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L^p ESTIMATES FOR THE BERGMAN PROJECTION ON SOME REINHARDT DOMAINS

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Abstract. We obtain L^p regularity for the Bergman projection on some Reinhardt domains. We start with a bounded initial domain Ω with some symmetry properties and generate successor domains in higher dimensions. We prove: If the Bergman kernel on Ω satisfies appropriate estimates, then the Bergman projection on the successor is L^p bounded. For example, the Bergman projection on successors of strictly pseudoconvex initial domains is bounded on L^p for $1 < p < \infty$. The successor domains need not have smooth boundary nor be strictly pseudoconvex.

1. Introduction

The purpose of this paper is to establish L^p regularity for the Bergman projection on certain domains. In [Huo17], the author began with an initial domain with certain symmetry properties. From this initial domain the author constructed various successor domains and computed (explicitly) the Bergman kernel on them in terms of the Bergman kernel on the initial domain.

Let Ω be an initial domain in $\mathbb{C}^n$. We consider two kinds of estimates on the Bergman kernel K_Ω. A first estimate implies L^p regularity of the Bergman projection on Ω. If, also, a second estimate holds, then we obtain L^p regularity of the Bergman projection on the successor domain. See Theorem 1.2. We use a variant of Schur's lemma to establish L^p regularity. We state the crucial estimates in Theorem 3.3 and give the proof in Section 4.

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. The Bergman projection is the orthogonal projection from $L^2(\Omega)$ onto the closed subspace of square-integrable holomorphic functions, and thus is bounded on L^2. It is natural to ask when this operator is bounded on L^p for $p \neq 2$. Using known estimates for the Bergman kernel, various authors have obtained L^p regularity results for $1 < p < \infty$ in the following settings:

(1) Ω is bounded, smooth, and strongly pseudoconvex. See [Fef74, PS77].
(2) $\Omega \subseteq \mathbb{C}^2$ is a domain of finite type. See [McN89, McN94a, NRSW88].
(3) $\Omega \subseteq \mathbb{C}^n$ is a convex domain of finite type. See [McN94a, McN94b, MS94].
(4) $\Omega \subseteq \mathbb{C}^n$ is a domain of finite type with locally diagonalizable Levi form. See [CD06].

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Progress has also been made on some domains with weaker assumptions on boundary regularity. In some cases, the Bergman projection is L^p bounded for $1 < p < \infty$. See [EL08, LS12]. For other domains, the projection has only a finite range of mapping regularity. See [Zey13, CZ16, EM16, EM17, Che17]. There are also smooth bounded domains where the projection has limited L^p range. See [BŞ12].

We start with a bounded complete Reinhardt domain Ω in $\mathbb{C}^n$ with a defining function ρ and analyze the L^p regularity of the Bergman projection on the successor domains $U^\alpha(\Omega)$ defined by

$$U^\alpha(\Omega) = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{B}^k : \left(\frac{z_1}{(1-\|w\|^2)^{\frac{\alpha_1}{2}}}, \dots, \frac{z_n}{(1-\|w\|^2)^{\frac{\alpha_n}{2}}}\right) \in \Omega\right\}. \tag{1.1}$$

Here $\mathbb{B}^k$ is the unit ball in $\mathbb{C}^k$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ with each α_j greater than 0. We will often use U^α to denote $U^\alpha(\Omega)$.

For each multi-index β, let D_z^β denote the differential operator $(\frac{\partial}{\partial z_1})^{\beta_1} \cdots (\frac{\partial}{\partial z_n})^{\beta_n}$. Given functions of several variables f and g, we use $f \lesssim g$ to denote that $f \leq Cg$ for a constant C. If $f \lesssim g$ and $g \lesssim f$, then we say f is comparable to g and write $f \simeq g$.

Next we introduce the estimates needed for the derivatives of the Bergman kernel on Ω.

Definition 1.1. Let Ω be a domain in $\mathbb{C}^n$. Let h be a positive function on Ω. A kernel K on $\Omega \times \Omega$ is h-regular of type l if there exists $a > 0$ such that for all $\epsilon \in (0, a)$, we have

$$\int_\Omega |K(z;\zeta)|\, h^{-\epsilon}(\zeta) dV(\zeta) \lesssim h^{-\epsilon-l}(z). \tag{1.2}$$

Now we are ready to state our main theorem:

Theorem 1.2. *Let ρ be a defining function for $\Omega \subseteq \mathbb{C}^n$ and let $U^\alpha \subseteq \mathbb{C}^{n+k}$ be as in* (1.1). *Suppose the Bergman kernel K_Ω satisfies the following two properties:*

(1) *K_Ω is $(-\rho)$-regular of type* 0.
(2) *$D_z^\beta K_\Omega(z;\bar{\zeta})$ is $(-\rho)$-regular of type $|\beta|$ whenever $|\beta| \leq k$.*

Then the Bergman projection is bounded on $L^p(U^\alpha)$ for $p \in (1, \infty)$.

We note that assumption (1) implies that the Bergman projection on Ω is bounded in L^p for $1 < p < \infty$. See Schur's lemma in Section 3. Using estimates for derivatives of the Bergman kernel from [McN94b, McN89, NRSW88, PS77, CD06], one can show that $D_z^\beta K_\Omega$ is $(-\rho)$-regular of type $|\beta|$ for all $\beta \in \mathbb{N}^n$ in classes of domains previously mentioned. In Theorem 1.2, we only require $D_z^\beta K_\Omega$ to be $(-\rho)$-regular of type $|\beta|$ for all β such that $|\beta| \leq k$.

In Section 2, we recall the technique in [Huo17] relating the Bergman kernels of initial domains to those of their successors. In Section 3, we discuss several lemmas and state Theorem 3.3. This result is used to prove Theorem 1.2 via Schur's lemma. We prove Theorem 3.3 in Section 4.

2. A formula for computing the Bergman kernel

In this section we recall a construction from [Huo17] that produces the Bergman kernel of various higher dimensional successors of an initial domain. We start with an initial domain Ω and construct a class of domains $U^\alpha(\Omega)$ by introducing new parameters α to Ω.

