PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 2749–2764
http://dx.doi.org/10.1090/proc/13896
Article electronically published on March 14, 2018

ON THE TRACE FORMULA FOR HECKE OPERATORS ON CONGRUENCE SUBGROUPS

ALEXANDRU A. POPA

(Communicated by Ken Ono)

Dedicated to Don Zagier on the occasion of his 65th birthday

ABSTRACT. We give a new, simple proof of the trace formula for Hecke operators on modular forms for finite index subgroups of the modular group. The proof uses algebraic properties of certain universal Hecke operators acting on period polynomials of modular forms, and it generalizes an approach developed by Don Zagier and the author for the modular group. This approach leads to a very simple formula for the trace on the space of cusp forms plus the trace on the space of modular forms. As applications, we investigate what happens when one varies the weight or the level in the trace formula.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Γ be a finite index subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and let χ be a character of Γ with kernel of finite index in Γ. We denote by $M_k(\Gamma, \chi)$, $S_k(\Gamma, \chi)$ the spaces of modular forms, respectively, cusp forms for Γ of weight $k \geqslant 2$ and Nebentypus χ. For Σ a double coset of Γ inside its commensurator, we denote by $[\Sigma]$ the associated operator acting on modular forms. In this paper, we give a simple formula for the combination of traces

$$\mathrm{Tr}([\Sigma], M_k(\Gamma,\chi) + S_k^c(\Gamma,\chi)) := \mathrm{Tr}([\Sigma], M_k(\Gamma,\chi)) + \mathrm{Tr}([\Sigma], S_k^c(\Gamma,\chi)), \tag{1.1}$$

under the assumption $|\Gamma\backslash\Sigma| = |\Gamma_1\backslash\Gamma_1\Sigma|$, where $S_k^c(\Gamma,\chi)$ denotes the space of antiholomorphic cusp forms. We note that the assumption on Σ is rather mild, being satisfied by the usual Hecke and Atkin–Lehner operators for the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ (see Remark 3.4).

The proof is entirely algebraic, and it is based on the fact that (1.1) is the trace of a universal Hecke element acting on the space of (vector) period polynomials associated to modular forms. For the full modular group, a method for computing the trace of this Hecke operator was sketched by Don Zagier more than 20 years ago [15]. We sharpened this approach in the joint work [10], whose main result, Theorem 2.5 below, we take for granted in this paper. It is surprising that the same

Received by the editors June 14, 2017, and, in revised form, July 16, 2017.
2010 *Mathematics Subject Classification.* Primary 11F11, 11F25, 11F67.
Key words and phrases. Trace formula, Hecke operators, holomorphic modular forms, period polynomials.
This work was partly supported by the European Community grant PIRG05-GA-2009-248569 and by the CNCS grant PN-II-RU-TE-2011-3-0259. Part of this work was completed during several visits at MPIM in Bonn, whose support the author gratefully acknowledges.

Hecke element—which is independent of the weight, congruence subgroup, Nebentypus and double coset—is the key to proving the trace formula for an arbitrary congruence subgroup.

Compared to the full modular group case, in this paper we take a novel point of view and obtain a general trace formula for double coset operators acting on the period subspace of all Γ_1-modules $\mathcal{V}$ that admit a Γ_1-invariant, nondegenerate pairing (Theorem 2.6). The period subspace is closely related to the parabolic cohomology group $H^1_P(\Gamma_1, \mathcal{V})$, and our results are then obtained by taking $\mathcal{V}$ to be the module induced from the Γ-module $\mathrm{Sym}^{k-2}\,\mathbb{C}^2$, twisted by χ, and using the Eichler–Shimura isomorphism and the Shapiro lemma, which are reviewed in Section 3. We also use the theory of period polynomials for finite index subgroups developed together with Vicenţiu Paşol in [8], and formulated in a more general context in Sections 2 and 3 below. Modulo this background material, the proof of the trace formula for modular forms is an immediate application of the cohomological trace formula in Theorem 2.6, and it is given in Section 4.

