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OPERATORS WITH CLOSED NUMERICAL RANGES IN NEST ALGEBRAS

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ABSTRACT. In the present paper, we continue our research on numerical ranges of operators. With newly developed techniques, we show that

Let $\mathcal{N}$ be a nest on a Hilbert space $\mathcal{H}$ and $T \in \mathcal{T}(\mathcal{N})$, where $\mathcal{T}(\mathcal{N})$ denotes the nest algebra associated with $\mathcal{N}$. Then for given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T+K \in \mathcal{T}(\mathcal{N})$ and the numerical range of $T+K$ is closed.

As applications, we show that the statement of the above type holds for the class of Cowen-Douglas operators, the class of nilpotent operators and the class of quasinilpotent operators.

1. INTRODUCTION

Throughout this paper, let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$ and the ideal of compact operators on $\mathcal{H}$, respectively.

Recall that the *numerical range* of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$W(T) = \big\{ \langle Tx, x \rangle : x \in \mathcal{H},\ \|x\| = 1 \big\}.$$

Clearly, $W(T)$ is a nonempty bounded subset of $\mathbb{C}$. The classical Toeplitz-Hausdorff theorem (see [9, 16]) asserts that the numerical range of an operator is always convex.

Let $T \in \mathcal{B}(\mathcal{H})$. Denote by T^* the adjoint of T, by $\sigma(T)$ the spectrum of T, by $\operatorname{conv}\sigma(T)$ the convex hull of $\sigma(T)$, by $\overline{W(T)}$ the closure of $W(T)$. It is well known that $\operatorname{conv}\sigma(T) \subseteq \overline{W(T)}$ (see [8, page 115]). In particular, if T is normal (i.e. $T^*T = TT^*$), then $\overline{W(T)} = \operatorname{conv}\sigma(T)$ (see [7, Theorem 1.4-4]). In addition, $W(T) \subseteq \mathbb{R}$ if and only if T is self-adjoint (i.e. $T^* = T$) (see [7, Theorem 1.2-2]). Moreover, it is well known that the numerical range of a direct sum is the convex hull of the numerical ranges of the summands. Some other properties of numerical range can be found in [7, 8].

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As we all know, operators on finite dimensional Hilbert spaces always have closed numerical ranges. But it may fail for operators acting on infinite dimensional Hilbert spaces, such as the unilateral shift operator (see [8, page 317]), while operators with closed numerical ranges are dense in $\mathcal{B}(\mathcal{H})$ with respect to the uniform (norm) topology. In fact, Bourin ([3, Proposition 1.3]) proved that for given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there must exist a compact operator K with $\|K\| < \varepsilon$ such that $W(T+K)$ is closed. That is, every operator in $\mathcal{B}(\mathcal{H})$ has a closed numerical range under small compact perturbations.

Inspired by Bourin's theorem, in [13], we introduced the concept of being strongly numerically closed. But to be more precise, we realize that such an operator class should be said to be approximately strongly numerically closed. Here we make a correction. An operator class $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ is said to be *approximately strongly numerically closed*, if for any $T \in \mathcal{A}$ and any $\varepsilon > 0$ there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{A}$ and $W(T+K)$ is closed.

In our previous work [13], we proved that the class of triangular operators, the class of hyponormal operators and the class of weighted shift operators, etc., are all approximately strongly numerically closed. As we can see, all these results obtained are about operator classes which maybe not operator algebras. Naturally, if we pay attention to operator algebras, the research will become more interesting.

Note that nest algebras are the natural analogues of upper triangular matrix algebras in infinite dimensional spaces. And what's more, nest algebras own algebraic structures as well as topological structures. So nest algebras naturally catch our attention. In addition, nest algebras are an important object of the class of nonself-adjoint operator algebras and play an important role in operator theory and operator algebras. So it is more interesting and more significant to discuss whether any nest algebra is approximately strongly numerically closed.

In this paper, with newly developed techniques and some theory of nest algebras, we will show that any nest algebra is approximately strongly numerically closed (see Theorem 2.1). And with applications of Theorem 2.1 and the techniques used in its proof, we obtain that the class of nilpotent operators, the class of Cowen-Douglas operators and the class of quasinilpotent operators are also approximately strongly numerically closed.

The rest of this paper is organized as follows. In section 2, we will give our main result by showing that any nest algebra is approximately strongly numerically closed. In section 3 we show some applications of the main result and the techniques. This section is divided into three subsections to deal with the class of nilpotent operators, the class of Cowen-Douglas operators and the class of quasinilpotent operators. And we will prove that these operator classes are also approximately strongly numerically closed.

2. Nest algebras

The following theorem is our main result.

Theorem 2.1. *For every nest $\mathcal{N}$, nest algebra $\mathcal{T}(\mathcal{N})$ associated with $\mathcal{N}$ is approximately strongly numerically closed.*

Before proving Theorem 2.1, we first review some terminologies and notation.

Given a collection $\{M_\alpha\}$ of subspaces of a Hilbert space $\mathcal{H}$, $\bigvee M_\alpha$ denotes the closed linear span and $\bigwedge M_\alpha$ denotes intersection. Then these two operations make the set of subspaces of $\mathcal{H}$ into a lattice.

Recall that a nest $\mathcal{N}$ is a chain (by inclusion) of closed subspaces of a Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under intersection and closed span. The nest algebra associated with $\mathcal{N}$, denoted by $\mathcal{T}(\mathcal{N})$, is the family of all operators defined by

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N \text{ for all N in } \mathcal{N}\}.$$

It is well known that $\mathcal{T}(\mathcal{N})$ is a weak operator closed subalgebra of $\mathcal{B}(\mathcal{H})$ (see [5, Proposition 2.2]).

