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STRONGNESS OF COMPANION BASES FOR CLUSTER-TILTED ALGEBRAS OF FINITE TYPE

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Abstract. For every cluster-tilted algebra of simply-laced Dynkin type we provide a companion basis which is strong, i.e., gives the set of dimension vectors of the finitely generated indecomposable modules for the cluster-tilted algebra. This shows in particular that every companion basis of a cluster-tilted algebra of simply-laced Dynkin type is strong. Thus we give a proof of Parsons's conjecture.

1. Introduction

Cluster algebras were introduced and first investigated by Fomin and Zelevinsky [8] in order to better understand the dual canonical basis of the quantised enveloping algebra of a finite dimensional semisimple Lie algebra. Today cluster algebras are connected to various subjects including representation theory of finite dimensional algebras, Poisson geometry, algebraic geometry, knot theory, Teichmüller theory, dynamical systems, mathematical physics, and combinatorics. Cluster categories were introduced in [3] (also for type A_n in [7]) as a categorical model for a better understanding of the cluster algebras. The cluster-tilted algebras were introduced by Buan, Marsh and Reiten [4] have a key role in the study of cluster categories. Also an important connection between cluster algebras and cluster-tilted algebras was established in [5] and [6]. More precisely it was proved in [5] and [6] that the quivers of the cluster-tilted algebras of a given simply-laced Dynkin type are precisely the quivers of the exchange matrices of the cluster algebras of that type.

Fomin and Zelevinsky [9] classified the cluster algebras of finite type. Their classification is identical to the Cartan-Killing classification of semisimple Lie algebras. The conditions of their classification is hard to check in general. For solving this problem, Barot, Geiss and Zelevinsky [2] considered the positive quasi-Cartan matrices which are related to the Cartan matrices. The main theorem of [2] shows that the exchange matrix associated to the quiver of any cluster-tilted algebra of simply-laced Dynkin type have a positive quasi-Cartan companion. More precisely they showed that there exists a $\mathbb{Z}$-basis of roots of the integral root lattice of the corresponding root system of simply-laced Dynkin type such that the matrix of inner products associated to this basis creates the positive quasi-Cartan companion.

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Parsons in [11] and [12] called such a $\mathbb{Z}$-basis a companion basis. Parsons used companion bases to study indecomposable modules over cluster-tilted algebras. Note that the knowledge of the indecomposable modules over cluster-tilted algebras is not only important in the representation theory of cluster-tilted algebras but also has applications in the corresponding cluster algebra. Nakanishi and Stella [10] showed that the set of all dimension vectors of the indecomposable modules over a cluster-tilted algebra of finite type, the set of the non-initial d-vectors of the corresponding cluster algebra, and the set of positive c-vectors of the corresponding cluster algebra coincide.

For any companion basis of a cluster-tilted algebra of simply-laced Dynkin type, Parsons defined a collection of positive vectors. Any positive root of the corresponding root system of simply-laced Dynkin type is a linear combination of elements of the companion basis with integer coefficients. That collection is the set of the absolute values of these coefficients. A companion basis of a cluster-tilted algebra of simply-laced Dynkin is called strong [11], [12] if the associated collection of positive vectors are precisely the dimension vectors of the finitely generated indecomposable modules over the given cluster-tilted algebra. Parsons proved that for any cluster-tilted algebra of Dynkin type A, any companion basis is strong. He also conjectured that any companion basis of any cluster-tilted algebra of simply-laced Dynkin type is strong (Conjecture 6.3 of [11]).

On the other hand Ringel [13] studied finitely generated indecomposable modules over cluster-concealed algebras (i.e., cluster-tilted algebras where the associated cluster-tilting object corresponds to a preprojective tilting module). Note that cluster-tilted algebras of simply-laced Dynkin type are cluster-concealed algebras, but there are many cluster-concealed algebras which are tame or wild. Ringel [13] used a theorem due to Assem et al. [1] and Zhu [14], which gives an equivalent definition of cluster-tilted algebras. According to that theorem, any cluster-tilted algebra is isomorphic to the relation-extension of some tilted algebra. By using the tilting functor Ringel defined a linear bijection map g between the Grothendieck group of the corresponding hereditary algebra $K_0(H)$ and the Grothendieck group of the tilted algebras $K_0(B)$. He proved that the dimension vectors of indecomposable modules over a cluster-concealed algebra are precisely the vectors $\mathrm{abs}(x)$ of absolute values with $x \in \Phi_B$, where Φ_B is the image of the root system Φ_H of H under g and where for each $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, $\mathrm{abs}(x) = (|x_1|, \cdots, |x_n|)$. He also proved that indecomposable modules over cluster-tilted algebras of simply-laced Dynkin type are uniquely determined by their dimension vectors.