In a special case, Theorem 2.6 can be stated as an "Euler–Poincaré trace formula" that may hold in much greater generality than proved here. Taking V to be the Γ-module $\mathrm{Sym}^{k-2}\,\mathbb{C}^2$ twisted by χ and using the Shapiro lemma, Theorem 2.6 yields

$$\sum_i (-1)^i \operatorname{Tr}([\Sigma], H^i(\Gamma, V)) \;=\; -\sum_{X\subset\overline{\Sigma}} \varepsilon_\Gamma(X)\cdot \operatorname{Tr}(M_X, V)\,, \tag{1.2}$$

where the sum on the right is over conjugacy classes X in the projectivization $\overline{\Sigma}$ of Σ with representatives $M_X \in \Sigma$, and $\varepsilon_\Gamma(X)$ are simple conjugacy class invariants defined in (1.7) below. The cohomology groups $H^i(\Gamma, V)$ are nontrivial for $i = 0, 1$ in our situation since Γ has cusps, but the formula makes sense for arbitrary Fuchsian groups of the first kind and modules V, and even for higher rank groups for appropriately defined coefficients $\varepsilon_\Gamma(X)$. This raises the question of whether a cohomological approach exists in much greater generality than proved in this paper, leading to the trace formula (1.2). We hope to return to this question in future work.

The resulting trace formula can be easily applied to investigate what happens when varying the weight or the level; see Theorems 2 and 3. Here we give two examples. Let $\Sigma = \Gamma$ be the trivial double coset, and assume that Γ is a finite index subgroup of Γ_1 with no elliptic elements. Only the conjugacy class of the identity contributes in formula (1.9), and we immediately obtain the dimension formula:

$$\dim M_k(\Gamma,\chi) + \dim S_k(\Gamma,\chi) = \frac{k-1}{6}\cdot[\overline{\Gamma}_1 : \overline{\Gamma}] + \delta_{k,2},$$

assuming $\chi(-1) = (-1)^k$ if $-I \in \Gamma$. For example if $\Gamma = \Gamma_0(11)$ and $k \neq 2$, the sum of dimensions above equals $2(k-1)$ for all 5 characters χ with $\chi(-1) = (-1)^k$. The dimension formula for the cuspidal subspace also follows from the Riemann–Roch theorem [13], but we obtain the formula above directly, without needing to compute the number of cusps and the genus of the associated modular surface. This is consistent with one theme of this paper, that the simplest formulas hold for the linear combination of traces (1.1).

As a generalization of the dimension formula, let T_ℓ be the Hecke operator for a prime ℓ for the principal congruence subgroup Γ_n of level n. Then Theorem 3 immediately gives

$$\lim_{\substack{n\to\infty\\(n,\ell(\ell-1))=1}} \frac{\mathrm{Tr}(T_\ell, M_k(\Gamma_n)+S_k(\Gamma_n))}{\varphi(n)} = \frac{\ell^{k-1}-1}{2}. \tag{1.3}$$

A similar limit for $\Gamma_0(N)$ in the level and weight aspect has been used by Serre to determine the distribution of the eigenvalue of a fixed T_ℓ of Hecke eigenforms of varying weight and level [12]. It also answers a question posed by Florin Rădulescu, who considers similar limits over a sequence of subgroups approaching the identity in [11, Cor. 4]. Similar limits for $\mathrm{Tr}(T_n, S_k(\Gamma_1(N)))$ as $N\to\infty$ are computed in the sequel to this paper [9], and we find that they are independent of n and k as long as $(N, n-1)=1$.

Our approach for proving the trace formula is related to the theory of modular symbols. The Hecke operators acting on period polynomials are adjoints of the Hecke operators on modular symbols introduced by Merel [7], and our approach may also be interpreted as computing the trace of Hecke operators on the space of modular symbols.

One could also apply our method to the module $\mathrm{Sym}^{k-2}\,\mathbb{C}^2\otimes\Psi$, for a finite dimensional representation Ψ of Γ, obtaining trace formulae for vector valued modular forms, but for simplicity we restrict ourselves to classical modular forms. Since we only use the structure of $\mathrm{PSL}_2(\mathbb{Z})$ as a free group with two elliptic generators and having one cusp, the same method easily applies to prove trace formulas for other Hecke groups.

