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CLASSIFICATION OF ENTIRE SOLUTIONS OF $(-\Delta)^N u + u^{-(4N-1)} = 0$ WITH EXACT LINEAR GROWTH AT INFINITY IN $\mathbf{R}^{2N-1}$

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Abstract. In this paper, we study global positive C^{2N}-solutions of the geometrically interesting equation $(-\Delta)^N u + u^{-(4N-1)} = 0$ in $\mathbf{R}^{2N-1}$. Using the sub poly-harmonic property for positive C^{2N}-solutions of the differential inequality $(-\Delta)^N u < 0$ in $\mathbf{R}^{2N-1}$, we prove that any C^{2N}-solution u of the equation having linear growth at infinity must satisfy the integral equation

$$u(x) = \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy$$

up to a multiple constant and hence take the following form:

$$u(x) = (1+|x|^2)^{1/2}$$

in $\mathbf{R}^{2N-1}$ up to dilations and translations. We also provide several non-existence results for positive C^{2N}-solutions of $(-\Delta)^N u = u^{-(4N-1)}$ in $\mathbf{R}^{2N-1}$.

1. Introduction

In this paper, we are interested in classification of entire solutions of the geometrically interesting equation

$$(-\Delta)^N u + u^{-(4N-1)} = 0 \tag{1.1}$$

in $\mathbf{R}^{2N-1}$ with $N \geqslant 2$. In order to understand the significance of studying (1.1) and the reason why we work on this equation, let us briefly exploit its root in conformal geometry. Loosely speaking, equations of the form (1.1) come from the problem of prescribing Q-curvature on $\mathbb{S}^{2N-1}$, which is associated with the conformally covariant GJMS operator with the principle part $(-\Delta_g)^{2N-1}$, discovered by Graham–Jenne–Mason–Sparling [GJMS92]. This operator is a high-order elliptic operator analogue with the well-known conformal Laplacian in the problem of prescribing scalar curvature.

Given a dimensional constant $n \geqslant 3$, let us consider the model $(\mathbb{S}^n, g_{\mathbb{S}^n})$ equipped with the standard metric $g_{\mathbb{S}^n}$. In this case, it is well-known that the GJMS operator of order $2N$ with $N \geqslant 2$ is given by

$$P_{2N, g_{\mathbb{S}^n}}(\cdot) = \prod_{k=1}^{N} \Big(-\Delta_{g_{\mathbb{S}^n}} + \Big(\frac{n}{2} - k\Big)\Big(\frac{n}{2} + k - 1\Big)\Big). \tag{1.2}$$

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(Note that the sign of $P_{2N,g_{\mathbb{S}^n}}$ given by (1.2) is different from the one in [Juh13, page 1353] by a factor $(-1)^N$.) The GJMS operator (1.2) is conformally covariant in the sense that if we conformally change the standard metric $g_{\mathbb{S}^n}$ to a new metric $\widetilde{g}$ via $\widetilde{g} = v^{4/(n-2N)} g_{\mathbb{S}^n}$ for some smooth function v on $\mathbb{S}^n$, then the two operators $P_{2N,\widetilde{g}}$ and $P_{2N,g_{\mathbb{S}^n}}$ are related via

$$P_{2N,\widetilde{g}}(\varphi) = v^{-\frac{n+2N}{n-2N}} P_{2N,g_{\mathbb{S}^n}}(v\varphi) \tag{1.3}$$

for any smooth, positive function φ on $\mathbb{S}^n$. In (1.3) if we set $\varphi \equiv 1$, then we obtain

$$P_{2N,g_{\mathbb{S}^n}}(v) = P_{2N,\widetilde{g}}(1) v^{\frac{n+2N}{n-2N}}.$$

Thanks to [Juh13, eq. (1.12)] and our convention for the sign of $P_{2N,g_{\mathbb{S}^n}}$, we know that

$$P_{2N,\widetilde{g}}(1) = \Big(\frac{n}{2} - N\Big) Q_{2N,\widetilde{g}}$$

for some scalar function $Q_{2N,\widetilde{g}}$ knowing that the Q-curvature is associated with the GJMS operator $P_{2N,\widetilde{g}}$. From this we obtain the equation

$$P_{2N,g_{\mathbb{S}^n}}(v) = \Big(\frac{n}{2} - N\Big) Q_{2N,\widetilde{g}}\, v^{\frac{n+2N}{n-2N}}. \tag{1.4}$$

Let us now limit ourselves to the case $n = 2N-1$. Then up to a multiple of positive constants, (1.4) becomes

$$P_{2N,g_{\mathbb{S}^{2N-1}}}(v) = -Q_{2N,\widetilde{g}}\, v^{-(4N-1)}. \tag{1.5}$$

Toward understanding the structure of the solution set of (1.5), let us only consider the case when $Q_{2N,\widetilde{g}}$ is constant. Upon a suitable scaling, we may assume that $Q_{2N,\widetilde{g}} = \pm 1$. Therefore, (1.5) becomes

$$P_{2N,g_{\mathbb{S}^{2N-1}}}(v) = \mp v^{-(4N-1)}. \tag{1.6}$$

Let us now denote by $\pi : \mathbb{S}^{2N-1} \to \mathbf{R}^{2N-1}$ the stereographic projection and set

$$u(x) = v(\pi^{-1}(x)) \Big(\frac{1+|x|^2}{2}\Big)^{1/2} \tag{1.7}$$

for $x \in \mathbf{R}^{2N-1}$. Thanks to [Gra07, Proposition 1], we can project (1.2) with $n = 2N-1$ from $\mathbb{S}^{2N-1}$ to $\mathbf{R}^{2N-1}$ to get

$$\Big(\frac{2}{1+|x|^2}\Big)^{-\frac{4N-1}{2}} (-\Delta)^N u(x) = (P_{2N,g_{\mathbb{S}^{2N-1}}}(v) \circ \pi^{-1})(x). \tag{1.8}$$

Therefore, via the stereographic projection π and up to a multiplication of positive constant, combining (1.8) and (1.6) gives

$$(-\Delta)^N u = \mp u^{-(4N-1)}.$$

In the preceding equation, if we consider the minus sign, the resulting equation leads us to (1.1), while for the plus sign, we arrive at the equation

$$(-\Delta)^N u = u^{-(4N-1)} \tag{1.9}$$

in $\mathbf{R}^{2N-1}$.