For each $N \in \mathcal{N}$, denote

$$N_- = \bigvee \{N' \in \mathcal{N} : N' < N\}, \ N_+ = \bigwedge \{N' \in \mathcal{N} : N' > N\}$$

and define $\{0\}_- = \{0\}, \mathcal{H}_+ = \mathcal{H}$. If $N_- \neq N$, N_- is called the immediate predecessor to N and $N = (N_-)_+$ is called the immediate successor of N_-. In this case, the subspace $N \ominus N_-$ is called an atom of $\mathcal{N}$. If the atoms of $\mathcal{N}$ span $\mathcal{H}$, then $\mathcal{N}$ is said to be atomic. If there are no atoms in $\mathcal{N}$, $\mathcal{N}$ is continuous. If $N_1 < N_2$ in $\mathcal{N}$, the subspace $E = N_2 \ominus N_1$ is called an interval of $\mathcal{N}$. We refer the reader to Davidson's monograph [5] for the theory of nest algebras.

In addition, we need the following auxiliary results.

Lemma 2.2 ([8])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Denote by P the orthogonal projection onto $\mathcal{M}$. Then $W(PT|_{\mathcal{M}}) \subseteq W(T)$.*

Recall that the *essential numerical range* of $T \in \mathcal{B}(\mathcal{H})$, denoted by $W_e(T)$, is the nonempty set

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(T+K)}.$$

It is apparent that $W_e(T)$ is convex, closed and invariant under compact perturbations. That is, $W_e(T+K) = W_e(T)$ for any $K \in \mathcal{K}(\mathcal{H})$. Moreover, by [6, Theorem 5.1], $\lambda \in W_e(T)$ if and only if there exists a orthonormal sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\langle Tx_n, x_n\rangle \to \lambda$. For references on essential numerical ranges, see [2, 6].

Lemma 2.3 ([15, Theorem 1])**.** *If $T \in \mathcal{B}(\mathcal{H})$, then $\overline{W(T)} = \operatorname{conv}\{W(T) \cup W_e(T)\}$.*

As an application of Lemma 2.3, to prove that $\overline{W(T)} = W(T)$, it suffices to show that $W_e(T) \subseteq W(T)$.

In this paper, we denote by $\operatorname{int}\Omega$ the interior of a set Ω of $\mathbb{C}$ and by $\partial\Omega$ the boundary of Ω. Moreover, for convex sets of $\mathbb{C}$, we have the following observation. We record this as a lemma for the convenience of future reference.

Lemma 2.4. *If Ω is a bounded convex subset of $\mathbb{C}$ and $0 \in \operatorname{int}\Omega$, then for given $0 < \varepsilon < \min\{d, \frac{1}{2}\}$, where $d = \operatorname{dist}(0, \partial\Omega)$, there exist finitely many points $u_1, u_2, \ldots, u_n \in \Omega$ and a positive integer N such that $\Omega \subseteq \operatorname{conv}\{(1 + \frac{3\varepsilon}{4})\lambda_k : 1 \leq k \leq n\}$, whenever λ_k satisfies $|\lambda_k - u_k| < \frac{\varepsilon^2}{N}$ for each k, $1 \leq k \leq n$.*

The following result about nest algebras plays an important role in our proof of the main theorem.

Lemma 2.5 ([5, Theorem 3.10])**.** *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{N}$ be a nest on $\mathcal{H}$. Then there is a sequence $\{R_n\}_{n=1}^{\infty}$ of finite rank contractions in $\mathcal{T}(\mathcal{N})$ such that R_n converges to the identity in the strong operator topology.*

Lemma 2.6 ([5, Lemma 3.7])**.** *Let $x \otimes y$ be a rank one operator in $\mathcal{T}(\mathcal{N})$. Then there is an element N of $\mathcal{N}$ such that x belongs to N and y belongs to $(N_-)^{\perp}$.*

Lemma 2.7. *Let $\mathcal{N}$ be an atomic nest. If $T \in \mathcal{T}(\mathcal{N})$ and $\overline{W(T)}$ is a closed line segment of $\mathbb{C}$, then for given $\varepsilon > 0$, there is a compact operator K with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{T}(\mathcal{N})$ and $W(T + K)$ is closed.*

Proof. First, suppose that $W(T)$ is not closed. And without loss of generality, we can assume that $\overline{W(T)} = [0, 1]$. Hence T is a self-adjoint operator in $\mathcal{T}(\mathcal{N})$.

Noting that $\mathcal{N}$ is atomic, there are two possibilities:

(i) there exists an atom A_0 of $\mathcal{N}$ such that $\overline{W\big(P(A_0)T|_{A_0}\big)} = [0, 1]$,

(ii) for $\varepsilon > 0$, there exist two atoms A_1 and A_2 of $\mathcal{N}$ such that

$$\operatorname{dist}\Big(0, W\big(P(A_1)T|_{A_1}\big)\Big) < \frac{\varepsilon}{8} \quad \text{and} \quad \operatorname{dist}\Big(1, W\big(P(A_2)T|_{A_2}\big)\Big) < \frac{\varepsilon}{8},$$

where $P(A_k)$ denotes the orthogonal projection onto A_k.