There is vast literature on the trace formula for Hecke operators for congruence subgroups, and previous authors computed the trace on the cuspidal subspace alone. One insight of the present paper which is apparent in the dimension formula above, and was first observed by Zagier [14, 15] in the case of $\mathrm{SL}_2(\mathbb{Z})$, is that the trace formula is much simpler for the linear combination (1.1). From our point of view this is reflected in the fact that formula (1.2) involves only scalar, elliptic, and split hyperbolic conjugacy classes, those for which $\varepsilon_\Gamma(X)\neq 0$. One can also extract the cuspidal contribution from (1.1) by computing the trace on the Eisenstein part, but since it is rather technical and would double the size of this paper, we leave this computation to the sequel [9]. As a consequence of the main result of this paper, there we obtain explicit formulas in terms of class numbers for the trace of a composition of Hecke and Atkin–Lehner operators for $\Gamma_0(N)$. The formulas obtained are among the simplest in the literature, and unlike in previous work, we need no restrictions on the index of the operators involved. We refer to [9] for a bibliography of previous results on the trace formula.

In the remainder of the introduction, we state the main theorem and give two applications.

1.1. **Statement of results.** We start with some definitions and notation in use throughout the paper. Let Γ be a finite index subgroup of Γ_1, and let Σ be a double coset inside the commensurator $\widetilde{\Gamma}$, that is, Σ is a left and right Γ-invariant set such that the number of left cosets $|\Gamma\backslash\Sigma|$ is finite. Let $\widetilde{\chi}$ be a multiplicative function on the semigroup generated by Γ and Σ inside $\widetilde{\Gamma}$ such that $\widetilde{\chi}|_\Gamma=\chi^{-1}$, namely

$$\widetilde{\chi}(\gamma\sigma\gamma')=\chi^{-1}(\gamma\gamma')\widetilde{\chi}(\sigma) \quad \text{for all } \gamma\in\Gamma, \sigma\in\Sigma\,. \tag{1.4}$$

A modular form $f \in M_k(\Gamma,\chi)$ satisfies $f|_k\gamma = \chi(\gamma)f$, and the double coset Σ defines an operator $[\Sigma]$ on $M_k(\Gamma,\chi)$ by

$$f|[\Sigma] = \sum_{\sigma\in\Gamma\backslash\Sigma} \det\sigma^{k-1}\cdot\widetilde{\chi}(\sigma)\cdot f|_k\sigma\ , \tag{1.5}$$

where $f|_k\gamma(z) = f(\gamma z)(c_\gamma z+d_\gamma)^{-k}$, and we write $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma\end{smallmatrix}\right)$ throughout the paper. In (1.5) and in similar sums over right cosets, it is understood (and usually obvious) that the sum is independent of the coset representatives chosen.

Example. Let Γ be the congruence subgroup $\Gamma_0(N) := \{\gamma\in\Gamma_1 : N|c_\gamma\}$, and let χ be a character modulo N viewed as a character of $\Gamma_0(N)$ by $\chi(\gamma)=\chi(d_\gamma)$. The usual Hecke operators T_n on $M_k(\Gamma,\chi)$ are associated to the double coset

$$\Delta_n := \{\sigma\in M_2(\mathbb{Z})\ :\ \det\sigma = n,\ N|c_\sigma,\ (a_\sigma,N)=1\}\ , \tag{1.6}$$

and $\widetilde{\chi}(\sigma)=\chi(a_\sigma)$ for $\sigma\in\Delta_n$.

For any subset $\mathcal{S}$ of $\mathrm{GL}_2^+(\mathbb{R})$, we denote by $\overline{\mathcal{S}} = (\mathcal{S}\cup-\mathcal{S})/\{\pm1\}\subset \mathrm{GL}_2^+(\mathbb{R})/\{\pm1\}$ its projectivization.

For a $\overline{\Gamma}$-conjugacy class $X\subset \mathrm{GL}_2^+(\mathbb{R})/\{\pm1\}$, we let $M_X\in\mathrm{GL}_2^+(\mathbb{R})$ be the lift of any representative. We denote by $\Delta(X)=\mathrm{Tr}(M_X)^2-4\det(M_X)$ the discriminant of the quadratic form associated to M_X, and by $|\operatorname{Stab}_{\overline{\Gamma}} M_X|$ the (possibly infinite) cardinality of the stabilizer of M_X under conjugation by $\overline{\Gamma}$. We introduce the conjugacy class invariant

$$\varepsilon_\Gamma(X) = \begin{cases} \dfrac{|\Gamma\backslash\mathcal{H}|}{2\pi} & \text{if } M_X \text{ scalar,}\\[2ex] \dfrac{\operatorname{sgn}\Delta(X)}{|\operatorname{Stab}_{\overline{\Gamma}} M_X|} & \text{otherwise,}\end{cases} \tag{1.7}$$