Given a cluster-tilted algebra of simply-laced Dynkin type, according to Ringel's theorem there exist $\{x_1, x_2, \cdots, x_n\} \subseteq \Phi_H$ such that $\mathrm{abs}(g(x_i)) = e_i$ for each i, where $\{e_1, \cdots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$. The goal of this paper is to show that the set $\{x_1, x_2, \cdots, x_n\}$ is a companion basis for the cluster-tilted algebra and that it is strong. This result not only illustrates the connection between Ringel's results [13] and Parsons's results [11], [12], it also proves Parsons's conjecture (Conjecture 6.3 of [11]). Before proving our results in Section 3 we provide the necessary background in the following section.

2. Preliminaries

Let $\mathbb{F} = \mathbb{Q}(u_1, \cdots, u_n)$ be the field of rational functions in n indeterminates. Let $\mathbf{x} \subseteq \mathbb{F}$ be a transcendence basis over $\mathbb{Q}$, and let $C = (c_{xy})_{x,y \in \mathbf{x}}$ be an $n \times n$

sign-skew-symmetric integer matrix with rows and columns indexed by $\mathbf{x}$. The pair $(\mathbf{x}, C)$ is called a seed. Given such a seed, and an element $z \in \mathbf{x}$, define a new element $z' \in \mathbb{F}$ via the exchange relation:

$$zz' = \prod_{x \in \mathbf{x}, c_{xz} > 0} x^{c_{xz}} + \prod_{x \in \mathbf{x}, c_{xz} < 0} x^{-c_{xz}}.$$

Let $\mathbf{x}' = \mathbf{x} \cup \{z'\} \backslash \{z\}$; it is a new transcendence basis of $\mathbb{F}$. Let C' be the mutation of the matrix C in direction z:

$$c'_{xy} = \begin{cases} -c_{xy} & \text{if } x = z \text{ or } y = z, \\ c_{xy} + \frac{|c_{xz}|c_{zy} + c_{xz}|c_{zy}|}{2} & \text{otherwise.} \end{cases}$$

The row and column labeled z in C are relabeled z' in C'. The pair $(\mathbf{x}', C')$ is called the mutation of the seed $(\mathbf{x}, C)$ in direction z. Let $\mathcal{S}$ be the set of seeds obtained by iterated mutation of $(\mathbf{x}, C)$. Then the set of cluster variables is, by definition, the union χ of all the elements of the transcendence bases appearing in the seeds in $\mathcal{S}$. These bases are known as clusters, and the cluster algebra $\mathcal{A}(\mathbf{x}, C)$ is the subring of $\mathbb{F}$ generated by χ. Up to isomorphism of cluster algebras, it does not depend on the initial choice $\mathbf{x}$ of transcendence basis, so it is just denoted by $\mathcal{A}_C$. If χ is finite, the cluster algebra $\mathcal{A}_C$ is said to be of finite type.

Let H be a hereditary finite dimensional k-algebra, where k is an algebraically closed field and let $\mathcal{D} = D^b(\text{mod}H)$ be the bounded derived category of finitely generated left H-modules with shift functor [1]. Also, let τ be the AR-translation in $\mathcal{D}$. The cluster category is defined as the orbit category $\mathcal{C}_H = \mathcal{D}/F$, where $F = \tau^{-1}[1]$. The objects of $\mathcal{C}_H$ are the same as the objects of $\mathcal{D}$, but maps are given by $\text{Hom}_{\mathcal{C}_H}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(X, F^i Y)$. An object $\widetilde{T}$ in $\mathcal{C}_H$ is called cluster-tilting provided for any object X of $\mathcal{C}_H$, we have $\text{Ext}^1_{\mathcal{C}_H}(\widetilde{T}, X) = 0$ if and only if X lies in the additive subcategory $\text{add}(\widetilde{T})$ of $\mathcal{C}_H$ generated by $\widetilde{T}$. Let $\widetilde{T}$ be a cluster-tilting object in $\mathcal{C}_H$. The cluster-tilted algebra associated to $\widetilde{T}$ is the algebra $\text{End}_{\mathcal{C}_H}(\widetilde{T})^{op}$.