As far as we know, several special cases of (1.1) have already been studied in the literature. To be precise, when $N = 2$, the equation

$$\Delta^2 u + u^{-7} = 0 \tag{1.10}$$

in $\mathbf{R}^3$ was studied by Choi and Xu in [CX09] as well as by McKenna and Reichel in [KR03]. The main result in [CX09] is that if u solves (1.10) with exact linear growth at infinity in the sense that $\lim_{|x|\to+\infty} u(x)/|x|$ exists, then u solves the integral equation

$$u(x) = \int_{\mathbf{R}^3} |x-y| u(y)^{-7} dy.$$

From this integral representation, by a beautiful classification of positive solutions of integral equations by Li [Li04] and Xu [Xu05], it is widely known that $u(x) = (1+|x|^2)^{1/2}$ up to dilations and translations. When $N = 3$, (1.1) leads us to the equation

$$\Delta^3 u = u^{-11} \tag{1.11}$$

in $\mathbf{R}^5$. Its associated integral equation becomes

$$u(x) = \int_{\mathbf{R}^5} |x-y| u(y)^{-11} dy.$$

This integral equation was studied by Feng and Xu in [FX13]. The main result in [FX13] tell us that the only entire positive solution of (1.11) is $u(x) = (1+|x|^2)^{1/2}$ up to dilations and translations. As a counterpart of (1.11), the triharmonic Lane–Emden equation

$$\Delta^3 u + |u|^{p-1} u = 0$$

in $\mathbf{R}^n$ with $p > 1$ was recently studied by Luo, Wei, and Zou [LWZ16]; see also [GW08]. We take this chance to remember a work by Ma and Wei in [MW08] where the authors studied the equation

$$\Delta u = u^\tau$$

with $\tau < 0$. Clearly, this equation has a form similar to that of (1.1) with $N = 1$.

In the present paper, following the main question posted in [CX09, Gue12], we initiate our study of the structure of the solution set of (1.1) and (1.9). To be precise, for (1.1), we are able to classify all solutions with exact linear growth at infinity. The following theorem is the content of this result.

Theorem 1.1. *All solutions of partial differential equation* (1.1) *which satisfy*

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|} = \alpha \quad \textit{uniformly} \tag{1.12}$$

for some non-negative *finite constant α verify the following integral equation:*

$$u(x) = c_0 \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy.$$

Consequently, up to dilations and translations, the only entire solution of (1.1) *satisfying* (1.12) *is*

$$u(x) = (1+|x|^2)^{1/2}$$

in $\mathbf{R}^{2N-1}$.

As already discussed in [CX09], a major reason for imposing assumption (1.12) in studying (1.1) follows from the fact that entire solutions of (1.1) with exact linear growth at infinity correspond to complete conformal metrics on $\mathbb{S}^{2N-1}$, thanks to (1.7). We expect that (1.1), if frozen from geometric interpretation, also admits entire solutions with different growth at infinity. This is supported by considering (1.1) when $N = 2$; see [Gue12, DN17].

For (1.9), we prove that in fact this equation does not admit solutions with exact linear growth at infinity.

Theorem 1.2. *There is no positive C^{2N}-solution to* (1.9) *which satisfies*

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|} = \alpha \qquad \textit{uniformly}$$

for some positive *finite constant α.*

We note that a similar non-existence result for solutions of (1.9) was obtained by Xu and Yang in [XY02, Lemma 4.3]. To be exact, it was proved in [XY02] that there is no C^4-solution u of (1.9) with $N = 2$ in $\mathbf{R}^3$ which is bounded from below away from zero with the following conditions: $\int_{\mathbf{R}^3} u^{-6}dx < +\infty$, $\int_{\mathbf{R}^3}(\Delta u)^2 dx < +\infty$. In the following result, we generalize this result for solutions of (1.9).

Theorem 1.3. *There is no positive C^{2N}-solution u to* (1.9) *which satisfies:*

(1) $\int_{\mathbf{R}^{2N-1}} u^{-(4N-2)}dx < \infty$,
(2) $u \geqslant 1$ *and* $u(0) = 1$, *and*
(3) $(-\Delta)^i u \in L^2(\mathbf{R}^{2N-1})$ *for all* $i = 1, 2, \ldots, N-1$.

As in [XY02], the main ingredients in the proof of Theorem 1.3 are mean value properties for biharmonic functions and the Liouville theorem. Note that in the proof of Theorem 1.2, we exploit the super poly-harmonic property for solutions of (1.9) under the linear growth assumption. In the proof of Theorem 1.3 we also exploit the super poly-harmonic property for solutions of (1.9) without using the linear growth property.

In the next section, several fundamental estimates for solutions of (1.1) are provided. These estimates are useful for obtaining an integral representation for all $(-\Delta)^k u$ for k from $N-1$ down to 0. Once we have an integral representation for u, we are able to classify solutions. In the last part of the paper, we prove Theorems 1.1, 1.2, and 1.3.

2. Elementary estimates

In this section, we set up some notation and provide elementary estimates necessary to deal with elliptic equations with poly-harmonic operators. We note that although our approach is similar to the one used in [CX09], in several places, we have to introduce new ideas to deal with high-order elliptic equations.