where $|\Gamma\backslash\mathcal{H}|$ is the area of a fundamental domain for Γ with respect to the standard hyperbolic metric, and we use the convention that $1/\infty=0$. Any double coset $\Sigma\subset\widetilde{\Gamma}$ contains only finitely many conjugacy classes X with $\varepsilon_\Gamma(X)\neq0$. These are the elliptic, scalar, and split hyperbolic classes (those that contain an element fixing two distinct cusps of Γ, for which $\varepsilon_\Gamma(X)=1$).

Let $p_w(t,n)$ be the Gegenbauer polynomial defined by the power series expansion

$$(1-tx+nx^2)^{-1} = \sum_{w\geqslant0} p_w(t,n)x^w\ . \tag{1.8}$$

We can now state the main theorem of this paper.

Theorem 1. *Let Γ be a finite index subgroup of Γ_1, let $k\geqslant 2$ be an integer, let χ be a character of Γ with kernel of finite index in Γ, and let Σ be a double coset of Γ such that $|\Gamma\backslash\Sigma| = |\Gamma_1\backslash\Gamma_1\Sigma|$. Assuming $\chi(-1)=(-1)^k$ if $-1\in\Gamma$, we have*

$$\begin{aligned}\mathrm{Tr}([\Sigma], M_k(\Gamma,\chi)+S_k^c(\Gamma,\chi)) \;=\; &\sum_{X\subset\overline{\Sigma}} p_{k-2}(\mathrm{Tr}\, M_X,\det M_X)\ \widetilde{\chi}(M_X)\ \varepsilon_\Gamma(X)\\ &+\delta_{k,2}\delta_{\chi,\mathbf{1}}\sum_{\sigma\in\Gamma\backslash\Sigma}\widetilde{\chi}(\sigma)\ ,\end{aligned} \tag{1.9}$$

where the sum is over $\overline{\Gamma}$-conjugacy classes X in $\overline{\Sigma}$ with representative $M_X\in\Sigma$. The symbol $\delta_{a,b}$ is 1 if $a=b$ and 0 otherwise.

If Γ, Σ and χ are invariant under conjugation by an order 2 element of determinant -1, then we can replace the space of antiholomorphic cusp forms $S_k^c(\Gamma,\chi)$ by $S_k(\Gamma,\chi)$ in the theorem, as well as in all the trace formulas in the paper (see Remark 3.2).

We state the theorem under the assumptions used in the proof, but we expect the same formula to hold for any Fuchsian subgroup of the first kind Γ, and any double coset $\Sigma \subset \widetilde{\Gamma}$, with the term on the second line multiplied by 2 if Γ has no cusps.

What we prove is an equivalent version of Theorem 1 in which the sum is over Γ_1-conjugacy classes X, and we set

$$\varepsilon(X) := \varepsilon_{\Gamma_1}(X)\,. \tag{1.10}$$

Explicitly, if $M_X \in X$ is any representative, then $\varepsilon(X)$ is equal to: $1/6$ if M_X is scalar; $-1/|\operatorname{Stab}_{\overline{\Gamma}_1} M_X|$ if M_X is elliptic; 1 if M_X is hyperbolic fixing two cusps of Γ_1; and 0 otherwise.

Theorem 1 (Second version)**.** *Under the assumptions in the first version, we have*

$$\begin{aligned}\operatorname{Tr}([\Sigma], M_k(\Gamma,\chi)+S_k^c(\Gamma,\chi)) \;=\; &\sum_X p_{k-2}(\operatorname{Tr} M_X, \det M_X)\, \mathcal{C}^{\chi}_{\Gamma,\Sigma}(M_X)\, \varepsilon(X)\\ &+\delta_{k,2}\delta_{\chi,\mathbf{1}} \sum_{\sigma\in\Gamma\backslash\Sigma} \widetilde{\chi}(\sigma)\,,\end{aligned} \tag{1.11}$$

where the sum is over $\overline{\Gamma}_1$-conjugacy classes $X \subset \overline{\Gamma_1\Sigma\Gamma_1}$ with representative $M_X \in \Gamma_1\Sigma\Gamma_1$, and[1]

$$\mathcal{C}^{\chi}_{\Gamma,\Sigma}(M) \;:=\; \sum_{\substack{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1\\ \pm AMA^{-1}\in\Sigma}} (\pm1)^k \widetilde{\chi}(\pm AMA^{-1})\,. \tag{1.12}$$

Theorem 1, as well as the equivalence of the two versions, is proved in Section 4. The coefficient $\mathcal{C}^{\chi}_{\Gamma,\Sigma}(M) = (-1)^k \mathcal{C}^{\chi}_{\Gamma,\Sigma}(-M)$ also depends on the parity of k, but for simplicity we suppress this dependence from the notation.