The technique in [Huo17] consists of the following 4 steps:

(1) Start with the kernel function K_Ω on the initial domain.
(2) Construct a function on $U^\alpha(\Omega) \times U^\alpha(\Omega)$ by evaluating K_Ω at a point off the diagonal.
(3) Define a specific differential operator (depending on α).
(4) Apply the operator in step (3) to the function in step (2), obtaining $K_{U^\alpha(\Omega)}$.

The points at which we evaluate in step (2) and define the operator in step (3) are independent of the initial domain Ω, but they depend on the parameters α.

We recall in the definition below the notion of "complete Reinhardt" for the symmetry property the initial domain must satisfy.

Definition 2.1. A domain $\Omega \subseteq \mathbb{C}^n$ is called complete Reinhardt in $(z_1, \ldots, z_n)$ if the containment $(z_1, \ldots, z_n) \in \Omega$ implies the containment

$$\{(\lambda_1 z_1, \ldots, \lambda_n z_n) : |\lambda_j| \leq 1 \text{ for } 1 \leq j \leq n\} \subseteq \Omega.$$

Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain in $(z_1, \ldots, z_n)$. For $\alpha \in \mathbb{R}^n_+$ and $w \in \mathbb{B}^k$, set

$$f_\alpha(z, w) = \left(\frac{z_1}{(1 - \|w\|^2)^{\frac{\alpha_1}{2}}}, \ldots, \frac{z_n}{(1 - \|w\|^2)^{\frac{\alpha_n}{2}}} \right). \tag{2.1}$$

The successor $U^\alpha(\Omega)$ is defined by

$$U^\alpha(\Omega) = \{(z, w) \in \mathbb{C}^n \times \mathbb{B}^k : f_\alpha(z, w) \in \Omega, \|w\| < 1\}. \tag{2.2}$$

For fixed $w \in \mathbb{B}^k$, let $U^\alpha_w(\Omega)$ denote the slice domain $\{z \in \mathbb{C}^n : (z, w) \in U^\alpha\}$ of U^α. We will often write U^α_w to denote $U^\alpha_w(\Omega)$. Since the mapping $f_\alpha(\cdot, w) : z \mapsto f_\alpha(z, w)$ is a biholomorphism from $U^\alpha_w(\Omega)$ onto Ω, the kernel on $U^\alpha_w(\Omega)$ can be obtained from K_Ω.

The main result in [Huo17] relates the Bergman kernel on $U^\alpha_w(\Omega)$ to K_{U^α}. To state this result, we need a few more notational definitions. Let I denote the identity operator. We define D_{U^α} to be the differential operator:

$$D_{U^\alpha} = \frac{(1 - \|\eta\|^2)^{|\alpha|}}{\pi^k (1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} \prod_{l=1}^{k} \left(lI + \sum_{j=1}^{n} \alpha_j \left(I + z_j \frac{\partial}{\partial z_j} \right) \right). \tag{2.3}$$

Let $h(z, w, \eta)$ denote the following:

$$h(z, w, \eta) = \left(z_1 \left(\frac{1 - \|\eta\|^2}{1 - \langle w, \eta \rangle} \right)^{\alpha_1}, \ldots, z_n \left(\frac{1 - \|\eta\|^2}{1 - \langle w, \eta \rangle} \right)^{\alpha_n} \right). \tag{2.4}$$

The formula for K_{U^α} in [Huo17] can be expressed as follows:

Theorem 2.2. *For $(z, w; \zeta, \eta) \in U^\alpha \times U^\alpha$, let D_{U^α} and $h(z, w, \eta)$ be as in* (2.3) *and* (2.4). *Then*

$$K_{U^\alpha}\left(z, w; \bar{\zeta}, \bar{\eta}\right) = D_{U^\alpha} K_{U^\alpha_\eta}\left(h(z, w, \eta); \bar{\zeta}\right). \tag{2.5}$$

3. Lemmas and Theorem 3.3

The proof of Theorem 1.2 uses the following variant of Schur's lemma. See [EM16] for its proof.

Lemma 3.1 (Schur's lemma). *Let Ω be a domain in $\mathbb{C}^n$ and let K be a non-negative measurable function on $\Omega \times \Omega$. Let $\mathcal{K}$ be the integral operator with kernel K. Suppose there exists a positive auxiliary function h on Ω and a number $a > 0$ such that for all $\epsilon \in (0, a)$, the following two inequalities hold:*

(1) $\mathcal{K}(h^{-\epsilon})(z) = \int_\Omega K(z,\zeta)h(\zeta)^{-\epsilon}dV(\zeta) \lesssim h^{-\epsilon}(z)$,
(2) $\mathcal{K}(h^{-\epsilon})(\zeta) = \int_\Omega K(z,\zeta)h(z)^{-\epsilon}dV(z) \lesssim h^{-\epsilon}(\zeta)$.

Then $\mathcal{K}$ is a bounded operator on $L^p(\Omega)$, for all $p \in (1, \infty)$.

We will take the function $K(z,\zeta)$ from Lemma 3.1 to be the absolute Bergman kernel $|K_\Omega(z;\bar{\zeta})|$. Inequalities (1) and (2) in the lemma are equivalent since $K_\Omega(z;\bar{\zeta}) = \overline{K_\Omega(\zeta,\bar{z})}$. The L^p boundedness of the corresponding operator $\mathcal{K}$ then implies the L^p boundedness of the Bergman projection. To show that the Bergman projection on Ω is L^p bounded for $p \in (1,\infty)$, it suffices to find an auxiliary function h as in Lemma 3.1 and show that K_Ω is h-regular of type 0. In many cases, one can choose h to be the distance function to the boundary.