Case (i). In this case, since $W(T)$ is not closed and $\overline{W(T)} = [0, 1]$, there are three possibilities:

$$W\big(P(A_0)T|_{A_0}\big) = (0, 1),\ W\big(P(A_0)T|_{A_0}\big) = (0, 1] \ \text{ and } \ W\big(P(A_0)T|_{A_0}\big) = [0, 1).$$

If $W\big(P(A_0)T|_{A_0}\big) = (0, 1)$, then using Lemma 2.3, we have $W_e\big(P(A_0)T|_{A_0}\big) = [0, 1]$. By Bourin's theorem, for given $\varepsilon > 0$, there exists a compact operator $K \in \mathcal{T}(\mathcal{N})$ with $K = P(A_0)KP(A_0)$ and $\|K\| < \varepsilon$, such that $T + K \in \mathcal{T}(\mathcal{N})$ and $W\big(P(A_0)(T + K)|_{A_0}\big)$ is closed. Then by Lemma 2.2, we have

$$[0, 1] = W_e\big(P(A_0)T|_{A_0}\big) \subseteq W\big(P(A_0)(T + K)|_{A_0}\big) \subseteq W(T + K).$$

Hence

$$W_e(T + K) = W_e(T) \subseteq [0, 1] \subseteq W(T + K).$$

It follows from Lemma 2.3 that $W(T + K)$ is closed.

If $W\big(P(A_0)T|_{A_0}\big) = (0, 1]$, then there exists a unit vector $x_1 \in A_0$ such that

$$\langle Tx_1, x_1\rangle = \langle P(A_0)T|_{A_0}x_1, x_1\rangle = 1.$$

By Lemma 2.3, we have $0 \in W_e(P(A_0)T|_{A_0})$. Hence for given $\varepsilon > 0$, there must exist a unit vector $x_2 \in A_0$ with $x_2 \perp x_1$ such that

$$\langle Tx_2, x_2\rangle = \langle P(A_0)T|_{A_0}x_2, x_2\rangle < \frac{\varepsilon}{2}.$$

Set $K = \frac{3\varepsilon}{4}x_1 \otimes x_1 - \frac{3\varepsilon}{4}x_2 \otimes x_2$. A simple calculation shows that $\|K\| < \varepsilon$. And by Lemma 2.6, we have $K \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}(\mathcal{H})$. Hence $T + K \in \mathcal{T}(\mathcal{N})$. Moreover, note that

$$\langle P(A_0)(T + K)|_{A_0}x_1, x_1\rangle = \langle (T + K)x_1, x_1\rangle = 1 + \frac{3\varepsilon}{4}$$

and

$$\begin{aligned}\langle P(A_0)(T + K)|_{A_0}x_2, x_2\rangle &= \langle (T + K)x_2, x_2\rangle \\ &= \langle Tx_2, x_2\rangle - \frac{3\varepsilon}{4} \\ &< -\frac{\varepsilon}{4}.\end{aligned}$$

Hence

$$W_e(T) \subseteq [0, 1] \subseteq W\big(P(A_0)(T + K)|_{A_0}\big) \subseteq W(T + K).$$

Then it follows from Lemma 2.3 that $W(T + K)$ is closed.

With the same argument as that in the case $W\big(P(A_0)T|_{A_0}\big) = (0,1]$, one can show that the result is true for the case where $W\big(P(A_0)T|_{A_0}\big) = [0,1)$. This completes the proof of Case (i).

Case (ii). In this case, there must exist two unit vectors $e_1 \in A_1$ and $e_2 \in A_2$ such that

$$0 \le \langle Te_1, e_1\rangle \le \frac{\varepsilon}{4} \quad \text{and} \quad 1 - \frac{\varepsilon}{4} \le \langle Te_2, e_2\rangle \le 1.$$

Set $K = -\frac{3\varepsilon}{4}(e_1 \otimes e_1 - e_2 \otimes e_2)$. It follows from Lemma 2.6 that $K \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}(\mathcal{H})$. Hence $T + K \in \mathcal{T}(\mathcal{N})$. Moreover, as $A_1 \perp A_2$, we have $e_1 \perp e_2$. Then a computation shows that $\|K\| < \varepsilon$. Furthermore, note that

$$\begin{aligned}\langle (T+K)e_1, e_1\rangle &= \langle Te_1, e_1\rangle + \langle Ke_1, e_1\rangle \\ &= \langle Te_1, e_1\rangle - \frac{3\varepsilon}{4} \\ &\le -\frac{\varepsilon}{2}\end{aligned}$$

and

$$\begin{aligned}\langle (T+K)e_2, e_2\rangle &= \langle Te_2, e_2\rangle + \langle Ke_2, e_2\rangle \\ &= \langle Te_2, e_2\rangle + \frac{3\varepsilon}{4} \\ &\ge 1 + \frac{\varepsilon}{2}.\end{aligned}$$

Hence

$$W_e(T+K) = W_e(T) \subseteq [0,1] \subseteq W(T+K).$$

By Lemma 2.3, we deduce that $W(T+K)$ is closed. □

Similarly, for any continuous nest, we have the following result.

Lemma 2.8. *Let $\mathcal{N}$ be a continuous nest. If $T \in \mathcal{T}(\mathcal{N})$ and $\overline{W(T)}$ is a closed line segment of $\mathbb{C}$, then for given $\varepsilon > 0$, there is a compact operator K with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{T}(\mathcal{N})$ and $W(T+K)$ is closed.*

To prove Lemma 2.8, we need the following lemma.