We will denote the sphere in $\mathbf{R}^{2N-1}$ of radius r and center x_0 by $\partial B(x_0, r)$ and its included solid ball in $\mathbf{R}^{2N-1}$ by $B(x_0, r)$. We introduce the average of a function f on $\partial B(x_0, r)$ by

$$\overline{f}(x_0, r) = \frac{1}{\omega_{2N-1} r^{2N-2}} \int_{\partial B(x_0,r)} f(x) d\sigma_x = \fint_{\partial B(x_0,r)} f(x) d\sigma_x,$$

which depends only on the radius r. Here by ω_{2N-1} we mean the volume of the unit sphere $\partial B(x_0, 1)$ centered at x_0 sitting in $\mathbf{R}^{2N-1}$. (Note that $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ for all n.) Throughout the paper, if $x_0 = O$, then we drop O in the notation $\overline{f}(O, r)$ for simplicity.

We also denote various dimensional constants

$$\left\{ \begin{aligned} c_{N-1} &= \omega_{2N-1}^{-1}, \\ c_{N-k-1} &= \frac{c_{N-k}}{2k(2N-2k-3)} \quad \text{for } 1 \leqslant k \leqslant N-2. \end{aligned} \right. \tag{2.1}$$

Clearly $c_k > 0$ for all $1 \leqslant k \leqslant N-2$. We also let $c_0 > 0$ be

$$c_0 = \frac{c_1}{2N-2}. \tag{2.2}$$

Keep in mind that $-c_{N-1}|x-y|^{-(2N-3)}$ is the Green function of the operator Δ in $\mathbf{R}^{2N-1}$.

We list here the following useful inequality whose proof is exactly the same as [CX09, Lemma 2.1] in $\mathbf{R}^3$.

Lemma 2.1. *For any point x_0 in $\mathbf{R}^{2N-1}$ and any $q, r > 0$, there holds*

$$\Big(\fint_{\partial B(x_0,r)} f d\sigma\Big)^{-q} \leqslant \fint_{\partial B(x_0,r)} f^{-q} d\sigma.$$

Using Lemma 2.1, we obtain from (1.1) the differential inequality

$$(-\Delta)^N \overline{u} + \overline{u}^{-(4N-1)} \leqslant 0. \tag{2.3}$$

In particular, there holds $(-\Delta)^N \overline{u} < 0$ everywhere in $\mathbf{R}^{2N-1}$. The next lemma, which is known as the sub-poly-harmonic property of u, is of crucial importance as it allows us to deal with high-order equations.

Lemma 2.2. *All positive solutions u of* (1.1) *with the growth* (1.12) *satisfy*

$$(-\Delta)^k u < 0$$

everywhere in $\mathbf{R}^{2N-1}$ for each $k = 1, \dots, N-1$.

This lemma can be proved by using a general result from [Ngo17, Theorem 2]; hence we omit the details. In the rest of this section, we show how important Lemma 2.2 is by exploiting further properties of solutions of (1.1). First, we recall the following well-known result in [CMM93, Example 2.3].

Lemma 2.3. *Let w be a radially symmetric function satisfying*

$$(-\Delta)^k w \leqslant 0$$

everywhere in $\mathbf{R}^n$ for each $k = 0, \dots, m$ with $n > 2m$. Then necessarily we have

$$rw'(r) + (n-2m)w(r) \leqslant 0, \quad rw''(r) + (n+1-2m)w'(r) \geqslant 0$$

everywhere in $\mathbf{R}^n$.

Using Lemma 2.3 we can prove that $\overline{u}''$ has a sign. Such a result has some role in our analysis and cannot be obtained directly from the inequality $\Delta\overline{u} > 0$. In particular, this helps us to deduce that any solution of (1.1) must grow at least linearly at infinity; see Lemma 2.6 below.

Lemma 2.4. *All positive solutions u of* (1.1) *with the growth* (1.12) *satisfy*

$$\overline{u}''(r) \geqslant 0$$

for any $r > 0$.

Proof. Thanks to Lemma 2.2 and (1.1), with $v = -\Delta\overline{u}$ there holds

$$(-\Delta)^k v < 0$$

everywhere in $\mathbf{R}^{2N-1}$ for any $k = 0, \dots, N-1$. Since $2N-1 > 2(N-1)$, we can apply Lemma 2.3 to get

$$rv'(r) + v(r) \leqslant 0, \quad rv''(r) + 2v'(r) \geqslant 0.$$

Using the formula $-r^{2-2N}(r^{2N-2}\overline{u}')' = v$ and the inequality $rv'+v<0$, we deduce that

$$\begin{aligned}-(r^{2N-2}\overline{u}')'' =&(r^{2N-2}v)' = (2N-2)r^{2N-3}v + r^{2N-2}v' \\ =&r^{2N-3}\big(rv'+v\big) + (2N-3)r^{2N-3}v \\ \leqslant&(2N-3)r^{2N-3}v.\end{aligned}$$

Thus, we have just proved that

$$-(r^{2N-2}\overline{u}')'' \leqslant (2N-3)r^{2N-3}v = (2N-3)r^{-1}(-r^{2N-2}\overline{u}')'.$$

Therefore, if we set $w = r^{2N-2}\overline{u}'$, then we obtain

$$(-rw' + (2N-2)w)' = -rw'' + (2N-3)w' \leqslant 0.$$

By definition, the function $rw' - (2N-2)w$ vanishes at $r=0$ and is strictly increasing on $(0,+\infty)$. It follows that

$$r(r^{2N-2}\overline{u}')' \geqslant (2N-2)r^{2N-2}\overline{u}' \tag{2.4}$$

for any $r \geqslant 0$, which is equivalent to

$$r^{2N-1}\overline{u}'' + (2N-2)r^{2N-2}\overline{u}' \geqslant (2N-2)r^{2N-2}\overline{u}'$$

for any $r \geqslant 0$. Hence, there holds

$$r^{2N-1}\overline{u}'' \geqslant 0. \tag{2.5}$$

Hence $\overline{u}''(r) \geqslant 0$ for all $r>0$ as claimed. □

In the following lemma, we study the asymptotic behavior of $(-\Delta)^k\overline{u}$ at infinity. Such a result is useful when we apply the Liouville theory to get an integral representation for $(-\Delta)^k u$.