From now on we let Ω be a smooth bounded complete Reinhardt domain in $\mathbb{C}^n$. On such a domain Ω, a defining function with several useful symmetry properties can be chosen.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a smooth complete Reinhardt domain. Then there exists a defining function ρ of Ω satisfying the following properties:*

(a) *ρ is smooth in a neighborhood of the boundary $\mathbf{b}\Omega$.*
(b) *If $|z_j| = |\zeta_j|$ for $1 \le j \le n$, then $\rho(z) = \rho(\zeta)$.*
(c) *If $|z_j| \le |\zeta_j|$ for $1 \le j \le n$, then $\rho(z) \le \rho(\zeta)$.*
(d) *For $1 \le j \le n$, $z_j\rho_{z_j}(z) \ge 0$.*
(e) *If $z \in \mathbf{b}\Omega$, then $\sum_{j=1}^n z_j\rho_{z_j}(z) > 0$.*

Proof. Set ρ to be the function defined by the distance between z and $\mathbf{b}\Omega$:

$$\rho(z) = \begin{cases} \text{-dist}(z, \mathbf{b}\Omega) & z \in \Omega, \\ \text{dist}(z, \mathbf{b}\Omega) & z \notin \Omega. \end{cases}$$

Then property (a) is true for any domain Ω with smooth boundary. Properties (b) and (c) also hold since Ω is complete Reinhardt. Consider polar coordinates $z_j = t_je^{i\theta_j}$ for $1 \le j \le n$. Since ρ is invariant under the rotation in each coordinate, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta_j}\rho\left(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n}\right) \\ &= i\left(z_j\rho_{z_j}(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n}) - \bar{z}_j\rho_{\bar{z}_j}(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n})\right). \end{aligned} \tag{3.1}$$

The monotonicity of ρ in $|z_j|$ implies that

(3.2)
$$0 \le t_j\frac{\partial}{\partial t_j}\rho\left(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n}\right) = z_j\rho_{z_j}(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n}) + \bar{z}_j\rho_{\bar{z}_j}(t_1e^{i\theta_1}, \dots, t_ne^{i\theta_n}).$$

Combining these two formulas yields property (d).

To prove property (e), it suffices to show that $\sum_{j=1}^n z_j\rho_{z_j}(z) \neq 0$ on $\mathbf{b}\Omega$. Suppose not. Then there exists some $z \in \mathbf{b}\Omega$ such that $z_j\rho_{z_j}(z) = 0$ for all j. Let $\mathcal{A}$ denote the set of indices j such that $z_j = 0$ and let $\mathcal{B}$ denote the complement of $\mathcal{A}$ in $\{1, \dots, n\}$. Then $\rho_{z_j}(z)$ equals 0 for all $j \in \mathcal{A}$. Since the gradient of ρ does not vanish on $\mathbf{b}\Omega$, there exists an index $j_0 \in \mathcal{B}$ such that $\rho_{z_{j_0}}(z) \neq 0$. Thus z_{j_0} equals 0. The fact that $z_{j_0} = 0$ and property (c) then imply that z is a local min for $\rho(z)$ in the z_{j_0} direction. This contradicts $\rho_{z_{j_0}}(z) \neq 0$. Therefore the sum $\sum_{j=1}^n z_j \rho_{z_j}(z)$ does not vanish on the boundary. □

The crucial estimates for Theorem 1.2 arise from the following theorem:

Theorem 3.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a smooth complete Reinhardt domain with a defining function ρ. For $\alpha \in \mathbb{R}_+^n$, let f_α and U^α be as in* (2.1) *and* (2.2). *If $D_z^\beta K_\Omega$ is $(-\rho)$-regular whenever $|\beta| \leq k$, then K_{U^α} is $\big((1-\|w\|^2)(-\rho \circ f_\alpha)\big)$-regular of type* 0.

We give a proof for Theorem 3.3 in Section 4. Theorem 3.3 implies Theorem 1.2. Indeed, the kernel K_{U^α} being $\big((1-\|w\|^2)(-\rho \circ f_\alpha)\big)$-regular of type 0 implies that the Bergman projection on U^α is bounded in L^p for $p \in (1, \infty)$.

We end this section by referencing several estimates needed in the proof of Theorem 3.3. See for example [Zhu05].

Lemma 3.4. *Let σ denote Lebesgue measure on the unit sphere $\mathbb{S}^k \subset \mathbb{C}^k$. For $\epsilon < 1$ and $w \in \mathbb{B}^k$, let*

$$a_{\epsilon,\delta}(w) = \int_{\mathbb{B}^k} \frac{(1-\|\eta\|^2)^{-\epsilon}}{|1-\langle w, \eta\rangle|^{1+k-\epsilon-\delta}} dV(\eta), \tag{3.3}$$

and let

$$b_\delta(w) = \int_{\mathbb{S}^k} \frac{1}{|1-\langle w, \eta\rangle|^{k-\delta}} d\sigma(\eta). \tag{3.4}$$

Then

1. *For $\delta > 0$, both $a_{\epsilon,\delta}$ and b_δ are bounded on $\mathbb{B}^k$.*
2. *For $\delta = 0$, both $a_{\epsilon,\delta}(w)$ and $b_\delta(w)$ are comparable to the function $-\log(1-\|w\|^2)$.*
3. *For $\delta < 0$, both $a_{\epsilon,\delta}(w)$ and $b_\delta(w)$ are comparable to the function $(1-\|w\|^2)^\delta$.*

4. Proof of Theorem 3.3

Proof of Theorem 3.3. Recall that for each multi-index β, D_z^β is the differential operator $(\frac{\partial}{\partial z_1})^{\beta_1} \cdots (\frac{\partial}{\partial z_n})^{\beta_n}$. Then D_{U^α} in the previous section can be regarded as a sum of D_z^β:

$$D_{U^\alpha} = \frac{(1-\|\eta\|^2)^{|\alpha|}}{\pi^k(1-\langle w, \eta\rangle)^{1+k+|\alpha|}} \left(\sum_{|\beta|\leq k} c_\beta z^\beta D_z^\beta \right), \tag{4.1}$$

where c_β are fixed constants.