Next we give two applications of the trace formula, in which we vary the weight or the level. When varying the weight k, we obtain that the generating series of the traces of Hecke operators is a rational function, as observed in special cases in [3, 14].

Theorem 2. *Set $\mathbf{T}^{\chi}_{\Gamma,\Sigma}(k) = \operatorname{Tr}([\Sigma], M_k(\Gamma,\chi)+S_k^c(\Gamma,\chi))$. When $-1\notin\Gamma$, the formula in Theorem 1 is equivalent to the following power series identity:*

$$\sum_{k\geqslant 2} \mathbf{T}^{\chi}_{\Gamma,\Sigma}(k)\cdot x^{k-2} = \sum_{X\subset\overline{\Sigma}} \frac{\widetilde{\chi}(M_X)\,\varepsilon_\Gamma(X)}{1-\operatorname{Tr}(M_X)x+\det(M_X)x^2} + \delta_{\chi,\mathbf{1}} \sum_{\sigma\in\Gamma\backslash\Sigma}\widetilde{\chi}(\sigma)\,,$$

where the sum is over Γ-conjugacy classes $X\subset\overline{\Sigma}$ with representatives $M_X\in\Sigma$. When $-1\in\Gamma$ a similar formula holds, with the first summand on the right side replaced by

$$\frac{\varepsilon_\Gamma(X)\widetilde{\chi}(M_X)}{2}\Big(\frac{1}{1-\operatorname{Tr}(M_X)x+\det(M_X)x^2} + \frac{\chi(-1)}{1+\operatorname{Tr}(M_X)x+\det(M_X)x^2}\Big).$$

[1] The same sign is chosen in all three places in (1.12). If $-1\notin\Gamma$ at most one choice of signs is possible for each A, while if $-1\in\Gamma$ both choices yield the same value for the summand.

When varying the level, the simplest formula is obtained when Γ varies through the principal congruence subgroups $\Gamma_n = \{\gamma \in \Gamma_1 : \gamma \equiv I \pmod{n}\}$. The next theorem seems to be the first explicit trace formula for Γ_n.

Theorem 3. *Fix $\sigma = \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\ell > 1$, and let $n > 2\ell + 2$ coprime to ℓ. Setting $n' = n/\gcd(n, \ell-1)$, we have*

$$\operatorname{Tr}([\Gamma_n \sigma \Gamma_n], M_k(\Gamma_n) + S_k(\Gamma_n)) = \frac{(\ell^{k-1} - 1)}{2} \cdot \frac{\varphi(n')\varphi_2(n)}{\varphi_2(n')} + \delta_{k,2}\varphi_1(\ell),$$

where $\varphi_1(\ell) = [\Gamma_1 : \Gamma_0(\ell)]$, $\varphi_2(n) = [\overline{\Gamma}_1 : \overline{\Gamma}_n]$, and φ is Euler's phi function.

In particular, for ℓ prime the operator in the theorem is the usual Hecke operator T_ℓ, and we obtain the limit formula (1.3). The proof of Theorem 3 is given in Section 5.

2. A trace formula on the parabolic cohomology of the modular group

Let $\mathcal{V}$ be a Γ-module, with Γ denoting $\mathrm{SL}_2(\mathbb{Z})$. We give a formula for the trace of Hecke operators on the parabolic cohomology group $H^1_P(\Gamma, \mathcal{V})$ by relating it to the trace of a certain operator $\widetilde{T}_n$ on the *period subspace* $\mathcal{W}$ of $\mathcal{V}$ (Corollary 2.3). The trace on $\mathcal{W}$ is computed in Theorem 2.6, using a special operator $\widetilde{T}_n$, studied in detail in [10].