Lemma 2.5. *All positive solutions u of* (1.1) *with the growth* (1.12) *satisfy*

$$\lim_{r\to+\infty}(-\Delta)^k\overline{u}(r) = 0$$

for each $k = 1,\dots,N-1$.

Proof. Fix $k \in \{1,\dots,N-1\}$ and denote

$$v_k(r) := (-\Delta)^k\overline{u}.$$

For clarity, we also set $v_0 = \overline{u}$. Our aim is to prove that $v_k \to 0$ at infinity for each $k>0$. In view of Lemma 2.2, there holds $v_k<0$. Observe that in $\mathbf{R}^{2N-1}$ we have

$$r^{2-2N}(r^{2N-2}v_k')' = \Delta v_k = -(-\Delta)^{k+1}\overline{u} > 0,$$

which implies that $v_k'>0$. Therefore, v_k has a limit at infinity.

To prove the desired limit, let us start with $k=1$. Upon using our convention and the monotone decreasing of $-v_1$, we clearly have

$$r^{2N-2}v_0'(r) = -\int_0^r s^{2N-2}v_1(s)ds \geqslant -\frac{r^{2N-1}}{2N-1}v_1(r),$$

which yields

$$v_0(r) \geqslant v_0(0) - Cr^2v_1(r) \geqslant v_0(0)$$

for some constant $C>0$. Since v_0 has linear growth at infinity, we deduce that $v_1(r)\to 0$ as $r\to+\infty$. The above argument can be repeatedly used to conclude the desired limits. Indeed, suppose that $v_{k-1}(r) \to 0$ as $r \to +\infty$; we will show that $v_k(r) \to 0$ as $r \to +\infty$. To this purpose, we observe that

$$r^{2N-2} v'_{k-1}(r) = -\int_0^r s^{2N-2} v_k(s) ds \geqslant -\frac{r^{2N-1}}{2N-1} v_k(r),$$

which implies that

$$v_{k-1}(r) \geqslant v_{k-1}(0) - Cr^2 v_k(r) \geqslant v_{k-1}(0)$$

for some constant $C > 0$ which depends only on N. Dividing both sides by r^2, we obtain

$$\frac{v_{k-1}(r)}{r^2} \geqslant \frac{v_{k-1}(0)}{r^2} + C_1(-v_k(r)) \geqslant \frac{v_{k-1}(0)}{r^2}.$$

We now send $r \to +\infty$ to get the desired result. □

Lemma 2.6. *Let $u > 0$ satisfy* (1.1) *with the linear growth* (1.12). *Then $\alpha > 0$ where the constant α is given in* (1.12).

Proof. In view of Lemma 2.4, the inequality $\overline{u}''(r) \geqslant 0$ implies that $\overline{u}'(r) \geqslant \overline{u}'(1) > 0$ for any $r \geqslant 1$. From this we obtain

$$\overline{u}(r) \geqslant \overline{u}'(1)(r-1) + \overline{u}(1)$$

for all $r \geqslant 1$. The above inequality tells us that u grows at least linearly at infinity. Moreover, if the limit $\lim_{|x| \to +\infty} u(x)/|x| = \alpha \geqslant 0$ exists uniformly, it must hold that $\alpha > 0$ thanks to $\overline{u}'(1) > 0$. □

3. A classification result: Proof of Theorem 1.1

The main purpose in this section is to provide a proof of Theorem 1.1. First we set

$$U(x) = c_0 \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy,$$

with the constant $c_0 > 0$ given by (2.2). Note that by the definition of the constants c_i in (2.1), there holds

$$c_{N-k-1} \Delta_x (|x-y|^{2k-2N+3}) = -c_{N-k} |x-y|^{2k-2N+1}.$$

Therefore, an easy calculation shows that

$$(-\Delta)^k U(x) = -c_k \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2k-1}} dy \tag{3.1}$$

for $k = 1, \ldots, N-1$ with the constant $c_k > 0$ given by (2.1) and

$$(-\Delta)^N U(x) = -u^{-(4N-1)}.$$

In particular,

$$(-\Delta)^k U(x) < 0$$

everywhere on $\mathbf{R}^{2N-1}$. Recall that the function u solves $\Delta^N u = (-1)^{N-1} u^{-(4N-1)}$ in $\mathbf{R}^{2N-1}$. For simplicity, we set

$$U_k(x) = (-\Delta)^k U(x).$$

We now prove the following important properties for U_k.

Lemma 3.1. *For each fixed $k \in \{1, \dots, N-1\}$, the function U_k satisfies*

$$U_k(x) \to 0$$

as $|x| \to +\infty$.