The main goal in this proof is to show the following inequality:

$$\int_{U^\alpha} \left|K_{U^\alpha}(z, w; \bar\zeta, \bar\eta)\right| \left(-\rho\left(f_\alpha(\zeta, \eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV \lesssim \left(-\rho\left(f_\alpha(z, w)\right)(1-\|w\|^2)\right)^{-\epsilon}. \tag{4.2}$$

To estimate the integral

$$\int_{U^\alpha} \left|K_{U^\alpha}(z, w; \bar\zeta, \bar\eta)\right| \left(-\rho\left(f_\alpha(\zeta, \eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV, \tag{4.3}$$

we use the formula in Theorem 2.2. Substituting (2.5) into the integral in (4.3) yields

$$\begin{aligned}&\int_{U^\alpha} \left|K_{U^\alpha}(z,w;\bar\zeta,\bar\eta)\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV \\ &= \int_{U^\alpha} \left|D_{U^\alpha} K_{U^\alpha_\eta}\left(h(z,w,\eta);\bar\zeta\right)\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV. \end{aligned} \tag{4.4}$$

We set

$$I_\beta = \frac{c_\beta(1-\|\eta\|^2)^{|\alpha|}}{(1-\langle w,\eta\rangle)^{1+k+|\alpha|}} z^\beta D_z^\beta$$

and

$$J_\beta = \int_{U^\alpha} \left|I_\beta K_{U^\alpha_\eta}\left(h(z,w,\eta);\bar\zeta\right)\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV. \tag{4.5}$$

By the triangle inequality, we have

$$\int_{U^\alpha} \left|D_{U^\alpha} K_{U^\alpha_\eta}\left(h(z,w,\eta);\bar\zeta\right)\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV \le \sum_{|\beta|\le k} J_\beta. \tag{4.6}$$

Therefore it suffices to prove that $J_\beta \lesssim \left(-\rho\left(f_\alpha(z,w)\right)(1-\|w\|^2)\right)^{-\epsilon}$ for each β.

The integral J_β equals

$$c_\beta \int_{U^\alpha} \left|\frac{(1-\|\eta\|^2)^{|\alpha|}}{(1-\langle w,\eta\rangle)^{1+k+|\alpha|}} z^\beta D_z^\beta K_{U^\alpha_\eta}\left(h(z,w,\eta);\bar\zeta\right)\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV. \tag{4.7}$$

In order to use $(-\rho)$-regularity assumptions of $D^\beta K_\Omega$ for estimating (4.7), we need to write $D_z^\beta K_{U^\alpha_\eta}$ in (4.7) in terms of $D_z^\beta K_\Omega$ and transform (4.7) into an integral on $\mathbb{B}^k \times \Omega$.

Recall the mapping $f_\alpha(\cdot,\eta)$ from (2.1) defined by

$$f_\alpha(\cdot,\eta) : z \mapsto \left(\frac{z_1}{(1-\|\eta\|^2)^{\alpha_1/2}}, \dots, \frac{z_n}{(1-\|\eta\|^2)^{\alpha_n/2}}\right). \tag{4.8}$$

It is a biholomorphism from U^α_η onto Ω. Hence we can write the kernel function $K_{U^\alpha_\eta}$ in terms of K_Ω using the biholomorphic transformation formula

$$K_{U^\alpha_\eta}(z;\bar\zeta) = (1-\|\eta\|^2)^{-|\alpha|} K_\Omega(f_\alpha(z,\eta), \overline{f_\alpha(\zeta,\eta)}). \tag{4.9}$$

Applying (4.9) to (4.7) yields

$$J_\beta = c_\beta \int_{U^\alpha} \left|\frac{z^\beta D_z^\beta K_\Omega\left(h'(z,w,\eta);\overline{f_\alpha(\zeta,\eta)}\right)}{|1-\langle w,\eta\rangle|^{1+k+|\alpha|}}\right| \left(-\rho\left(f_\alpha(\zeta,\eta)\right)(1-\|\eta\|^2)\right)^{-\epsilon} dV, \tag{4.10}$$

where $h'(z,w,\eta) = \left(\frac{z_1(1-\|\eta\|^2)^{\alpha_1/2}}{(1-\langle w,\eta\rangle)^{\alpha_1}}, \dots, \frac{z_n(1-\|\eta\|^2)^{\alpha_n/2}}{(1-\langle w,\eta\rangle)^{\alpha_n}}\right)$.

By substituting $t_j = \frac{\zeta_j}{(1-\|\eta\|^2)^{\alpha_j/2}}$ for $1 \le j \le n$ in (4.10), we transform J_β into an integral on $\mathbb{B}^k \times \Omega$:

$$J_\beta = c_\beta \int_{\mathbb{B}^k} \int_\Omega \left|\frac{z^\beta D_z^\beta K_\Omega\left(h'(z,w,\eta);\bar t\right)}{(1-\|\eta\|^2)^{\epsilon-|\alpha|}|1-\langle w,\eta\rangle|^{1+k+|\alpha|}}\right| (-\rho(t))^{-\epsilon} dV(t) dV(\eta). \tag{4.11}$$

For $1 \le j \le n$, let D_j denote the partial derivative $\frac{\partial}{\partial z_j}$. Since

$$\begin{aligned} D_j K_\Omega\left(h'(z,w,\eta);\bar t\right) &= \frac{\partial h'_j}{\partial z_j}\frac{\partial}{\partial h'_j} K_\Omega\left(h'(z,w,\eta);\bar t\right) \\ &= \frac{(1-\|\eta\|^2)^{\alpha_j/2}}{(1-\langle w,\eta\rangle)^{\alpha_j}}\frac{\partial}{\partial h'_j} K_\Omega\left(h'(z,w,\eta);\bar t\right), \end{aligned} \tag{4.12}$$

applying the $(-\rho)$-regularity of $D_z^\beta K_\Omega$ to the inner integral in (4.11) yields

$$J_\beta \lesssim \int_{\mathbb{B}^k} \left| \frac{z^\beta \left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon-|\beta|}}{(1-\|\eta\|^2)^{\epsilon-|\alpha|-\alpha\cdot\beta/2}|1-\langle w,\eta\rangle|^{1+k+\alpha\cdot(\mathbf{1}+\beta)}} \right| dV(\eta). \tag{4.13}$$