Lemma 2.9. *Let $\mathcal{N}$ be a continuous nest and $T \in \mathcal{T}(\mathcal{N})$. If T is self-adjoint with $\sigma(T) \subseteq [0,1]$ and $0, 1 \in \sigma(T)$, then for given $0 < \varepsilon < \frac{1}{2}$, there exist two intervals E_1 and E_2 of $\mathcal{N}$ with $E_1 \perp E_2$ such that*

$$\sigma\big(P(E_1)T|_{E_1}\big) \cap [0, \varepsilon] \ne \emptyset \quad \text{and} \quad \sigma\big(P(E_2)T|_{E_2}\big) \cap [1-\varepsilon, 1] \ne \emptyset,$$

where $P(E_k)$ denotes the orthogonal projection onto E_k.

Proof. For each $n \ge 1$, set $\mathcal{H}_k = N_{\frac{k}{2^n}} \ominus N_{\frac{k-1}{2^n}}$, $1 \le k \le 2^n$, where $N_0 = \{0\}$ and $N_1 = \mathcal{H}$. As $T \in \mathcal{T}(\mathcal{N})$ is self-adjoint, T admits the following matrix representation:

$$T = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{2^n} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_{2^n} \end{matrix},$$

where each T_k is self-adjoint and each omitted entry is 0.

We claim that for given $0 < \varepsilon < \frac{1}{2}$, there exists a positive integer n_0 such that

$$\sigma(T_{i_1}) \cap [0, \varepsilon] \neq \emptyset \quad \text{and} \quad \sigma(T_{i_2}) \cap [1-\varepsilon, 1] \neq \emptyset$$

for some i_1, i_2, $1 \leq i_1 \neq i_2 \leq 2^{n_0}$.

Otherwise, there exists an $\varepsilon \in (0, \frac{1}{2})$ such that for each $n \geq 1$, there exists an i_n, $1 \leq i_n \leq 2^n$ such that

$$0, 1 \in \sigma(T_{i_n}) \quad \text{and} \quad \sigma(T_k) \subseteq [\varepsilon, 1-\varepsilon] \text{ for each } k \neq i_n, 1 \leq k \leq 2^n.$$

Let P_{i_n} denote the orthogonal projection onto $N_{\frac{i_n}{2^n}} \ominus N_{\frac{i_n - 1}{2^n}}$. Then $\{I - P_{i_n}\}_{n=1}^{\infty}$ is an increasing sequence of orthogonal projections and $I - P_{i_n}$ converges to I in the strong operator topology as $n \to +\infty$.

Set $A_n = (I - P_{i_n})T|_{(I-P_{i_n})\mathcal{H}}$ for all $n \geq 1$. It is clear that each A_n is self-adjoint. And it is not hard to check that $\overline{W(T)} = \overline{\bigcup_{n=1}^{\infty} W(A_n)}$.

Moreover, noting that

$$\sigma(A_n) = \bigcup \big\{\sigma(T_k), 1 \leq k \leq 2^n, k \neq i_n\big\},$$

we deduce that $\sigma(A_n) \subseteq [\varepsilon, 1-\varepsilon]$ for all $n \geq 1$. Since each A_n is self-adjoint,

$$\overline{W(A_n)} = \operatorname{conv} \sigma(A_n) \subseteq [\varepsilon, 1-\varepsilon]$$

for all $n \geq 1$. Therefore

$$\overline{W(T)} = \overline{\bigcup_{n=1}^{\infty} W(A_n)} \subseteq [\varepsilon, 1-\varepsilon].$$

This contradicts the fact that

$$\overline{W(T)} = \operatorname{conv} \sigma(T) = [0, 1].$$

So the claim is proved. This also completes the proof of the lemma. □

Proof of Lemma 2.8. First, suppose that $W(T)$ is not closed. And without loss of generality, we can assume that $\overline{W(T)} = [0, 1]$. Then T is a self-adjoint operator in $\mathcal{T}(\mathcal{N})$. Hence $\operatorname{conv} \sigma(T) = \overline{W(T)} = [0, 1]$. This implies that $0, 1 \in \sigma(T)$ and $\sigma(T) \subseteq [0, 1]$.

For fixed $\varepsilon > 0$, without loss of generality, one can assume that $0 < \varepsilon < \frac{1}{2}$. Then by Lemma 2.9, there exist two intervals $E_1 = N_2 \ominus N_1$ and $E_2 = N_4 \ominus N_3$ satisfying $E_1 \perp E_2$ for some $N_k \in \mathcal{N}, 1 \leq k \leq 4$, such that

$$\sigma\big(P(E_1)T|_{E_1}\big) \cap [0, \frac{\varepsilon}{8}] \neq \emptyset \quad \text{and} \quad \sigma\big(P(E_2)T|_{E_2}\big) \cap [1 - \frac{\varepsilon}{8}, 1] \neq \emptyset.$$

Hence there exist two unit vectors $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$0 \leq \langle Tx_1, x_1 \rangle \leq \frac{\varepsilon}{4} \quad \text{and} \quad 1 - \frac{\varepsilon}{4} \leq \langle Tx_2, x_2 \rangle \leq 1.$$

Furthermore, there exist two elements N_1' and N_2' in $\mathcal{N}$ with $N_1 < N_1' < N_2$ and $N_3 < N_2' < N_4$ such that

$$\|x_1^{(1)}\| = \|x_1^{(2)}\| = \frac{\sqrt{2}}{2} \quad \text{and} \quad \|x_2^{(1)}\| = \|x_2^{(2)}\| = \frac{\sqrt{2}}{2},$$

where

$$x_1^{(1)} = P(N_1' \ominus N_1)x_1, \quad x_1^{(2)} = P(N_2 \ominus N_1')x_1$$

and

$$x_2^{(1)} = P(N_2' \ominus N_3)x_2, \quad x_2^{(2)} = P(N_4 \ominus N_2')x_2.$$

Since $E_1 \perp E_2$, $x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}$ are pairwise orthogonal.

