu = 0.$$

Hence $f = 0$. □

4.3. $E = \mathcal{O}_X(U)$**.** Let $(X, \mathcal{O}_X)$ be a separable analytic space, and $U \subset X$ any open set. Recall that $\mathcal{O}_X(U)$ has a structure of a Fréchet space defined by the topology of compact convergence,

$$\mathbb{F}({}^*\mathcal{O}_X(U)) = \{f \in {}^*\mathcal{O}_X(U) : f(\mathrm{ns}({}^*U)) \subset \mathbb{F}\},$$

$$\mu_\mathbb{F}(0) = \{f \in {}^*\mathcal{O}_X(U) : f(\mathrm{ns}({}^*U)) \subset {}^i\mathbb{F}\}.$$

The space $\widehat{\mathcal{O}_X(U)} = \mathbb{F}({}^*\mathcal{O}(U))/\mu_\mathbb{F}(0)$ is the $\mathbb{F}$-nonstandard hull of $\mathcal{O}_X(U)$. Moreover, the mapping $U \mapsto \widehat{\mathcal{O}_X(U)}$ defines a separated presheaf on X as any element $f \in \widehat{\mathcal{O}_X(U)}$ gives a pointwise mapping $\widehat{f} : \mathrm{ns}({}^*U)/{}^i\mathbb{F}^n \to \widehat{\mathbb{F}}$.

More generally, let $\mathcal{G}$ be a Fréchet sheaf on a topological space X with a countable topology, that is, $\mathcal{G}$ is a sheaf of vector spaces on X such that $\mathcal{G}(U)$ is a Fréchet space for every $U \subset X$, an open subset of X, and for every $V \subset U$, the restriction-homomorphism $\mathcal{G}(U) \to \mathcal{G}(V)$ is continuous. Let $\widehat{\mathcal{G}(U)}$ be the $\mathbb{F}$-nonstandard hull of $\mathcal{G}(U)$. If $\widehat{\mathbb{F}}$ has a nontrivial real-valued valuation compatible with its order, then the space $\widehat{\mathcal{G}(U)}$ is complete with respect to a countable sequence of ultra-seminorms. Hence, using Theorem 3.19, we deduce that the mapping $U \mapsto \widehat{\mathcal{G}(U)}$ gives a Fréchet presheaf on X.

A fundamental example of a Fréchet sheaf is given by a coherent $\mathcal{O}_X$-module, where $(X, \mathcal{O}_X)$ is a complex space.

4.4. $E = \mathbb{Z}$ **the ring of integers equipped with the** $p-$**adic norm.** Let us consider $\mathbb{Z}$, the ring of the integers equipped with $|.|_p$, the $p-$adic norm. Recall that if $n \in \mathbb{Z} \setminus \{0\}$, $|n|_p = p^{-\nu_p(n)}$, where ν_p denotes the $p-$adic valuation for $\mathbb{Z}$. Let $\mathbb{F}$ be a proper convex subring of ${}^*\mathbb{R}$, that is, ${}^b\mathbb{R} \subset \mathbb{F} \subsetneq {}^*\mathbb{R}$.

Then the set of $\mathbb{F}$-bounded and $\mathbb{F}$-infinitesimal elements are given by

$$\mathbb{F}({}^*\mathbb{Z}) = \{n \in {}^*\mathbb{Z} \,:\, |n|_p \in \mathbb{F}\} = {}^*\mathbb{Z},$$

$$\mu_{\mathbb{F}}(0) = \{n \in {}^*\mathbb{Z} \,:\, |n|_p \in {}^i\mathbb{F}\}.$$

One can easily check that $\mu_{\mathbb{F}}(0)$ is an external prime ideal in ${}^*\mathbb{Z}$.

The following proposition provides a generalization of Theorem 18.4.1 in Goldblatt [6], where the author considered the case $\mathbb{F} = {}^b\mathbb{R}$.

Proposition 4.12. *Let n be a nonzero hyperinteger $n \in {}^*\mathbb{Z}$. The following are equivalent:*

(i) $|n|_p \in {}^i\mathbb{F}$.
(ii) $\nu_p(n) \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.
(iii) *n is divisible by p^k for all $k \in {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.*
(iv) *n is divisible by p^K for some $K \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.*

The proof is a direct consequence of the following elementary lemma.

Lemma 4.13. $\log({}^\infty\mathbb{F}_+) \cap {}^*\mathbb{N} = {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.

Proof (Proposition).
(i) $\iff$ (ii): $|n|_p \in {}^i\mathbb{F} \iff p^{\nu_p(n)} \in {}^\infty\mathbb{F}_+ \iff \nu_p(n) \in \log({}^\infty\mathbb{F}_+) \cap {}^*\mathbb{N}$.
(ii) $\iff$ (iii): $p^k/n \iff k \leq \nu_p(n)$.
(iii) $\iff$ (vi): follows from the overflow principle; see Theorem A.2. □

Let us consider θ_p, the following homomorphism of rings,

$$\theta_p : {}^*\mathbb{Z} \longrightarrow \varprojlim_{k \in {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z}$$

defined by $\theta_p(n) = (n \mod p^k)_{k \in {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}}$. Using Proposition 4.12, we deduce that $\ker \theta_p = \mu_{\mathbb{F}}(0)$. Hence

$$\widehat{\mathbb{Z}}^{\mathbb{F}} \hookrightarrow \varprojlim_{k \in {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z},$$

where $\widehat{\mathbb{Z}}^{\mathbb{F}}$ denotes the $\mathbb{F}$-nonstandard hull ${}^*\mathbb{Z}$, that is, $\widehat{\mathbb{Z}}^{\mathbb{F}} = \mathbb{F}({}^*\mathbb{Z})/\mu_{\mathbb{F}}(0) = {}^*\mathbb{Z}/\mu_{\mathbb{F}}(0)$.