Here we use the notation $\alpha\cdot\beta$ to denote $\sum_{j=1}^n \alpha_j\beta_j$ and use the notation $\mathbf{1}$ to denote the multi-index $(1,\dots,1)\in\mathbb{N}^n$. When $\beta = \mathbf{0}$, we have

$$J_{\mathbf{0}} \lesssim \int_{\mathbb{B}^k} \left| \frac{\left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon}}{(1-\|\eta\|^2)^{\epsilon-|\alpha|}|1-\langle w,\eta\rangle|^{1+k+|\alpha|}} \right| dV(\eta). \tag{4.14}$$

Since $w,\eta\in\mathbb{B}^k$, the triangle inequality and Cauchy-Schwarz inequality imply that

$$\left|\frac{z_j(1-\|\eta\|^2)^{\alpha_j/2}}{(1-\langle w,\eta\rangle)^{\alpha_j}}\right| \le \left|\frac{z_j(1-\|\eta\|^2)^{\alpha_j/2}}{(1-\|w\|^2)^{\alpha_j/2}(1-\|\eta\|^2)^{\alpha_j/2}}\right| = \left|\frac{z_j}{(1-\|w\|^2)^{\alpha_j/2}}\right|.$$

Therefore, property (c) in Lemma 3.2 implies that

$$\begin{aligned} J_{\mathbf{0}} &\lesssim \int_{\mathbb{B}^k} \left| \frac{\left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon}}{(1-\|\eta\|^2)^{\epsilon-|\alpha|}|1-\langle w,\eta\rangle|^{1+k+|\alpha|}} \right| dV(\eta) \\ &\le \left(-\rho(f_\alpha(z,w))\right)^{-\epsilon}\int_{\mathbb{B}^k}\frac{(1-\|\eta\|^2)^{-\epsilon+|\alpha|}}{|1-\langle w,\eta\rangle|^{1+k+|\alpha|}}dV(\eta). \end{aligned} \tag{4.15}$$

For $w,\eta\in\mathbb{B}^k$, we have

$$\frac{1-\|\eta\|^2}{|1-\langle w,\eta\rangle|} \le \frac{1-\|\eta\|^2}{1-|\langle w,\eta\rangle|} < \frac{1-\|\eta\|^2}{1-\|\eta\|} < 2. \tag{4.16}$$

Applying this inequality and Lemma 3.4 to (4.15) yields the inequality we need for J_0:

$$\begin{aligned} J_{\mathbf{0}} &\lesssim \left(-\rho(f_\alpha(z,w))\right)^{-\epsilon}\int_{\mathbb{B}^k}\frac{(1-\|\eta\|^2)^{-\epsilon}}{|1-\langle w,\eta\rangle|^{1+k}}dV(\eta) \\ &\lesssim \left(-\rho(f_\alpha(z,w))\right)^{-\epsilon}(1-\|w\|^2)^{-\epsilon}. \end{aligned} \tag{4.17}$$

For the case $\beta \ne \mathbf{0}$, we recall the integral we need to estimate:

$$\int_{\mathbb{B}^k} \left| \frac{z^\beta \left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon-|\beta|}}{(1-\|\eta\|^2)^{\epsilon-|\alpha|-\alpha\cdot\beta/2}|1-\langle w,\eta\rangle|^{1+k+\alpha\cdot(\mathbf{1}+\beta)}} \right| dV(\eta). \tag{4.18}$$

After rewriting the integral in spherical coordinates $\eta = rt$ with $r\in[0,1]$ and $t\in\mathbb{S}^k$, we would like to write $\left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon-|\beta|}$ in terms of the $|\beta|$-th order derivative of $\left(-\rho\left(h'(z,w,rt)\right)\right)^{-\epsilon}$ in r. These derivatives vanish at the point $\eta = w$ and hence are relatively small when compared with $(-\rho)^{-\epsilon-|\beta|}$. To deal with this problem, we need to move the vanishing point $\eta = w$ to the origin.

When $w = \mathbf{0}$, we keep (4.18) the same. When $w \in \mathbb{B}^k - \{\mathbf{0}\}$, we set

$$\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w \rangle},$$

where $s_w = \sqrt{1 - \|w\|^2}$, $P_w(z) = \frac{\langle z,w\rangle}{\|w\|^2} w$, and $Q_w(z) = z - \frac{\langle z,w\rangle}{\|w\|^2} w$. Then φ_w is the automorphism of $\mathbb{B}^k$ that sends $\mathbf{0}$ to w and satisfies $\varphi_w \circ \varphi_w = id$. We use this φ_w to send the point $\eta = w$ to the origin. Setting $\tau = \varphi_w(\eta)$, we then have

$$\eta = \varphi_w(\tau), \tag{4.19}$$

$$1 - \langle \eta, w \rangle = \frac{1 - \|w\|^2}{1 - \langle \tau, w \rangle}, \tag{4.20}$$

$$1 - \|\eta\|^2 = \frac{(1 - \|w\|^2)(1 - \|\tau\|^2)}{|1 - \langle \tau, w \rangle|^2}, \tag{4.21}$$

$$dV(\eta) = \left(\frac{1 - \|w\|^2}{|1 - \langle \tau, w \rangle|^2} \right)^{k+1} dV(\tau). \tag{4.22}$$