Proof. It follows from (1.12) that there exists $R > 0$ such that if $|x| > R$, then $u(x) > \alpha|x|/2$. This implies that

$$\int_{\mathbf{R}^{2N-1}} |x-y|^{2k} u^{-(4N-1)}(y)dy < +\infty$$

for all $k = 1, \dots, N-1$. In particular, we have $\int_{\mathbf{R}^{2N-1}} u^{-(4N-1)}(y)dy < +\infty$ is finite and $u^{-(4N-1)}(x)$ is a bounded function, say by $M > 0$. By (3.1), we have

$$U_k(x) = -c_k \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2k-1}} dy.$$

For given $\varepsilon > 0$, there exists some $\delta > 0$ small enough such that

$$\int_{|x-y| \leqslant \delta} \frac{u^{-(4N-1)}(y)}{|x-y|^{2k-1}} dy \leqslant CM \int_0^\delta s^{2N-2k-1} ds < \frac{\varepsilon}{2}$$

for any $x \in \mathbf{R}^{2N-1}$. In the region $\{|x-y| \geqslant \delta\}$, we can use the dominated convergence theorem to conclude that

$$\lim_{|x| \to +\infty} \int_{|x-y| > \delta} \frac{u^{-(4N-1)}(y)}{|x-y|^{2k-1}} dy = 0.$$

Therefore,

$$\int_{|x-y| > \delta} \frac{u^{-(4N-1)}(y)}{|x-y|^{2k-1}} dy < \frac{\varepsilon}{2}$$

for any large $x \in \mathbf{R}^{2N-1}$. This shows that $U_k(x)$ has the limit zero at infinity. □

Following the method used in [CX09], to prove our main theorem, we need to establish an integral representation for $\Delta^k u$ for any $k \in \{1, \dots, N-1\}$. First, for $\Delta^{N-1} u$, we prove the following result.

Lemma 3.2. *Let u satisfy* (1.1) *with the linear growth* (1.12)*. Then the representation*

$$(-\Delta)^{N-1} u(x) = -c_{N-1} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-3}} dy \tag{3.2}$$

holds with the constant $c_{N-1} > 0$ given in (2.1).

Proof. Upon using the notation for U_k mentioned at the beginning of this section, U_{N-1} is exactly the right hand side of (3.2), that is,

$$U_{N-1}(x) = -c_{N-1} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-3}} dy.$$

We also denote an upper bound of $u^{-(4N-1)}$ by M. By Lemma 3.1, we know that U_{N-1} is bounded. Note that $-c_{N-1}|x-y|^{-(2N-3)}$ is the Green function of Δ in $\mathbf{R}^{2N-1}$; therefore an easy calculation shows that

$$\Delta U_{N-1}(x) = \int_{\mathbf{R}^{2N-1}} \Delta_x \Big(\frac{-c_{N-1}}{|x-y|^{2N-3}} \Big) u^{-(4N-1)}(y)dy = u^{-(4N-1)}(x).$$

Now it follows from the equations satisfied by U_{N-1} and u that

$$\Delta((-\Delta)^{N-1}u - U_{N-1}) = 0$$

in $\mathbf{R}^{2N-1}$. Since U_{N-1} is bounded and $(-\Delta)^{N-1}u$ is non-positive, we deduce that $(-\Delta)^{N-1}u - U_{N-1}$ is a harmonic function which is bounded from above. Thus the Liouville theorem can be applied to conclude that

$$(-\Delta)^{N-1}u = U_{N-1} + b_{N-1} \tag{3.3}$$

for some constant b_{N-1}. To get rid of the constant b_{N-1}, we take the spherical average of both sides of (3.3) to get

$$v_{N-1}(r) = \overline{U}_{N-1}(r) + b_{N-1},$$

where v_{N-1} is defined in the proof of Lemma 2.5. Taking the limit as $r \to +\infty$ we deduce that $b_{N-1} = 0$, thanks to Lemmas 2.5 and 3.1. □

By repeating the argument used in the proof of Lemma 3.2, we easily obtain the following result for $\Delta^k u$ for each $k \in \{1, \dots, N-2\}$.

Lemma 3.3. *Let u satisfy* (1.1) *with the linear growth* (1.12). *Then for each $k = 1, \dots, N-1$, the representation*

$$(-\Delta)^{N-k}u(x) = -c_{N-k} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-1-2k}} dy \tag{3.4}$$

holds with the constant $c_{N-k} > 0$ given in (2.1).

Proof. We prove (3.4) by induction on k. Clearly (3.4) holds for $k = 1$ by Lemma 3.2. Suppose that (3.4) holds for k, that is,

$$(-\Delta)^{N-k}u(x) = -c_{N-k} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-1-2k}} dy.$$

We prove (3.4) for $k+1$, that is,

$$(-\Delta)^{N-k-1}u(x) = -\, c_{N-k-1} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-3-2k}} dy.$$

Notice that

$$U_{N-k-1}(x) = -\, c_{N-k-1} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-3-2k}} dy.$$

Clearly, the function U_{N-k-1} is bounded by means of Lemma 3.1. Hence

$$\Delta U_{N-k-1}(x) = -c_{N-k-1} \int_{\mathbf{R}^{2N-1}} u^{-(4N-1)}(y) \Delta_x \Big(\frac{1}{|x-y|^{2N-3-2k}} \Big) dy.$$

Note that by the definition of the constants c_i in (2.1), there holds

$$c_{N-k-1} \Delta_x (|x-y|^{2k-2N+3}) = -c_{N-k} |x-y|^{2k-2N+1}.$$

Therefore,

$$\Delta((-\Delta)^{N-k-1}u - U_{N-k-1}) = 0$$

in $\mathbf{R}^{2N-1}$. Since U_{N-k-1} is bounded and $(-\Delta)^{N-k-1}u$ is non-positive, we deduce that $(-\Delta)^{N-k-1}u - U_{N-k-1}$ is a harmonic function which is bounded from above. Thus the Liouville theorem can be applied to conclude that

$$(-\Delta)^{N-k-1}u = U_{N-k-1} + b_{N-k-1} \tag{3.5}$$

for some constant b_{N-k-1}. Taking the spherical average of both sides of (3.5) we get

$$v_{N-k-1}(r) = \overline{U}_{N-k-1}(r) + b_{N-k-1},$$

where v_{N-k-1} is defined in the proof of Lemma 2.5. Taking the limit as $r \to +\infty$ we deduce that $b_{N-k-1} = 0$, thanks to Lemmas 2.5 and 3.1. This completes the present proof. □

Using Lemma 3.3, we obtain the following representation of Δu:

$$\Delta u(x) = c_1 \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|} dy \tag{3.6}$$

with the constant c_1 given in (2.1). Then using (3.6), we obtain a representation for u as follows.