Let Q be a quiver of simply-laced Dynkin type with underlying graph Δ and n vertices. Let $H = kQ$ and $\widetilde{T}$ be a basic cluster-tilting object in the cluster category $\mathcal{C}_H$. Let $\Lambda = \text{End}_{\mathcal{C}_H}(\widetilde{T})^{op}$ be the corresponding cluster-tilted algebra. Let $\mathcal{A}$ be the cluster algebra of type Δ and let $(\mathbf{x}, C)$ be the seed in $\mathcal{A}$ corresponding to $\widetilde{T}$. According to Theorem 3.1 of [6] (also Section 6 of [5]), $\Gamma = \Gamma(C)$ is the quiver of Λ, where $\Gamma(C)$ is the quiver with vertices corresponding to the rows and columns of C, and c_{ij} arrows from the vertex i to the vertex j whenever $c_{ij} > 0$. Let V be the Euclidean space with positive definite symmetric bilinear form $(-, -)$ and let $\Phi_H \subseteq V$ be the root system of Dynkin type Δ.

Definition 2.1 (Definitions 4.1, 5.1 and 5.2 of [11]).

(1) The subset $\Psi = \{\gamma_i | 1 \leq i \leq n\} \subseteq \Phi_H$ is called a companion basis for $\Gamma = \Gamma(C)$ if it satisfies the following properties:
  (i) $\{\gamma_i | 1 \leq i \leq n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi_H$,
  (ii) $|(\gamma_i, \gamma_j)| = |c_{ij}|$ for all $i \neq j$, $1 \leq i, j \leq n$.

(2) Let $\alpha \in \Phi_H$ and suppose that $\alpha = \sum_{i=1}^{n} c_i \gamma_i$ with $c_i \in \mathbb{Z}$ for each i. d^{Ψ}_{α} is defined to be the vector $d^{\Psi}_{\alpha} = (|c_1|, \cdots, |c_n|)$.

(3) The companion basis $\Psi = \{\gamma_i | 1 \leq i \leq n\}$ of Γ is called strong companion basis if the vectors d_α^Ψ for $\alpha \in \Phi_H^+$ are the dimension vectors of the finitely generated indecomposable Λ-modules.

Let Λ be a cluster-tilted algebra of simply-laced Dynkin type and let $\Gamma = \Gamma(C)$ be the quiver of Λ. Parsons conjectured in Conjecture 6.3 of [11] that all companion bases for Γ are strong. He proved this conjecture for cluster-tilted algebras of simply-laced type A_n (Theorem 5.3 of [11]). Let $\Psi = \{\gamma_i | 1 \leq i \leq n\}$ and $\Omega = \{\gamma_i' | 1 \leq i \leq n\}$ be two arbitrary companion bases for Γ. By Proposition 6.2 of [11], $\{d_\alpha^\Psi | \alpha \in \Phi_H^+\} = \{d_\alpha^\Omega | \alpha \in \Phi_H^+\}$. This tells us that the existence of a strong companion basis for Γ shows that all companion bases of Γ are strong.

Assem, Brüstle and Schiffler [1] and Zhu [14] independently provided a characterization of cluster-tilted algebras. They proved that an algebra Λ is cluster-tilted if and only if there exists a tilted algebra B such that $\Lambda \cong B^c$, where B^c is trivial extension algebra $B^c = B \ltimes \mathrm{Ext}_B^2(DB, B)$, with $D = \mathrm{Hom}(-, k)$ the k-duality. Recall that a k-algebra B is said to be tilted provided B is the endomorphism ring of a tilting H-module T, where H is a finite dimensional hereditary k-algebra. A tilted algebra B is said to be concealed provided B is the endomorphism ring of a preprojective tilting H-module (i.e., a tilting H-module whose summands are all preprojective). If B is a concealed algebra, then B^c is called a cluster-concealed algebra.