Substituting (4.19), (4.20), (4.21), and (4.22) into the integral (4.18) yields

(4.23)

$$\int_{\mathbb{B}^k} \left| \frac{z^\beta \left(-\rho\left(h'(z,w,\eta)\right)\right)^{-\epsilon - |\beta|}}{(1 - \|\eta\|^2)^{\epsilon - |\alpha| - \alpha\cdot\beta/2} |1 - \langle w, \eta \rangle|^{1+k+\alpha\cdot(\mathbf{1}+\beta)}} \right| dV(\eta)$$

$$= \int_{\mathbb{B}^k} \frac{|z^\beta| \left(\frac{(1-\|w\|^2)(1-\|\tau\|^2)}{|1-\langle\tau,w\rangle|^2} \right)^{\alpha\cdot\beta/2 - \epsilon + |\alpha|}}{\left| \frac{1-\|w\|^2}{1-\langle\tau,w\rangle} \right|^{1+k+\alpha\cdot(\mathbf{1}+\beta)} \frac{(1-\|w\|^2)^{-k-1}}{|1-\langle\tau,w\rangle|^{-2(k+1)}}} \left(-\rho\left(h'(z,w,\varphi_w(\tau))\right)\right)^{-\epsilon-|\beta|} dV(\tau).$$

Canceling terms in the integral gives

(4.24)

$$\int_{\mathbb{B}^k} \frac{|z^\beta|(1 - \|\tau\|^2)^{\alpha\cdot\beta/2 - \epsilon + |\alpha|}}{|1 - \langle \tau, w \rangle|^{1+k-2\epsilon+|\alpha|}(1 - \|w\|^2)^{\alpha\cdot\beta/2+\epsilon}} \left(-\rho\left(h'(z,w,\varphi_w(\tau))\right)\right)^{-\epsilon-|\beta|} dV(\tau),$$

which is consistent with (4.18) when $w = \mathbf{0}$. Applying inequality (4.16) to (4.24) and using the fact that $\frac{|z^\beta|}{(1-\|w\|^2)^{\alpha\cdot\beta/2}}$ is bounded on Ω, we obtain the following inequality:

(4.25)

$$\int_{\mathbb{B}^k} \frac{|z^\beta|(1 - \|\tau\|^2)^{\alpha\cdot\beta/2 - \epsilon + |\alpha|}}{|1 - \langle \tau, w \rangle|^{1+k-2\epsilon+|\alpha|}(1 - \|w\|^2)^{\alpha\cdot\beta/2+\epsilon}} \left(-\rho\left(h'(z,w,\varphi_w(\tau))\right)\right)^{-\epsilon-|\beta|} dV(\tau)$$

$$\lesssim \int_{\mathbb{B}^k} \frac{(1 - \|\tau\|^2)^{\alpha\cdot\beta/2 - \epsilon + |\alpha|}}{|1 - \langle \tau, w \rangle|^{1+k-2\epsilon+|\alpha|}(1 - \|w\|^2)^{\epsilon}} \left(-\rho\left(h'(z,w,\varphi_w(\tau))\right)\right)^{-\epsilon-|\beta|} dV(\tau)$$

$$\lesssim \int_{\mathbb{B}^k} \frac{\left(-\rho\left(h'(z,w,\varphi_w(\tau))\right)\right)^{-\epsilon-|\beta|}}{|1 - \langle \tau, w \rangle|^{1+k-\epsilon}(1 - \|w\|^2)^{\epsilon}} dV(\tau).$$

We set $l(z,w,\tau) = (l_1(z,w,\tau), \ldots, l_n(z,w,\tau))$ where

$$l_j(z,w,\tau) = \left| h'_j\left(z,w,\varphi_w(\tau)\right) \right| = \left| \frac{z_j \left(\frac{(1-\|w\|^2)(1-\|\tau\|^2)}{|1-\langle\tau,w\rangle|^2} \right)^{\alpha_j/2}}{\left(\frac{1-\|w\|^2}{1-\langle\tau,w\rangle} \right)^{\alpha_j}} \right| = \frac{|z_j|(1 - \|\tau\|^2)^{\alpha_j/2}}{(1 - \|w\|^2)^{\alpha_j/2}}.$$

Then Lemma 3.2 implies that $\rho(h'(z,w,\varphi_w(\tau))) = \rho(l(z,w,\tau))$, and the integral in the last line of (4.25) becomes

$$\int_{\mathbb{B}^k} \frac{(-\rho\,(l(z,w,\tau)))^{-\epsilon-|\beta|}}{|1-\langle\tau,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}} dV(\tau). \tag{4.26}$$

Rewriting (4.26) using spherical coordinates $\tau = rt$ with $r \in [0,1)$ and $t \in \mathbb{S}^k$ yields

$$c_k \int_0^1 r^{2k-1} \int_{\mathbb{S}^k} \frac{(-\rho\,(l(z,w,rt)))^{-\epsilon-|\beta|}}{|1-\langle rt,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}} d\sigma(t)dr, \tag{4.27}$$

where c_k is a constant depending on the dimension k.

By property (e) in Lemma 3.2, there exists an open neighborhood $\mathcal{U}$ of $\mathbf{b}\Omega$ such that for any $z \in \mathcal{U}$,

$$\sum_{j=1}^{n} z_j \rho_{z_j}(z) > c, \tag{4.28}$$

for some positive c. For $\delta > 0$, let $\bar{\Omega}_\delta$ denote the set

$$\{z \in \mathbb{C}^n : \rho((1+\delta)^{\alpha_1/2} z_1, \ldots, (1+\delta)^{\alpha_n/2} z_n) \leq 0\}. \tag{4.29}$$

Then there exists a constant $\delta_0 > 0$ such that $\Omega - \mathcal{U} \subseteq \bar{\Omega}_{\delta_0}$. Since $\bar{\Omega}_{\delta_0}$ is compact in Ω, we have $(-\rho(z))^{-1} < C$ in $\bar{\Omega}_{\delta_0}$ for some constant C. Let $\mathcal{U}_0$ denote the set $\bar{\Omega}_{\delta_0}$, and let $\mathcal{U}_1$ denote the set $\Omega - \mathcal{U}_0$. Then on $\mathcal{U}_1$, inequality (4.28) still holds. For $t \in \mathbb{S}^k$ and $j = 0, 1$, set

$$U_j = \{r \in [0,1] : l(z,w,rt) \in \mathcal{U}_j\}.$$

Here the U_j's are well-defined for any $t \in \mathbb{S}^k$: for fixed z and w, the value of $l(z,w,rt)$ only depends on r and $\|t\|$. For each U_j, we set

$$\mathcal{I}_j^\beta = \int_{U_j} r^{2k-1} \int_{\mathbb{S}^k} \frac{(-\rho\,(l(z,w,rt)))^{-\epsilon-|\beta|}}{|1-\langle rt,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}} d\sigma(t)dr. \tag{4.30}$$