Lemma 3.4. *There exists a constant γ such that u has the following representation:*

$$u(x) = c_0 \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy + \gamma \tag{3.7}$$

with the constant c_0 given by (2.2)*.*

Proof. Denote by h the function

$$h(x) = c_0 \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy$$

and let

$$\beta = c_0 \int_{\mathbf{R}^{2N-1}} u^{-(4N-1)}(y) dy.$$

First of all, we have

$$|\nabla h|(x) = \left| c_0 \int_{\mathbf{R}^{2N-1}} \frac{x-y}{|x-y|} u^{-(4N-1)}(y) dy \right| \leqslant \beta.$$

By observing (2.2), we easily verify that $c_0 \Delta_x(|x-y|) = c_1 |x-y|^{-1}$. From this, it is immediate to see that $\Delta(u-h) = 0$. It follows from the dominated convergence theorem that

$$\lim_{|x| \to +\infty} \frac{h(x)}{|x|} = \beta.$$

Since both u and h are at most linear growth at infinity, we obtain by the generalized Liouville theorem that

$$u(x) = h(x) + \sum_{i=1}^{2N-1} b_i x_i + \gamma \tag{3.8}$$

for some constants b_i and γ. Denote $x/|x|$ and $(b_1, \ldots, b_{2N-1})$ by Θ and $\vec{b}$, respectively. It follows from (3.8) that

$$\frac{u(x)}{|x|} = \frac{h(x)}{|x|} + \vec{b} \cdot \Theta + \frac{\gamma}{|x|}. \tag{3.9}$$

Taking the limit as $|x| \to +\infty$ on both sides of (3.9) we get $\alpha = \beta$ and $\vec{b} = 0$. This finishes the proof of the lemma. □

In the last part of the section, we prove that $\gamma = 0$.

Lemma 3.5. *The constant γ in the representation formula* (3.7) *is zero.*

Proof. As an immediate consequence of Lemma 3.4, we obtain the representation for ∇u as follows:

$$\nabla u(x)=c_0\int_{\mathbf{R}^{2N-1}}\frac{x-y}{|x-y|}u^{-(4N-1)}(y)dy. \tag{3.10}$$

From this we obtain

$$x\cdot\nabla u(x)=c_0\int_{\mathbf{R}^{2N-1}}\frac{|x|^2-x\cdot y}{|x-y|}u^{-(4N-1)}(y)dy. \tag{3.11}$$

Now multiply (3.11) thoughout by $u^{-(4N-1)}$ and integrate the resulting equation over the ball centered at the origin with radius R to obtain

$$\begin{aligned}
&-\frac{1}{4N-2}\int_{B(0,R)}x\cdot\nabla u^{-(4N-2)}(x)dx\\
&\qquad=c_0\int_{\mathbf{R}^{2N-1}}\Big(\int_{B(0,R)}\frac{|x|^2-x\cdot y}{|x-y|}u^{-(4N-1)}(x)dx\Big)u^{-(4N-1)}(y)dy.
\end{aligned}$$

Now for the left hand side of the preceding equation, we integrate by parts to get

$$\begin{aligned}
&-\frac{1}{4N-2}\int_{B(0,R)}x\cdot\nabla u^{-(4N-2)}(x)dx\\
&=-\frac{1}{4N-2}\begin{bmatrix}R\displaystyle\int_{\partial B(0,R)}u^{-(4N-2)}(x)d\sigma_x\\ -(2N-1)\displaystyle\int_{B(0,R)}u^{-(4N-2)}(x)dx\end{bmatrix}\\
&=\frac{1}{2}\int_{B(0,R)}u^{-(4N-2)}(x)dx-\frac{R}{4N-2}\int_{\partial B(0,R)}u^{-(4N-2)}(x)d\sigma_x.
\end{aligned}\tag{3.12}$$

For the right hand side, we notice that $|x|^2-x\cdot y=\big(|x-y|^2+(x-y)\cdot(x+y)\big)/2$, which leads to

$$\begin{aligned}
&c_0\int_{\mathbf{R}^{2N-1}}\Big(\int_{B(0,R)}\frac{|x|^2-x\cdot y}{|x-y|}u^{-(4N-1)}(x)dx\Big)u^{-(4N-1)}(y)dy\\
&=\frac{c_0}{2}\int_{\mathbf{R}^{2N-1}}\Big(\int_{B(0,R)}\frac{|x-y|^2+|x|^2-|y|^2}{|x-y|}u^{-(4N-1)}(x)dx\Big)u^{-(4N-1)}(y)dy\\
&=\frac{1}{2}\int_{B(0,R)}(u(x)-\gamma)u^{-(4N-1)}(x)dx\\
&\quad+\frac{c_0}{2}\int_{\mathbf{R}^{2N-1}}\Big(\int_{B(0,R)}\frac{|x|^2-|y|^2}{|x-y|}u^{-(4N-1)}(x)dx\Big)u^{-(4N-1)}(y)dy.
\end{aligned}$$