2.1. **Double coset operators on cohomology.** To define the action of double coset operators on cohomology, let us consider in this section only an arbitrary group Γ and a right Γ-module $\mathcal{V}$, which is assumed to be a vector space over $\mathbb{C}$. Let Σ be a double coset of Γ contained in its commensurator, that is, $\Sigma = \Gamma\Sigma\Gamma$ and the number of left cosets $|\Gamma\backslash\Sigma|$ is finite. Assume that elements in Σ act on $\mathcal{V}$ in a way compatible with the action of Γ, that is,

$$P|(gM) = (P|g)|M, \quad P|(Mg) = (P|M)|g, \text{ for } P \in \mathcal{V}, g \in \Gamma, M \in \Sigma\,, \tag{2.1}$$

namely $\mathcal{V}$ is a module for the semigroup generated by Γ and Σ inside the commensurator of Γ. Fix representatives $M_K \in \Sigma$ for cosets $K \in \Gamma\backslash\Sigma$, and for $\gamma \in \Gamma$, let $\gamma_K \in \Gamma$ be the unique element such that $M_K\gamma^{-1} = \gamma_K^{-1} M_{K\gamma^{-1}}$. If $\phi : \Gamma \to \mathcal{V}$ is a cocycle, namely $\phi(gh) = \phi(g)|h + \phi(h)$, we define

$$\phi|[\Sigma](\gamma) = \sum_{K \in \Gamma\backslash\Sigma} \phi(\gamma_K)|M_K\,. \tag{2.2}$$

Then $\phi|[\Sigma]$ is a cocycle whose cohomology class is independent of the choice of representatives M_K. If Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$ and ϕ is a parabolic cocycle (that is, $\phi(\gamma) = v_\gamma|1-\gamma$ for all parabolic elements γ), so is $\phi|[\Sigma]$, which defines an action of $[\Sigma]$ on the parabolic cohomology group $H^1_P(\Gamma, \mathcal{V})$ [4, 13].

Remark 2.1. If Γ contains an element J in the center with $J^2 = 1$ (e.g., $J = -1$ for Γ a subgroup of $\mathrm{SL}_2(\mathbb{R})$), let $\mathcal{V}^J$ be the subspace of J-invariants in $\mathcal{V}$. For any cocycle $\phi : \Gamma \to \mathcal{V}$ and $\gamma \in G$ we have $\phi(\gamma)|1 - J = \phi(J)|1 - \gamma$, so $\gamma \mapsto \phi(\gamma)|1 - J$ is a coboundary. Therefore $\phi'(\gamma) = \frac{1}{2}\phi(\gamma)|(1 + J)$ is a cocycle in the same class as ϕ, which takes values in the module $\mathcal{V}^J$, and the map

$$H^1(\Gamma, \mathcal{V}) \simeq H^1(\Gamma, \mathcal{V}^J), \quad [\phi] \mapsto [\phi']\,,$$

is an isomorphism. The same is true for parabolic cohomology, if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$. Therefore we can restrict ourselves, without loss of generality, to modules on which J acts identically.

2.2. **Hecke operators on the period subspace.** In this section we set $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{V}$ be a right Γ-module on which -1 acts identically, and let Σ be a double coset of Γ whose elements act on $\mathcal{V}$ by an action denoted $|$, in a way compatible with the action of Γ as in (2.1). By linearity we also have an action of elements of $\mathcal{R}_\Sigma = \mathbb{Q}[\overline{\Sigma}]$ on $\mathcal{V}$.

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ be two generators of Γ, and let $U = TS$, an element of order 3 in $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$. To ease notation, we use the same notation for an element in $\overline{\Gamma}$ and for a lift of it in Γ. Any parabolic cocycle $\varphi : \Gamma \to \mathcal{V}$ can be modified by a coboundary so that $\varphi(T) = 0$, and then the element $P = \varphi(S) = \varphi(TS)$ belongs to the subspace

$$\mathcal{W} := \{P \in \mathcal{V} : P|(1+S) = P|(1+U+U^2) = 0\},$$

called *the period subspace.* Conversely, if $P \in \mathcal{W}$, the map $\varphi_P : \Gamma \to \mathcal{V}$ with $\varphi_P(T) = 0$, $\varphi_P(S) = P$ extends to a parabolic cocycle via $\varphi_P(gh) = \varphi_P(g)|h + \varphi_P(h)$. This gives an exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{W} \xrightarrow{P \mapsto [\varphi_P]} H^1_P(\Gamma, \mathcal{V}) \longrightarrow 0\,, \tag{2.3}$$

where $\mathcal{C} = \{P|(1-S) \ : \ P \in \mathcal{V},\ P|(1-T) = 0\} \subset \mathcal{W}$ is called the *coboundary subspace.*