Let $B = \mathrm{End}_H(T)$ be a concealed algebra and let $\Lambda = B^c$ be the corresponding cluster-concealed algebra and let $K_0(H)$ be the Grothendieck group of H. We identify $K_0(H)$ with $\mathbb{Z}^n$, where n is the number of the isomorphism class of simple H-modules. Let M be a finitely generated H-module. The dimension vector of M is the vector $\mathbf{dim}\, M \in \mathbb{Z}^n$, whose coefficients are the Jordan-Hölder multiplicities of M. We denote by $\langle -, - \rangle_H$ the bilinear form on $K_0(H)$ given by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_H = \sum_{i \geq 0} (-1)^i \dim_k \mathrm{Ext}_H^i(M, N)$$

for all H-modules M, N. The corresponding symmetric bilinear form $(-, -)_H$ is defined as $(\mathbf{dim}\, M, \mathbf{dim}\, N)_H = \langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_H + \langle \mathbf{dim}\, N, \mathbf{dim}\, M \rangle_H$.

Let $T_1, \cdots, T_n$ be indecomposable direct summands of T, one from each isomorphism class. Let $g : K_0(H) \to K_0(B)$ given by $g(x) = (\langle \mathbf{dim}\, T_i, x \rangle_H)_i$. Then for each i, we have $g(\mathbf{dim}\, T_i) = \mathbf{dim}\, G(T_i)$, where $G = \mathrm{Hom}_H(T, -) : \mathrm{mod} H \to \mathrm{mod} B$ is a tilting functor. Since $\{\mathbf{dim}\, T_1, \cdots, \mathbf{dim}\, T_n\}$ is a basis of $K_0(H)$ and $\{\mathbf{dim}\, G(T_1), \cdots, \mathbf{dim}\, G(T_n))\}$ is a basis of $K_0(B)$, g is a linear bijection.

Let Φ_H be the root system in $K_0(H)$ corresponding to the underlying graph of the quiver of H and $\Phi_B = g(\Phi_H)$. In Theorem 2 of [13], Ringel proved that the dimension vectors of the indecomposable Λ-modules are precisely the vectors $\mathrm{abs}(x)$ with $x \in \Phi_B$.

Remark 2.2. Let $\Phi_B^+ = g(\Phi_H^+)$, where Φ_H^+ is the set of positive roots of H. For every $x \in \Phi_B$ there exists $a \in \Phi_H$ such that $x = g(a)$. The root a is either positive or negative. If a is negative, then $-a \in \Phi_H^+$ and $x = -g(-a)$. Thus $-x \in \Phi_B^+$. Since $\mathrm{abs}(x) = \mathrm{abs}(-x)$, it is enough to consider positive roots. So according to Ringel's theorem, $\{\mathbf{dim}\, M | M \in ind\Lambda\} = \{\mathrm{abs}(x) | x \in \Phi_B^+\}$.

In the following example we illustrate the connection between Parsons's results [11] and Ringel's results [13].

Example 2.3. Consider the quiver Γ given by

$$\begin{tikzcd} & & 2 \arrow[dr, "\beta"] & \\ 4 \arrow[rrr, bend right, "\eta"'] & 3 \arrow[l, "\delta"'] \arrow[ur, "\alpha"] & & 1 \arrow[ll, "\gamma"'] \end{tikzcd}$$

In fact Γ is the mutation of Q at vertex 3, where Q is the quiver

$$\begin{tikzcd} & & 2 \arrow[dl] \\ 4 \arrow[r] & 3 \arrow[dr] & \\ & & 1 \end{tikzcd}$$