We obtain the following commutative diagram:

$$\begin{array}{ccc}
\widehat{\mathbb{Z}}^{\mathbb{F}} & \hookrightarrow & \varprojlim_{k\in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z} \\
\downarrow\!\!\downarrow & & \downarrow \\
\widehat{\mathbb{Z}} & \xrightarrow{\simeq} & \varprojlim_{k\in\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z}
\end{array}$$

Here, $\widehat{\mathbb{Z}}$ denotes the ${}^b\mathbb{R}$-nonstandard hull of ${}^*\mathbb{Z}$, which is isomorphic to $\mathbb{Z}_p$, the ring of p-adic integers; see Goldblatt [6].

Proposition 4.14.

$$\widehat{\mathbb{Z}}^{\mathbb{F}} = \widehat{\mathbb{Z}_p}^{\mathbb{F}}.$$

For any $n \in \widehat{\mathbb{Z}_p}^{\mathbb{F}}$, *and for any* $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}$, *we have*

$$n = \sum_{k\leq N} a_k p^k \mod \mu_{\mathbb{F}}(0) = \sum_{k\in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}} a_k p^k,$$

where $(a_k)_{k\in {}^*\mathbb{N}}$ *is an internal sequence of integers with values in* $[\![0, p-1]\!]$.

Appendix A. Spilling principles

We recall several spilling principles in terms of a *proper* convex subring $\mathbb{F}$ of ${}^*\mathbb{R}$. We note that the familiar underflow and overflow principles in nonstandard analysis follow as a particular case for $\mathbb{F} = {}^b\mathbb{R}$.

Theorem A.1 (Spilling principles [21]). *Let* $\mathbb{F}$ *be a proper convex subring of* ${}^*\mathbb{R}$ *and let* $\mathcal{A} \subset {}^*\mathbb{R}$ *be an internal set. Then:*

(i) *Overflow of* $\mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily large numbers in* $\mathbb{F}$, *then* $\mathcal{A}$ *contains arbitrarily small numbers in* ${}^*\mathbb{R} \setminus \mathbb{F}$. *In particular,*

$$\mathbb{F} \setminus {}^i\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap ({}^*\mathbb{R} \setminus \mathbb{F}) \neq \emptyset.$$

(ii) *Underflow of* $\mathbb{F} \setminus {}^i\mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily small numbers in* $\mathbb{F} \setminus {}^i\mathbb{F}$, *then* $\mathcal{A}$ *contains arbitrarily large numbers in* ${}^i\mathbb{F}$. *In particular,*

$$\mathbb{F} \setminus {}^i\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap {}^i\mathbb{F} \neq \emptyset.$$

(iii) *Overflow of* ${}^i\mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily large numbers in* ${}^i\mathbb{F}$, *then* $\mathcal{A}$ *contains arbitrarily small numbers in* $\mathbb{F} \setminus {}^i\mathbb{F}$. *In particular,*

$${}^i\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathbb{F} \setminus {}^i\mathbb{F}) \neq \emptyset.$$

(iv) *Underflow of* ${}^*\mathbb{R} \setminus \mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily small numbers in* ${}^*\mathbb{R} \setminus \mathbb{F}$, *then* A *contains arbitrarily large numbers in* $\mathbb{F}$. *In particular,*

$${}^*\mathbb{R} \setminus \mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathbb{F} \setminus {}^i\mathbb{F}) \neq \emptyset.$$

We should mention that these spilling principles *fail* if $\mathbb{F} = {}^*\mathbb{R}$ and ${}^i\mathbb{F} = \{0\}$.

Theorem A.2 (Principles of permanence of ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, [13]). *Let* A *be an internal subset of* ${}^*\mathbb{N}$ *and* $\mathbb{F}$ *and* $\mathbb{G}$ *be two convex subrings of* ${}^*\mathbb{C}$ *such that* $\mathbb{F} \subset \mathbb{G}$.

(i) *(The underflow principle) Let* $K \in {}^\infty\mathbb{N}$ *be an infinite integer. If every* $H \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *with* $H \leq K$ *belongs to* A, *then there is some* $k \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *such that* $[\![k..K]\!] \subset A$.

(ii) *(The overflow principle) Let* $k \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$*. If every* $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *with* $n \geq k$ *belongs to* A*, then there is some* $K \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *such that* $[\![k..K]\!] \subset A$*.*

Where

$$ {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} := \{\nu \in {}^*\mathbb{N} \,:\, \forall R \in \mathbb{F}_+,\ R^\nu \in \mathbb{G}\} $$

is the set of $(\mathbb{F}, \mathbb{G})$ naturals, and $\mathbb{F}_+ = \mathbb{F} \cap {}^*\mathbb{R}_+$.

Lemma A.3. *Let* $\mathbb{F}$ *be a convex subring of* ${}^*\mathbb{R}$ *generated by an asymptotic scale* (λ_n)*. Let* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence in* ${}^i\mathbb{F}_+$*. Then there exists* $\alpha \in {}^i\mathbb{F}_+$ *such that* $0 \leq \alpha_n \leq \alpha$ *for all* $n \in \mathbb{N}$*.*

Proof. For each $n \in \mathbb{N}$, consider $A_n = \{x \in {}^*\mathbb{R} : \alpha_n \leq x \leq \lambda_n\} = [\alpha_n, \lambda_n]$. A_n is a sequence of internal subsets of ${}^*\mathbb{R}$ satisfying the finite intersection property, hence $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$. □

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