We claim that $\mathcal{I}_j^\beta \lesssim \left((-\rho)(l(z,w,t))(1-|w|^2)\right)^{-\epsilon}$ for each j. Then by having

$$J_\beta \lesssim \mathcal{I}_0^\beta + \mathcal{I}_1^\beta \lesssim \left((-\rho)(l(z,w,t))(1-|w|^2)\right)^{-\epsilon}, \tag{4.31}$$

we complete the proof.

We first consider $\mathcal{I}_0^\beta$. Since $(-\rho(l(z,w,rt)))^{-1} < C$ for $r \in U_0$, we have

$$\mathcal{I}_0^\beta \lesssim \int_{U_0} r^{2k-1} \int_{\mathbb{S}^k} \frac{1}{|1-\langle rt,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}} d\sigma(t)dr. \tag{4.32}$$

Applying Lemma 3.4 to the inner integral of (4.32) yields

$$\int_{\mathbb{S}^k} \frac{1}{|1-\langle rt,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}} d\sigma(t) \lesssim (1-\|w\|^2)^{-\epsilon}(1-r^2\|w\|^2)^{\epsilon-1}. \tag{4.33}$$

Then inequality (4.33) gives the desired estimate for $\mathcal{I}_0^\beta$:

$$
\begin{aligned}
\mathcal{I}_0^\beta \lesssim & \int_{U_0} r^{2k-1}(1-\|w\|^2)^{-\epsilon}(1-r^2\|w\|^2)^{\epsilon-1}dr \\
\lesssim & \int_{U_0} r^{2k-1}(1-\|w\|^2)^{-\epsilon}(1-r^2)^{\epsilon-1}dr \\
\lesssim & (1-\|w\|^2)^{-\epsilon} \\
\lesssim & \left((-\rho)(l(z,w,t))(1-|w|^2)\right)^{-\epsilon}. \qquad (4.34)
\end{aligned}
$$

Now we turn to $\mathcal{I}_1^\beta$. When $r\in U_1$, we have $l(z,w,rt)\in\mathcal{U}_1$ and

$$
\sum_{j=1}^n l_j(z,w,rt)\rho_{z_j}(l(z,w,rt))>c. \qquad (4.35)
$$

For such an r, $\frac{\partial}{\partial r}\left((-\rho)^{-\epsilon-|\beta|+1}(l(z,w,rt))\right)$ is controlled from below by $(-\rho)^{-\epsilon-|\beta|}(l(z,w,rt))$:

(4.36)

$$
\begin{aligned}
&-\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon-|\beta|+1} \\
=&2(\epsilon+|\beta|-1)(-\rho)^{-\epsilon-|\beta|}(l(z,w,rt))\sum_{j=1}^n \frac{\alpha_j r|z_j|(1-r^2)^{\alpha_j/2-1}}{(1-\|w\|^2)^{\alpha_j/2}}\rho_{z_j}(l(z,w,rt)) \\
\gtrsim&\frac{r(-\rho)^{-\epsilon-|\beta|}(l(z,w,rt))}{(1-r^2)}\sum_{j=1}^n l_j(z,w,rt)\rho_{z_j}(l(z,w,rt)) \\
\gtrsim&\frac{r(-\rho)^{-\epsilon-|\beta|}(l(z,w,rt))}{(1-r^2)}.
\end{aligned}
$$

Applying (4.36), (4.16), and Lemma 3.4 to (4.30) then yields

$$
\begin{aligned}
\mathcal{I}_1^\beta \lesssim & -\int_{U_j} r^{2k-2}\int_{\mathbb{S}^k}\frac{(1-r^2)\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{|1-\langle rt,w\rangle|^{1+k-\epsilon}(1-\|w\|^2)^{\epsilon}}d\sigma(t)dr \\
\lesssim & -\int_{U_j} r^{2k-2}\int_{\mathbb{S}^k}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{|1-\langle rt,w\rangle|^{k-\epsilon}(1-\|w\|^2)^{\epsilon}}d\sigma(t)dr \\
\lesssim & -(1-\|w\|^2)^{-\epsilon}\int_{U_j} r^{2k-2}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}dr. \qquad (4.37)
\end{aligned}
$$

Since for fixed z and w, the point $l(z,w,0)$ is closest to $\mathbf{b}\Omega$, we may assume that $U_1=[0,r_0]$ where r_0 depends on both z and w.

When $k=1$, integrating the last line of (4.37) by parts yields

$$
\begin{aligned}
&-(1-\|w\|^2)^{-\epsilon}\int_{U_1} r^{2k-2}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}dr \\
=&-(1-\|w\|^2)^{-\epsilon}\int_0^{r_0}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}dr \\
=&-\frac{r^{2k-2}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}\Big|_0^{r_0}}{(1-\|w\|^2)^{\epsilon}}. \qquad (4.38)
\end{aligned}
$$

Noting that $k = 1$ also implies that $-\epsilon - |\beta| + 1 \geq -\epsilon - k + 1 = -\epsilon$, we have

$$\begin{aligned} &-(1-\|w\|^2)^{-\epsilon}\int_{U_1}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon}dr \\ &\leq \frac{\left(-\rho\left(l(z,w,0)\right)\right)^{-\epsilon}+\left(-\rho\left(l(z,w,r_0t)\right)\right)^{-\epsilon}}{(1-\|w\|^2)^{\epsilon}}. \end{aligned} \tag{4.39}$$