Thus Γ is a quiver of a cluster-tilted algebra Λ of type D_4. More precisely $\Lambda \cong k\Gamma/I$, where I is an admissible ideal of $k\Gamma$ generated by $\beta\gamma, \gamma\alpha, \eta\gamma, \gamma\delta, \alpha\beta - \delta\eta$. By Theorem 6.1 of [11], $\Psi = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_3, \alpha_3 + \alpha_4\}$ is a companion basis for Γ, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a simple system of Φ_H and $H = kQ$. Recall that the positive roots Φ_H^+ are $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_3 + \alpha_4\}$. In terms of Ψ, their dimension vectors are $\{d_\alpha^\Psi | \alpha \in \Phi_H^+\} = \{(1,0,0,0), (0,1,1,0), (0,0,1,0), (0,0,1,1), (1,0,1,0), (0,1,0,0), (0,0,0,1), (1,1,0,0), (1,0,0,1), (0,1,1,1), (1,1,1,1), (1,1,0,1)\}$. On the other hand $\Lambda \cong B \ltimes \mathrm{Ext}_B^2(DB, B)$, where $B = \mathrm{End}_H(T')$, $T' = P_1 \oplus P_2 \oplus P_3' \oplus P_4$, P_1, P_2, P_3, P_4 are indecomposable projective H-modules and P_3' is the indecomposable H-module with the dimension vector $(1,1,1,1)$ (in fact $H \cong \mathrm{End}_H(H)$ and T' is a mutation of H at 3). An easy calculation shows that $\Phi_B^+ = \{(1,0,0,0), (0,1,1,0), (0,0,-1,0), (0,0,1,1), (1,0,-1,0), (0,1,0,0), (0,0,0,1), (1,1,0,0), (1,0,0,1), (0,1,1,1), (1,1,1,1), (1,1,0,1)\}$ and so by Ringel's theorem the dimension vectors of the indecomposable Λ-modules are precisely $\{\mathrm{abs}(x) | x \in \Phi_B^+\} = \{(1,0,0,0), (0,1,1,0), (0,0,1,0), (0,0,1,1), (1,0,1,0), (0,1,0,0), (0,0,0,1), (1,1,0,0), (1,0,0,1), (0,1,1,1), (1,1,1,1), (1,1,0,1)\}$. This shows that Ψ is a strong companion basis for Γ, the vectors of absolute values of the images of the four roots of the companion basis are $\mathrm{abs}(g(\alpha_1)) = (1,0,0,0)$, $\mathrm{abs}(g(\alpha_2 + \alpha_3)) = (0,1,0,0)$, $\mathrm{abs}(g(\alpha_3)) = (0,0,1,0)$ and $\mathrm{abs}(g(\alpha_3 + \alpha_4)) = (0,0,0,1)$. In fact we have a companion basis for Γ such that its image under g is the canonical basis of $K_0(B) = \mathbb{Z}^4$. In the next section we prove this fact for arbitrary cluster-tilted algebras of simply-laced Dynkin type.

3. Main result

Let Λ be an arbitrary cluster-tilted algebra of finite type. It is known that there exists $H = kQ$, Q of Dynkin type Δ, and a cluster-tilting object $\widetilde{T}$ of $\mathcal{C}_H$ such that $\Lambda \cong \mathrm{End}_{\mathcal{C}_H}(\widetilde{T})$. Furthermore, it is known that there exists a tilting H-module T such that $\Lambda \cong B^c$ where $B = \mathrm{End}_H(T)$; cf. [1] or [14].

Let $\mathcal{A}$ be a cluster algebra of type Δ, and suppose that $(\mathbf{x}, C)$ is the seed in $\mathcal{A}$ corresponding to $\widetilde{T}$. Then $\Gamma = \Gamma(C)$ is the quiver of Λ ([5], [6]).

Let $g : K_0(H) \to K_0(B)$ be the linear bijection given by $g(x) = (\langle \mathbf{dim}\, T_i, x\rangle_H)_i$, and let $S_1, S_2, \cdots, S_n$ be the isomorphism class of simple Λ-modules. By Ringel's theorem there exist $y_1, \cdots, y_n \in \Phi_B^+$ such that $\mathrm{abs}(y_i) = \mathbf{dim}\, S_i = e_i$, for each $1 \leq i \leq n$. Also for $1 \leq i \leq n$ there exists $x_i \in \Phi_H^+$ such that $g(x_i) = y_i$.