Set $K = -\frac{3\varepsilon}{2}\big(x_1^{(1)} \otimes x_1^{(2)} - x_2^{(1)} \otimes x_2^{(2)}\big)$. A simple calculation shows that $\|K\| < \varepsilon$. And it follows from Lemma 2.6 that $K \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}(\mathcal{H})$. Hence $T + K \in \mathcal{T}(\mathcal{N})$.

Moreover, note that

$$\begin{aligned}\langle (T+K)x_1, x_1\rangle &= \langle Tx_1, x_1\rangle + \langle Kx_1, x_1\rangle \\ &\leq \frac{\varepsilon}{4} - \frac{3\varepsilon}{2}\|x_1^{(1)}\|^2\|x_1^{(2)}\|^2 = -\frac{\varepsilon}{8}\end{aligned}$$

and

$$\begin{aligned}\langle (T+K)x_2, x_2\rangle &= \langle Tx_2, x_2\rangle + \langle Kx_2, x_2\rangle \\ &\geq 1 - \frac{\varepsilon}{4} + \frac{3\varepsilon}{2}\|x_2^{(1)}\|^2\|x_2^{(2)}\|^2 = 1 + \frac{\varepsilon}{8}.\end{aligned}$$

Hence

$$W_e(T+K) = W_e(T) \subseteq \overline{W(T)} = [0,1] \subsetneq W(T+K).$$

It follows from Lemma 2.3 that $W(T+K)$ is closed. The proof is complete. □

Now we are in a position to give our proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that $T \in \mathcal{T}(\mathcal{N})$ and $W(T)$ is not closed. Then either $\operatorname{int} W(T) \neq \emptyset$ or $\operatorname{int} W(T) = \emptyset$.

Case 1. $\operatorname{int} W(T) \neq \emptyset$. Without loss of generality, one can assume that $\|T\| = 1$ and $0 \in \operatorname{int} W(T)$.

For fixed $\varepsilon > 0$, without loss of generality, one can assume that $0 < \varepsilon < \min\{r, \frac{1}{2}\}$, where $r = \operatorname{dist}\big(0, \partial W(T)\big)$. Then by Lemma 2.4, one can find a positive integer N and finitely many points $u_1, u_2, \ldots, u_n \in W(T)$ such that

$$W(T) \subseteq \operatorname{conv}\big\{(1 + \frac{3\varepsilon}{4})\lambda_k : 1 \leq k \leq n\big\}$$

whenever $\lambda_k \in \mathbb{C}$ satisfies $|\lambda_k - u_k| < \frac{\varepsilon^2}{N}$ for each $k, 1 \leq k \leq n$. As $u_k \in W(T)$, there exists a unit vector x_k such that $u_k = \langle Tx_k, x_k\rangle$.

Set $T_n = TR_n$ for all $n \geq 1$, where $\{R_n\}_{n=1}^\infty$ is the sequence in Lemma 2.5. Then each $T_n \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}(\mathcal{H})$ and T_n converges to T in the strong operator topology. Hence there is a sufficiently large integer n_0 such that

$$|\langle T_{n_0}x_k, x_k\rangle - u_k| < \frac{\varepsilon^2}{N} \text{ for each } k,\ 1 \leq k \leq n.$$

Denote

$$\lambda_k = \langle T_{n_0}x_k, x_k\rangle \text{ and } w_k = \frac{1}{1 + \frac{3\varepsilon}{4}}(u_k + \frac{3}{4}\varepsilon\lambda_k).$$

It is trivial that $|w_k - u_k| < \frac{\varepsilon^2}{N}$. Hence

$$\overline{W(T)} \subseteq \operatorname{conv}\big\{(1 + \frac{3\varepsilon}{4})w_k : 1 \leq k \leq n\big\} = \operatorname{conv}\big\{u_k + \frac{3}{4}\varepsilon\lambda_k : 1 \leq k \leq n\big\}.$$

Set $K = \frac{3}{4}\varepsilon T_{n_0}$. Then $K \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. Hence $T + K \in \mathcal{T}(\mathcal{N})$. Moreover, note that

$$\begin{aligned}\langle (T+K)x_k, x_k\rangle &= \langle Tx_k, x_k\rangle + \langle Kx_k, x_k\rangle \\ &= u_k + \frac{3}{4}\varepsilon\lambda_k.\end{aligned}$$

That is, $u_k + \frac{3}{4}\varepsilon\lambda_k \in W(T+K)$ for each $k, 1 \le k \le n$. Hence

$$W_e(T) \subseteq \overline{W(T)} \subseteq \operatorname{conv}\left\{u_k + \frac{3}{4}\varepsilon\lambda_k : 1 \le k \le n\right\} \subseteq W(T+K).$$

By Lemma 2.3, $W(T+K)$ is closed.

Case 2. $\operatorname{int} W(T) = \emptyset$. Since $W(T)$ is convex, $\overline{W(T)}$ is a closed line segment in $\mathbb{C}$. Without loss of generality, one can directly assume that $\overline{W(T)} = [0,1]$. This implies that T is a self-adjoint operator.