By its definition, the point $l(z, w, r_0t)$ is in $\mathcal{U}_0$. Therefore $(-\rho\left(l(z,w,r_0t)\right))^{-\epsilon-|\beta|+1} \lesssim 1$ and the desired estimate follows:

$$\mathcal{I}_1^\beta = -\int_{U_1}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon}}{(1-\|w\|^2)^{\epsilon}}dr \lesssim \left(-\rho\left(f_\alpha(z,w)\right)\right)^{-\epsilon}(1-\|w\|^2)^{-\epsilon}. \tag{4.40}$$

When $k > 1$, integrating the last line of (4.37) by parts yields

$$\begin{aligned} &-(1-\|w\|^2)^{-\epsilon}\int_{U_1} r^{2k-2}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}dr \qquad (4.41)\\ &=-(1-\|w\|^2)^{-\epsilon}\int_0^{r_0} r^{2k-2}\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}dr \\ &=-\frac{r^{2k-2}(-\rho\left(l(z,w,rt)\right))^{-\epsilon-|\beta|+1}\Big|_0^{r_0}}{(1-\|w\|^2)^{\epsilon}} \\ &\quad+\int_0^{r_0}(2k-2)r^{2k-3}\frac{\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr. \end{aligned}$$

The numerator of the term in the third line equals $r_0^{2k-2}(-\rho(l(z,w,r_0t)))^{-\epsilon-|\beta|+1}$, which is also controlled by a constant. Thus it remains to show that

$$\int_0^{r_0} r^{2k-3}\frac{\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr \lesssim \left(-\rho\left(f_\alpha(z,w)\right)\right)^{-\epsilon}(1-\|w\|^2)^{-\epsilon}. \tag{4.42}$$

Applying (4.36) to the left hand side of (4.42) gives

$$\int_0^{r_0} r^{2k-3}\frac{\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr \lesssim -\int_0^{r_0} r^{2k-4}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+2}}{(1-\|w\|^2)^{\epsilon}}dr.$$

This together with (4.41) implies that for $k > 1$,

$$\begin{aligned} &-\int_0^{r_0} r^{2k-2}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr \\ &\lesssim -\int_0^{r_0} r^{2k-4}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-|\beta|+2}}{(1-\|w\|^2)^{\epsilon}}dr. \end{aligned} \tag{4.43}$$

Since (4.43) holds whenever $|\beta| \leq k$, we have for $0 < s \leq k$,

$$\begin{aligned} &-\int_0^{r_0} r^{2k-2}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-s+1}}{(1-\|w\|^2)^{\epsilon}}dr \\ &\lesssim -\int_0^{r_0} r^{2k-4}\frac{\frac{\partial}{\partial r}\left(-\rho\left(l(z,w,rt)\right)\right)^{-\epsilon-s+2}}{(1-\|w\|^2)^{\epsilon}}dr. \end{aligned} \tag{4.44}$$

Repeated use of inequality (4.44) then gives

$$\begin{aligned} &-\int_0^{r_0} r^{2k-2}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr\\ &\lesssim -\int_0^{r_0} r^{2k-4}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon-|\beta|+2}}{(1-\|w\|^2)^{\epsilon}}dr\\ &\quad\vdots\\ &\lesssim -\int_0^{r_0} r^{2k-2|\beta|}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon}}{(1-\|w\|^2)^{\epsilon}}dr. \end{aligned}\tag{4.45}$$

Noting that $r^{2k-2|\beta|}$ is bounded on $[0,r_0]$, we have

$$-\int_0^{r_0} r^{2k-2|\beta|}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon}}{(1-\|w\|^2)^{\epsilon}}dr \le -\int_0^{r_0}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon}}{(1-\|w\|^2)^{\epsilon}}dr.\tag{4.46}$$

Applying inequality (4.40) to (4.46) then yields

$$\mathcal{I}_1^{\beta} = -\int_0^{r_0} r^{2k-2}\frac{\frac{\partial}{\partial r}(-\rho(l(z,w,rt)))^{-\epsilon-|\beta|+1}}{(1-\|w\|^2)^{\epsilon}}dr \lesssim (-\rho(f_\alpha(z,w)))^{-\epsilon}(1-\|w\|^2)^{-\epsilon},\tag{4.47}$$

which completes the proof. □

Remark. As in the proof of Thereom 3.3, we can obtain an L^p regularity result for the Bergman projection on more generalized domains which are generated from Ω by iterating the construction of U^α from (2.2).

Set $\alpha = (\alpha^{(1)},\dots,\alpha^{(l)}) \in \mathbb{R}^n_+\times\cdots\times\mathbb{R}^n_+$ where each $\alpha^{(j)}$ is in $\mathbb{R}^n_+$. Let $k_1,\dots,k_l$ be l positive integers. The successor $\mathbf{U}(\Omega)$ is defined by

$$\mathbf{U}(\Omega) = \{(z,w_1,w_2,\dots,w_l)\in\mathbb{C}^n\times\mathbb{B}^{k_1}\times\cdots\times\mathbb{B}^{k_l} : (\mathbf{f}_\alpha(z,w_1,\dots,w_l))\in\Omega\},\tag{4.48}$$

where

$$\mathbf{f}_\alpha(z,w_1,\dots,w_l) = \left(\frac{z_1}{\prod_{j=1}^l(1-\|w_j\|^2)^{\frac{\alpha_1^{(j)}}{2}}},\dots,\frac{z_n}{\prod_{j=1}^l(1-\|w_j\|^2)^{\frac{\alpha_n^{(j)}}{2}}}\right).\tag{4.49}$$

Suppose $\Omega\subseteq\mathbb{C}^n$ is a smooth complete Reinhardt domain with defining function ρ and $D_z^\beta K_\Omega(z;\bar\zeta)$ is $(-\rho)$-regular of type $|\beta|$ for $0\le|\beta|\le\sum_{j=1}^l k_j$. Then the Bergman projection on $\mathbf{U}(\Omega)$ is L^p bounded for all $1<p<\infty$. The proof of this statement is similar to the proof for the first successor. We omit it here.

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