Theorem 3.1. *The subset $\{x_1, \cdots, x_n\} \subseteq \Phi_H^+$ is a companion basis for Γ.*

Proof. First we show that $\{x_1, \cdots, x_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi_H$. Let $x \in \mathbb{Z}\Phi_H$; then $g(x) \in \mathbb{Z}\Phi_B$, and so $g(x) = d_1e_1 + \cdots + d_ne_n$, where $d_i \in \mathbb{Z}$ for each i and $\{e_1, \cdots, e_n\}$ is the canonical basis of $K_0(B)$. Since $e_i = \mathrm{abs}(g(x_i))$ for each i, $g(x) = d_1\varepsilon_1 g(x_1) + \cdots + d_n\varepsilon_n g(x_n)$ with $\varepsilon_i \in \{1, -1\}$ for each i. Then $x = d_1\varepsilon_1 x_1 + \cdots + d_n\varepsilon_n x_n$, since g is a linear bijection. Now we show that for any $1 \leq i, j \leq n$, $i \neq j$, we have $|(x_i, x_j)_H| = |c_{ij}|$. We have $|(x_i, x_j)_H| = |(g(x_i), g(x_j))_B| = |(\varepsilon_i e_i, \varepsilon_j e_j)_B|$ with $\varepsilon_i, \varepsilon_j \in \{1, -1\}$. Then $|(x_i, x_j)_H| = |(e_i, e_j)_B| = |\langle e_i, e_j\rangle_B + \langle e_j, e_i\rangle_B|$. Since H is hereditary, $\mathrm{gl.dim}\, B \leq 2$ and so $\langle e_i, e_j\rangle_B = \dim_k \mathrm{Hom}_B(S_i, S_j) - \dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_i, S_j)$. Thus $|(x_i, x_j)_H| = |\dim_k \mathrm{Hom}_B(S_i, S_j) - \dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_i, S_j) + \dim_k \mathrm{Hom}_B(S_j, S_i) - \dim_k \mathrm{Ext}^1_B(S_j, S_i) + \dim_k \mathrm{Ext}^2_B(S_j, S_i)|$. Since $i \neq j$, $\mathrm{Hom}_B(S_i, S_j) = \mathrm{Hom}_B(S_j, S_i) = 0$ and so $|(x_i, x_j)_H| = |-\dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_i, S_j) - \dim_k \mathrm{Ext}^1_B(S_j, S_i) + \dim_k \mathrm{Ext}^2_B(S_j, S_i)|$. It is known that the quivers of cluster-tilted algebras contain no 2-cycles (i.e., oriented cycles of length two). Therefore, $|c_{ij}|$ is equal to the number of arrows in $\Gamma = \Gamma(C)$ from i to j plus the number of arrows in $\Gamma = \Gamma(C)$ from j to i. But Γ is the quiver of B^c and according to Theorem 2.6 of [1] (see also the proof of this theorem), the number of arrows in the quiver of B^c, Q_{B^c} from i to j is equal to the number of arrows in the quiver of B, Q_B from i to j plus $\dim_k \mathrm{Ext}^2_B(S_j, S_i)$ additional arrows. Also it is known that the number of arrows in Q_B from i to j equals the $\dim_k \mathrm{Ext}^1_B(S_i, S_j)$. Hence $|c_{ij}| = \dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_j, S_i) + \dim_k \mathrm{Ext}^1_B(S_j, S_i) + \dim_k \mathrm{Ext}^2_B(S_i, S_j)$. We claim that $|-\dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_i, S_j) - \dim_k \mathrm{Ext}^1_B(S_j, S_i) + \dim_k \mathrm{Ext}^2_B(S_j, S_i)| = \dim_k \mathrm{Ext}^1_B(S_i, S_j) + \dim_k \mathrm{Ext}^2_B(S_j, S_i) + \dim_k \mathrm{Ext}^1_B(S_j, S_i) + \dim_k \mathrm{Ext}^2_B(S_i, S_j)$.

It is known that in Q_B there are no oriented cycles and in Q_{B^c} there are no multiple arrows. If $\mathrm{Ext}^1_B(S_i, S_j) \neq 0$, then we have an arrow from i to j in Q_B and so in Q_{B^c}. Then in this case there is no arrow from j to i and no other arrow from i to j in Q_{B^c}. Thus $\mathrm{Ext}^2_B(S_i, S_j) = \mathrm{Ext}^2_B(S_j, S_i) = \mathrm{Ext}^1_B(S_j, S_i) = 0$ and our claim follows. If $\mathrm{Ext}^1_B(S_j, S_i) \neq 0$, a similar argument shows that our claim holds. Finally in case $\mathrm{Ext}^1_B(S_i, S_j) = \mathrm{Ext}^1_B(S_j, S_i) = 0$ our claim is obvious.