Let P_a be the orthogonal projection onto $\bigvee\{E_\alpha : \alpha \in \Lambda\}$, where $\{E_\alpha\}_{\alpha\in\Lambda}$ is the set of atoms of $\mathcal{N}$. Denote $\mathcal{H}_a = \operatorname{ran} P_a$ and $\mathcal{H}_c = \operatorname{ran}(I - P_a)$. Since T is a self-adjoint operator in $\mathcal{T}(\mathcal{N})$, T admits the following matrix representation:

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_a \\ \mathcal{H}_c \end{matrix}$$

for some self-adjoint operators A and B. This implies that A is self-adjoint in $\mathcal{T}(\mathcal{N}_a)$ and B is self-adjoint in $\mathcal{T}(\mathcal{N}_c)$, where $\mathcal{N}_a = \{P_a N : N \in \mathcal{N}\}$ is an atomic nest and $\mathcal{N}_c = \{(I-P_a)N : N \in \mathcal{N}\}$ is a continuous nest.

Since $W(T) = \operatorname{conv}\{W(A) \cup W(B)\}$, $W(A) \subseteq [0,1]$ and $W(B) \subseteq [0,1]$. Then it suffices to consider the atomic case and the continuous case separately. For either case, in Lemma 2.7 or Lemma 2.8, we have given an affirmative answer. Now the proof is complete. □

Remark 2.10. From the construction of K in the proof of Case 1, we know that if T admits a strictly upper triangular matrix representation with respect to a nest $\mathcal{N}$, then K can be chosen to be arbitrarily small and also strictly upper triangular with respect to $\mathcal{N}$ such that $W(T+K)$ is closed.

3. Some applications

In this section, we will deal with the class of nilpotent operators, the class of Cowen-Douglas operators and the class of quasinilpotent operators. Using Theorem 2.1, especially the techniques used in the proof, we will prove that all these operator classes are also approximately strongly numerically closed.

3.1. **Class of nilpotent operators.** Let $\mathcal{N}_n(\mathcal{H})$ denote the set of all nilpotent operators of order at most n ($n = 1, 2, \ldots$) of $\mathcal{B}(\mathcal{H})$. That is, $\mathcal{N}_n(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T^n = 0\}$. For the class of nilpotent operators, we have the following theorem.

Theorem 3.1. *For each integer $n \ge 1$, $\mathcal{N}_n(\mathcal{H})$ is approximately strongly numerically closed.*

Proof. Suppose that $T \in \mathcal{N}_n(\mathcal{H})$ and $W(T)$ is not closed. Without loss of generality, we may assume that $T^n = 0$ and $T^{n-1} \neq 0$.

Denote $N_k = \ker T^k$ for each k, $0 \le k \le n$. Then $\mathcal{N} = \{N_k : 0 \le k \le n\}$ is an atomic nest and $T \in \mathcal{T}(\mathcal{N})$. Note that T admits an $n \times n$ strictly upper triangular matrix representation with respect to $\mathcal{N}$, Then by Theorem 2.1 and Remark 2.10, for given $\varepsilon > 0$, we can find a strictly upper triangular compact operator K with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{T}(\mathcal{N})$ and $W(T+K)$ is closed.

Moreover, note that $T+K$ also admits an $n \times n$ strictly upper triangular matrix representation. This implies that $T+K$ is a nilpotent operator of order n. The proof is complete. □

3.2. **Class of Cowen-Douglas operators.** Let Ω be a bounded connected open set of $\mathbb{C}$ and n be a positive integer. The set $B_n(\Omega)$ of *Cowen-Douglas operators* of index n is the set of operators $T \in \mathcal{B}(\mathcal{H})$ satisfying

(i) $\Omega \subseteq \sigma_p(T) \subseteq \sigma(T)$, where $\sigma_p(T)$ denotes the set of all eigenvalues of T;
(ii) $\dim \ker (\lambda - T) = n$ for each $\lambda \in \Omega$;
(iii) $\operatorname{ran}(T - \lambda) = \mathcal{H}$ for all $\lambda \in \Omega$;
(iv) $\bigvee\{\ker (\lambda - T) : \lambda \in \Omega\} = \mathcal{H}$.

Note that (iv) can be replaced by

(iv′) $\bigvee\{\ker(\lambda - T)^k : k \geq 1\} = \mathcal{H}$ for some $\lambda \in \Omega$.

From the definition of Cowen-Douglas operators, we can see that $\Omega \subseteq \sigma_p(T) \subseteq W(T)$ for all $T \in B_n(\Omega)$. For references on Cowen-Douglas operators, see [11,14,17].

Let $T \in \mathcal{B}(\mathcal{H})$. Recall that if $\operatorname{ran} T$ is closed and either $\dim \ker T$ or $\dim \ker T^*$ is finite, then T is called a *semi-Fredholm operator*. In this case, the *index* of T is defined by

$$\operatorname{ind} T = \dim \ker T - \dim \ker T^*.$$

In particular, if $-\infty < \operatorname{ind} T < +\infty$, then T is called a *Fredholm operator*. It is well known that if T is Fredholm, then the index of T is stable under compact perturbations. That is, $\operatorname{ind}(T + K) = \operatorname{ind} T$ for all $K \in \mathcal{K}(\mathcal{H})$. Moreover, if S is also Fredholm, then $\operatorname{ind}(TS) = \operatorname{ind} T + \operatorname{ind} S$. For references on Fredholm theory, see [1, 4].

By Fredholm theory and the definition of Cowen-Douglas operators, if $T \in B_n(\Omega)$, then $\operatorname{ind}(T - \lambda) = n$ and $\operatorname{ind}(T - \lambda)^k = kn$ for all $k \geq 1$ and $\lambda \in \Omega$.

The following theorem is the main result of this subsection.

Theorem 3.2. *For each integer $n \geq 1$, $B_n(\Omega)$ is approximately strongly numerically closed.*

Proof. First, without loss of generality, one can assume that $0 \in \Omega$. Suppose that $T \in B_n(\Omega)$ and $W(T)$ is not closed.