Here in the last step, we have used the representation formula for u established in Lemma 3.4. Letting $R\to+\infty$, since the integrand in the last term is absolutely integrable, this term becomes $\int_{\mathbf{R}^{2N-1}}\int_{\mathbf{R}^{2N-1}}$ with the same integrand. Hence, in the limit, this last term vanishes. Since u has exact linear growth at infinity and $N\geqslant 2$, the boundary term in (3.12) also vanishes. Hence, one gets

$$\frac{1}{2}\int_{\mathbf{R}^{2N-1}}u^{-(4N-2)}(x)dx=\frac{1}{2}\int_{\mathbf{R}^{2N-1}}u^{-(4N-2)}(x)dx-\frac{\gamma}{2}\int_{\mathbf{R}^{2N-1}}u^{-(4N-1)}(x)dx,$$

which implies $\gamma=0$. $\square$

Proof of Theorem 1.1. Now we prove Theorem 1.1. Suppose that u solves (1.1). Then the representation

$$u(x) = c_0 \int_{\mathbf{R}^{2N-1}} |x-y| u^{-(4N-1)}(y) dy$$

for some positive constant c_0 is simply a consequence of Lemmas 3.4 and 3.5. From this representation, we can apply a general classification result due to Li in [Li04], via the method of moving spheres, to conclude that u takes the form

$$u(x) = (1+|x|^2)^{1/2}$$

in $\mathbf{R}^{2N-1}$ up to dilations and translations. □

As can be seen from the above proof, the exact growth condition (1.12) is crucial. This leads us to raise a question whether or not Theorem 1.1 remains valid if we replace the exact growth condition (1.12) by a more reasonable growth condition

$$\alpha_1(1+|x|) \leqslant u(x) \leqslant \alpha_2(1+|x|) \tag{3.13}$$

for some positive constants α_1 and α_2.

We note that in the case $N=2$, McKenna and Reichel have already showed that a radially symmetric solution to (1.1) with linear growth exists. It turns out that this solution is unique and has exactly linear growth at infinity; see [KR03, Theorem 4.2(a)]; see also [Gue12, Theorem 1.3]. Therefore, Theorem 1.1 in the case $N=2$ with the new growth condition (3.13) remains valid in the class of radially symmetric solutions. However, to the best of our knowledge, there is no similar result in a larger class of solutions.

Let us recall that in this paper to classify solutions to (1.1) with exact linear growth, we essentially transform the differential equation to an integral equation. From this we obtain the classification as shown in Theorem 1.1. In view of McKenna and Reichel's result mentioned above, if we wish to replace the growth condition (1.12) by the growth condition (3.13), then we have to show that any positive C^{2N}-solution to (1.1) with linear growth (3.13) is radially symmetric. Hence, toward an answer for the above question, we need to answer the following:

- Does McKenna and Reichel's result still hold for any $N > 2$?
- Is any positive C^{2N}-solution to (1.1) with linear growth (3.13) radially symmetric?

Due to the limit of length, we leave this topic for future research.

4. Non-existence results: Proof of Theorems 1.2 and 1.3

4.1. **Proof of Theorem 1.2.** We prove the non-existence result in Theorem 1.2 by way of contradiction. Indeed, suppose that u solves (1.9) with exact linear growth $\alpha > 0$ at infinity. By the equation, we note that

$$(-\Delta)^N u > 0$$

everywhere in $\mathbf{R}^{2N-1}$. Therefore, as in Lemma 2.2, we can apply a general result from [Ngo17, Theorem 2] to get

$$(-\Delta)^k u < 0$$

everywhere in $\mathbf{R}^{2N-1}$ for each $k = 1, \dots, N-1$. In particular $\Delta u < 0$, which implies that

$$\overline{u}'(r) < 0$$

for any r. Since u has exact linear growth $\alpha > 0$ at infinity, we deduce that

$$u(x) \geqslant \frac{\alpha}{2}|x|$$

for $|x|$ large. Hence

$$\overline{u}(r) = \fint_{\partial B(0,r)} u(x) d\sigma_x \geqslant \frac{\alpha}{2} r$$

for large r. This gives us a contradiction since $\overline{u}' < 0$.

4.2. **Proof of Theorem 1.3.** We prove Theorem 1.3 by contradiction. First, by contradiction assumption, we recover the super poly-harmonic property for solutions of (1.9) without using the linear growth property as in Lemma 2.2. Indeed, suppose that u solves (1.9) which satisfies all assumptions in the theorem, that is,

$$u(x) \geqslant 1 = u(0) \tag{4.1}$$

for all $x \in \mathbf{R}^{2N-1}$,

$$\int_{\mathbf{R}^{2N-1}} u^{-(4N-2)} dx < +\infty, \tag{4.2}$$

and

$$\int_{\mathbf{R}^{2N-1}} |(-\Delta)^i u|^2 dx < +\infty \tag{4.3}$$

for $i = 1, \dots, N-1$. In the sequel, we prove that there exists a sequence of non-negative functions U_k and a sequence of positive numbers $q_k > 1$ such that

$$(-\Delta)^k u = U_k$$

for all $k = 1, \dots, N-1$ and that

$$U_k \in L^q(\mathbf{R}^{2N-1})$$

for all $q > q_k$. By induction, we first verify the statement for $k = N-1$. Set

$$U_{N-1}(x) = c_{N-1} \int_{\mathbf{R}^{2N-1}} \frac{u^{-(4N-1)}(y)}{|x-y|^{2N-3}} dy,$$

where c_{N-1} is given in (2.1). Thanks to (4.1) and (4.2), it is not hard to see that

$$\int_{\mathbf{R}^{2N-1}} u^{-q}(x) dx < +\infty$$

for all $q \geqslant 4N-2$; hence $U_{N-1} \in L^q(\mathbf{R}^{2N-1})$ for all $q > 1 =: q_{N-1}$. As in the proof of Lemma 3.2, there holds

$$\Delta((-\Delta)^{N-1} u - U_{N-1}) = 0. \tag{4.4}$$

On the other hand, for $r > 0$ and any $x \in \mathbf{R}^{2N-1}$, we have

$$\int_{B(x,r)} u^{-(4N-1)} dy = -r^{2N-2} \frac{\partial}{\partial r}\Big(r^{-(2N-2)} \int_{\partial B(x,r)} (-\Delta)^{N-1} u d\sigma\Big). \tag{4.5}$$