Therefore, we have $|(x_i, x_j)_H| = |c_{ij}|$ for any $1 \leq i, j \leq n$, $i \neq j$ and our result follows. $\square$

Now we are ready to prove our main theorem.

Theorem 3.2. *Let $\Lambda = \mathrm{End}_{\mathcal{C}_H}(\widetilde{T})$ be a cluster-tilted algebra of simply-laced Dynkin type. Suppose that $(\mathbf{x}, C)$ is the seed corresponding to $\widetilde{T}$. Then all companion bases for $\Gamma = \Gamma(C)$ are strong.*

Proof. By Proposition 6.2 of [11], it is enough to show that there exists a strong companion basis for Γ. According to Theorem 3.1, $\Psi = \{x_1, \cdots, x_n\} \subseteq \Phi_H^+$ is a companion basis for Γ. We show that Ψ is a strong companion basis for Γ. Let $\alpha \in \Phi_H^+$. $\alpha = c_1x_1 + \cdots + c_nx_n$, with $c_i \in \mathbb{Z}$ for each i. Then $g(\alpha) = c_1g(x_1) + \cdots + c_ng(x_n) = c_1\varepsilon_1e_1 + \cdots + c_n\varepsilon_ne_n$ with $\varepsilon_i \in \{1, -1\}$, for each i.

Also $g(\alpha) \in \Phi_B^+$ and so by Theorem 2 of [13], $\mathrm{abs}(g(\alpha)) = (|c_1|, \cdots, |c_n|)$ is a dimension vector of some indecomposable Λ-module. But $d_\alpha^\Psi = (|c_1|, \cdots, |c_n|)$ by definition and hence Ψ is a strong companion basis for Γ. $\square$

It is known that different tilted algebras B and B' may correspond to the same cluster-tilted algebra $B \ltimes \mathrm{Ext}_B^2(DB, B)$. In that case, the linear bijection $g : K_0(H) \to K_0(B)$ depends on the choice of the tilted algebra B. In the following example we show that different tilted algebras give a different companion basis.

Example 3.3. Consider the cluster-tilted algebra $\Lambda = B \ltimes \mathrm{Ext}_B^2(DB, B)$ of Example 2.3. Let B' be the tilted algebra given by the quiver

$$\begin{tikzcd}
4 & & \\
& 3 \arrow[ul, "\beta"'] \arrow[dl, "\gamma"] & 1 \arrow[l, "\alpha"'] \\
2 & &
\end{tikzcd}$$

bound by $\alpha\beta = 0$ and $\alpha\gamma = 0$. Then it is easy to see that $B \ltimes \mathrm{Ext}_B^2(DB, B) \cong B' \ltimes \mathrm{Ext}_{B'}^2(DB', B')$ and $B' = \mathrm{End}_{H'}(T'')$, where $T'' = (0,0,0,1) \oplus (1,0,0,0) \oplus (1,1,1,1) \oplus (0,1,0,0)$ and $H' = kQ'$, where Q' is the quiver

$$\begin{tikzcd}
& 2 & \\
1 & 3 \arrow[l] \arrow[u] & 4 \arrow[l]
\end{tikzcd}$$

Let $g' : K_0(H') \to K_0(B')$ be the linear bijection given by $g'(x) = (\langle \mathbf{dim}\, T_i'', x\rangle_{H'})_i$. An easy calculation shows that $\mathrm{abs}(g'(1,0,0,0)) = (0,1,0,0)$, $\mathrm{abs}(g'(0,1,0,0)) = (0,0,0,1)$, $\mathrm{abs}(g'(0,0,1,0)) = (1,0,0,0)$ and $\mathrm{abs}(g'(0,0,1,1)) = (0,0,1,0)$. Then by Theorems 3.1 and 3.2, the set $\Psi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_3 + \alpha_4\}$ is a strong companion basis for Λ.

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