Set $N_k = \ker T^k$ and $\mathcal{H}_k = N_k \ominus N_{k-1}$ for each $k \geq 1$, where $N_0 = \{0\}$. Since $0 \in \Omega$ and $T \in B_n(\Omega)$, $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$ and $\dim \mathcal{H}_k = n$ for all $k \geq 1$. Denote $\mathcal{N} = \{N_k, k \geq 0, \mathcal{H}\}$. Then $\mathcal{N}$ is an atomic nest and $T \in \mathcal{T}(\mathcal{N})$.

Let P_k denote the orthogonal projection onto N_k. As $\dim N_k = kn$ for each $k \geq 0$, then $\{P_k\}_{k=0}^{\infty}$ is an increasing sequence of finite rank orthogonal projections in $\mathcal{T}(\mathcal{N})$ and P_k converges to the identity I in the strong operator topology.

Note that $\Omega \subseteq \operatorname{int} W(T)$. Then by the proof of Theorem 2.1, for given $\varepsilon > 0$, there is a compact operator $K = \delta T P_{n_0}$ with $\|K\| < \varepsilon$ for some positive integer n_0 and $\delta > 0$, such that $T + K \in \mathcal{T}(\mathcal{N})$ and $W(T + K)$ is closed.

Now it only remains to show that $T + K$ is also a Cowen-Douglas operator.

Note that T admits a strictly upper triangular matrix representation

$$T = \begin{bmatrix} 0 & T_1 & * & * & \cdots \\ & 0 & T_2 & * & \cdots \\ & & 0 & T_3 & \cdots \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \vdots \end{matrix}$$

with respect to $\mathcal{N}$ and each T_k is invertible. By the choice of K, $T+K$ also admits a strictly upper triangular matrix representation of the form

$$T+K=\begin{bmatrix} 0 & T_1' & * & * & \cdots \\ & 0 & T_2' & * & \cdots \\ & & 0 & T_3' & \cdots \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \vdots \end{matrix},$$

where

$$T_k'=\begin{cases}(1+\delta)T_k & \text{if } 1\le k\le n_0-1,\\ T_k & \text{otherwise.}\end{cases}$$

For any $\lambda\in\Omega$, by the stability of the index, $\operatorname{ind}(T+K-\lambda)=\operatorname{ind}(T-\lambda)=n$. Noting that each T_k' is invertible, then a simple calculation shows that $\ker(T+K-\lambda)^*=\{0\}$. This implies that $\dim\ker(T+K-\lambda)=n$. Hence $\Omega\subseteq\sigma_p(T+K)\subseteq\sigma(T+K)$.

Moreover, one can directly show that $\ker T^k\subseteq\ker(T+K)^k$ for all $k\ge 1$. This, together with $\bigvee\{\ker T^k : k\ge 1\}=\mathcal{H}$, implies that $\bigvee\{\ker(T+K)^k : k\ge 1\}=\mathcal{H}$.

Summarizing the above arguments, we conclude that $T+K\in B_n(\Omega)$. Now the proof is complete. □

3.3. **Class of quasinilpotent operators.** Recall that an operator $T\in\mathcal{B}(\mathcal{H})$ is *quasinilpotent* if $\sigma(T)=\{0\}$. The main result of this subsection is the following theorem.

Theorem 3.3. *The class of quasinilpotent operators in $\mathcal{B}(\mathcal{H})$ is approximately strongly numerically closed.*

Before giving the proof of Theorem 3.3, we first make some preparation.

Let $T\in\mathcal{B}(\mathcal{H})$. The *Wolf spectrum* $\sigma_{lre}(T)$ of T is defined by

$$\sigma_{lre}(T)=\{\lambda\in\mathbb{C} : T-\lambda \text{ is not semi-Fredholm}\}.$$

It is clear that $\sigma_{lre}(T)\subseteq\sigma_e(T)$, where $\sigma_e(T)$ denotes the essential spectrum of T.

Denote by $\sigma_0(T)$ the set of *normal eigenvalues* of T, that is,

$$\sigma_0(T)=\{\lambda\in\mathbb{C} : \lambda \text{ is an isolalted point of } \sigma(T) \text{ and } \operatorname{ind}(\lambda-T)=0\}.$$

From the definition of normal eigenvalues, one can easily obtain that $\sigma_0(T)\subseteq\sigma_p(T)$.

Recall that the *Weyl spectrum* $\sigma_W(T)$ of $T\in\mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_W(T)=\bigcap\big\{\sigma(T+K) : K\in\mathcal{K}(\mathcal{H})\big\}.$$

And it is well known that

$$\sigma_W(T)=\{\lambda\in\mathbb{C} : \lambda-T \text{ is not a Fredholm operator of index } 0\}.$$

It follows that $\sigma_{lre}(T)\subseteq\sigma_W(T)$ and $\sigma_0(T)\cap\sigma_W(T)=\emptyset$ for all $T\in\mathcal{B}(\mathcal{H})$.

The reader is referred to [4, 10] for more details about the notation and terminologies.