After dividing both sides of (4.5) by r^{2N-2} and integrating the resulting equation over $[0, r]$, we obtain

$$\begin{aligned} &\int_0^r s_1^{-(2N-2)} \Big(\int_{B(x,s_1)} u^{-(4N-1)} dy\Big) ds_1 \\ &\qquad = -r^{-(2N-2)} \int_{\partial B(x,r)} (-\Delta)^{N-1} u d\sigma + \omega_{2N-1} (-\Delta)^{N-1} u(x). \end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by r^{2N-2} and integrating the resulting equation over $[0,r]$ we get

$$\begin{aligned}
&\int_0^r s_2^{2N-2}\Big(\int_0^{s_2} s_1^{-(2N-2)}\Big(\int_{B(x,s_1)} u^{-(4N-1)}dy\Big)ds_1\Big)ds_2\\
&\quad= -\int_{B(x,r)}(-\Delta)^{N-1}udy + \frac{\omega_{2N-1}}{2N-1}(-\Delta)^{N-1}u(x)r^{2N-1}\\
&\quad= r^{2N-2}\frac{\partial}{\partial r}\Big(r^{-(2N-2)}\int_{\partial B(x,r)}(-\Delta)^{N-2}ud\sigma\Big)\\
&\qquad+\frac{\omega_{2N-1}}{2N-1}(-\Delta)^{N-1}u(x)r^{2N-1}.
\end{aligned}\tag{4.7}$$

Repeating the above argument we get

$$\begin{aligned}
g(r) :=& \int_0^r s_3^{-(2N-2)}\Big(\int_0^{s_3} s_2^{2N-2}\Big(\int_0^{s_2} s_1^{-(2N-2)}\Big(\int_{B(x,s_1)} u^{-(4N-1)}dy\Big)ds_1\Big)ds_2\Big)ds_3\\
=& r^{-(2N-2)}\int_{\partial B(x,r)}(-\Delta)^{N-2}ud\sigma + \omega_{2N-1}(-\Delta)^{N-2}u(x)\\
&+\frac{\omega_{2N-1}}{2(2N-1)}(-\Delta)^{N-1}u(x)r^2.
\end{aligned}\tag{4.8}$$

Making use of the L'Hospital rule, we conclude that

$$\lim_{r\to+\infty}\frac{g(r)}{r^2}\leqslant C,$$

for some constant $C>0$ independent of x. We go back to (4.8) to conclude that $(-\Delta)^{N-1}u$ is bounded from above in $\mathbf{R}^{2N-1}$. Together with the fact that U_{N-1} is positive everywhere, by the Liouville theorem, we obtain from (4.4) that

$$(-\Delta)^{N-1}u - U_{N-1} = C$$

everywhere in $\mathbf{R}^{2N-1}$ for some constant C. Since $U_{N-1}\in L^q(\mathbf{R}^{2N-1})$ for any $q>q_{N-1}$, we claim that $\lim_{|x|\to\infty}U_{N-1}(x)=0$. This combines with the condition (4.3) to give $C=0$. That is equivalent to

$$(-\Delta)^{N-1}u = U_{N-1}\geqslant 0.$$

Now, we suppose that

$$(-\Delta)^{N-k}u = U_{N-k}$$

for some non-negative function U_{N-k} in $\mathbf{R}^{2N-1}$ with $U_{N-k}\in L^q(\mathbf{R}^{2N-1})$ for any $q>q_k$ for some positive constant q_k. Our next task is to prove that $(-\Delta)^{N-k-1}u$ has a similar property. To this purpose, we repeat the same calculation as above. Indeed, we set

$$U_{N-k-1}(x) = c_{N-1}\int_{\mathbf{R}^{2N-1}}\frac{U_{N-k}(y)}{|x-y|^{2N-3}}dy.$$

Hence, similar to the way we obtained (4.8), after several steps we arrive at

$$\begin{aligned}\int_0^r & s_3^{-(2N-2)}\Big(\int_0^{s_3} s_2^{2N-2}\Big(\int_0^{s_2} s_1^{-(2N-2)}\Big(\int_{B(x,s_1)} U_{N-k}(y)dy\Big)ds_1\Big)ds_2\Big)ds_3\\ &=r^{-(2N-2)}\int_{\partial B(x,r)}(-\Delta)^{N-k-2}ud\sigma+\omega_{2N-1}(-\Delta)^{N-k-2}u(x)\\ &\quad+\frac{\omega_{2N-1}}{2(2N-1)}(-\Delta)^{N-k-1}u(x)r^2.\end{aligned}$$

From this, it is not hard to see that the function $(-\Delta)^{N-k-1}u$ is bounded from above and

$$\Delta U_{N-k-1}(x) = -U_{N-k}(x) = -(-\Delta)^{N-k}u(x).$$

Therefore,

$$\Delta((-\Delta)^{N-k-1}u - U_{N-k-1}) = 0.$$

From the positivity of U_{N-k-1}, we get that $(-\Delta)^{N-k-1}u - U_{N-k-1}$ is also bounded from above. Therefore, by the Liouville theorem, there exists a constant C such that

$$(-\Delta)^{N-k-1}u - U_{N-k-1} = C$$

everywhere in $\mathbf{R}^{2N-1}$. Meanwhile, since $(-\Delta)^{N-k}u = U_{N-k} \in L^q(\mathbf{R}^{2N-1})$ for any $q > q_k$, we can conclude that there exists some $q_{k+1} > q_k$ such that $U_{N-k-1} \in L^q(\mathbf{R}^{2N-1})$ for any $q > q_{k+1}$. Hence, there holds $C = 0$, which completes the proof of the statement.

Let $k = N-1$; it follows that $-\Delta u$ is non-negative. However, we can also check that

$$\Delta\Big(\frac{1}{u}\Big) = -\frac{\Delta u}{u^2} + 2\frac{|\nabla u|^2}{u^3} \geqslant 0.$$

It follows that $1/u$ must be constant, which contradicts (4.2). The proof is complete.

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