Lemma 3.4 ([4, page 366])**.** *Let $T\in\mathcal{B}(\mathcal{H})$. Then $\partial\sigma(T)\subseteq\sigma_0(T)\cup\sigma_{lre}(T)$.*

Lemma 3.5. *If $T\in\mathcal{B}(\mathcal{H})$ admits a strictly upper triangular matrix representation and $\sigma_e(T)=\{0\}$, then T is quasinilpotent.*

Proof. As $\sigma_e(T) = \{0\}$, it follows from Lemma 3.4 that

$$\partial\sigma(T) \subseteq \sigma_0(T) \cup \sigma_{lre}(T) \subseteq \sigma_0(T) \cup \{0\}.$$

This implies that $\sigma(T) = \{0\} \cup \sigma_0(T)$. Hence $\operatorname{ind}(T - \lambda) = 0$ for any $\lambda \neq 0$.

Since T admits a strictly upper triangular matrix, a simple calculation shows that $\ker(T - \lambda)^* = \{0\}$ for all $\lambda \neq 0$. Hence $\ker(T - \lambda) = \{0\}$ for all $\lambda \neq 0$. This implies that $\sigma_0(T) = \emptyset$. Hence $\sigma(T) = \{0\}$. That is, T is quasinilpotent. □

Lemma 3.6. *If $T \in \mathcal{B}(\mathcal{H})$ is quasinilpotent, then*

$$\sigma(T + K) = \{0\} \cup \sigma_0(T + K), \ \forall\ K \in \mathcal{K}(\mathcal{H}).$$

Proof. As T is quasinilpotent, $\sigma_{lre}(T) = \sigma_e(T) = \{0\}$. Hence for any $K \in \mathcal{K}(\mathcal{H})$, $\sigma_{lre}(T + K) = \{0\}$. It follows from Lemma 3.4 that $\partial\sigma(T + K) \subseteq \sigma_0(T + K) \cup \{0\}$. This implies that $\sigma(T + K) = \sigma_0(T + K) \cup \{0\}$. The proof is complete. □

Recall that an operator T is *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections in $\mathcal{B}(\mathcal{H})$ with $P_n \to I$ in the strong operator topology such that $\|(I - P_n)TP_n\| \to 0$. It is well known that an operator $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular if and only if $\operatorname{ind}(T - \lambda) \geq 0$ for all $\lambda \notin \sigma_{lre}(T)$ (see [10, Theorem 6.4]).

Lemma 3.7 ([12, Theorem 2.3])**.** *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular and $\sigma(T) = \sigma_W(T)$. Let $\Gamma = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that*

(1) *$\lambda_n \in \sigma(T)$ for all $n \geq 1$,*
(2) *each clopen subset σ of $\sigma(T)$ satisfies* $\operatorname{card}\{n : \lambda_n \in \sigma\} = \aleph_0$.

Then given $\varepsilon > 0$, there exists a compact K with $\|K\| < \varepsilon$ such that $\sigma(T + K) = \sigma_W(T + K)$ and $T + K$ admits an upper triangular matrix representation

$$T + K = \begin{bmatrix} a_1 & * & * & \cdots \\ & a_2 & * & \cdots \\ & & a_3 & \cdots \\ & & & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}$$

with respect to some orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and the set of $\{a_n : n = 1, 2, \ldots\}$ coincides with Γ.

Lemma 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be quasinilpotent and $\varepsilon > 0$. Then there is a compact operator K with $\|K\| < \varepsilon$ such that $T + K$ is also quasinilpotent and $T + K$ is strictly upper triangular with respect to some orthonormal basis.*

Proof. Since $T \in \mathcal{B}(\mathcal{H})$ is quasinilpotent, $\sigma_{lre}(T) = \sigma_e(T) = \{0\}$ and $\operatorname{ind}(\lambda - T) = 0$ for any $\lambda \neq 0$. Hence T is quasitriangular and $\sigma(T) = \sigma_W(T) = \{0\}$. By Lemma 3.7, for given $\varepsilon > 0$, there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $\sigma(T + K) = \sigma_W(T + K)$ and $T + K$ is strictly upper triangular with respect to some orthonormal basis.

Moreover, by Lemma 3.6, we know $\sigma(T + K) = \{0\} \cup \sigma_0(T + K)$. This, together with $\sigma(T + K) = \sigma_W(T + K)$, implies that $\sigma(T + K) = \{0\}$. That is, T is quasinilpotent. The proof is complete. □

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Suppose that $T \in \mathcal{B}(\mathcal{H})$ is quasinilpotent and $W(T)$ is not closed.

As T is quasinilpotent, it follows from Lemma 3.8 that for given $\varepsilon > 0$, there exists a $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \varepsilon/2$ such that $T + K_1$ is also quasinilpotent and $T + K_1$ admits a strictly upper triangular matrix representation relative to some orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Without loss of generality, suppose that $T + K_1$ is nonzero. Otherwise, the proof is complete.

Let $N_n = \{e_k : 1 \leq k \leq n\}$ for all $n \geq 1$. Then $\mathcal{N} = \big\{\{0\}, N_n, n \geq 1, \mathcal{H}\big\}$ is an atomic nest and $T + K_1 \in \mathcal{T}(\mathcal{N})$. Note that $T + K_1$ admits a strictly upper triangular matrix representation with respect to $\mathcal{N}$. Then by Theorem 2.1 and Remark 2.10, for given $\varepsilon > 0$, we can find a strictly upper triangular compact operator K_2 with $\|K_2\| < \varepsilon/2$ such that $T + K_1 + K_2 \in \mathcal{T}(\mathcal{N})$ and $W(T + K_1 + K_2)$ is closed.

Set $K = K_1 + K_2$. Then $\|K\| < \varepsilon$. Note that $\sigma_e(T + K) = \sigma_e(T) = \{0\}$ and $T + K$ is strictly upper triangular. Hence by Lemma 3.5, $T + K$ is quasinilpotent. This completes the proof. □

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