The action of the double coset operator $[\Sigma]$ on cohomology was defined in the previous section, and now we show directly that the corresponding action on $\mathcal{W}$ is determined by the same elements introduced by Choie and Zagier to express the action of Hecke operators on period polynomials of modular forms. In each coset $K \in \Gamma\backslash\Sigma$ (identified as above with $\overline{\Gamma}\backslash\overline{\Sigma}$), we choose a representative M_K that fixes the cusp infinity, and we let

$$T^\infty_\Sigma = \sum_{K \in \Gamma\backslash\Sigma} M_K \in \mathcal{R}_\Sigma\,. \tag{2.4}$$

Then there exist elements $\widetilde{T}_\Sigma \in \mathcal{R}_\Sigma$ satisfying

$$(1-S)\widetilde{T}_\Sigma - T^\infty_\Sigma(1-S) \in (1-T)\mathcal{R}_\Sigma\,. \tag{A}$$

This was shown in [2] in the case $\Sigma = \mathcal{M}_n$, and the proof for general Σ is similar.

Proposition 2.2. *Any element $\widetilde{T}_\Sigma \in \mathcal{R}_\Sigma$ satisfying property* (A) *preserves the space $\mathcal{W}$, and its action on $\mathcal{W}$ corresponds to the action of $[\Sigma]$ on $H^1_P(\Gamma, \mathcal{V})$ via the map in* (2.3)*, namely the cocycles $\varphi_P|[\Sigma]$ and $\varphi_{P|\widetilde{T}_\Sigma}$ are in the same cohomology class, for any $P \in \mathcal{W}$.*

We use the following easy lemma, which we state in greater generality than needed here.

Lemma. *Let G be a group with generators $g_1, \ldots, g_r$, let V be a right G-module also viewed as a $\mathbb{Q}[G]$-module, and let $\varphi : G \to V$ be a cocycle. Then for each $g \in G$ there exist $X_i \in \mathbb{Q}[G]$ such that in the group algebra $\mathbb{Q}[G]$ we have $1-g = \sum_{i=1}^r (1-g_i)X_i$, and for any such choice of X_i we have*

$$\varphi(g) = \sum_{i=1}^r \varphi(g_i)|X_i\,.$$

Proof. Both the existence of X_i as above and the relation follow by induction on the length of g as a product in the generators g_i: we have $1-g_ig = 1-g+(1-g_i)g$, and $\varphi(g_ig) = \varphi(g_i)|g + \varphi(g)$. □

Proof of Proposition 2.2. We use the representatives M_K in (2.4) to define the action of $[\Sigma]$ on φ in (2.2).

Let $\varphi \in Z^1_P(\Gamma, \mathcal{V})$ with $\varphi(T) = 0$. Writing $M_K T^{-1} = T_K^{-1} M_{KT^{-1}}$, we have $T_K\infty = \infty$, so $\varphi(T_K) = 0$, and (2.2) shows that $\varphi|[\Sigma](T) = 0$. Therefore it remains to show that $\varphi|[\Sigma](S) = \varphi(S)|\widetilde{T}_\Sigma$ if $\widetilde{T}_\Sigma$ satisfies (A), which would show in particular that $\widetilde{T}_\Sigma$ preserves $\mathcal{W}$.

Let $\widetilde{T}_\Sigma = \sum_{K\in\Gamma\backslash\Sigma} X_K M_K$ satisfy (A), with $X_K \in \mathbb{Q}[\Gamma]$. Relation (A) implies

$$(1-S)X_K M_K \equiv M_K - M_{KS^{-1}}S = (1-S_K)M_K \pmod{(1-T)\mathcal{R}_n},$$

where $M_K S^{-1} = S_K^{-1} M_{KS^{-1}}$. We therefore have $1-S_K = (1-S)X_K + (1-T)Y_K$, and the lemma applied to the group $\overline{\Gamma}$ with generators S, T and to the cocycle φ with $\varphi(T) = 0$ gives

$$\varphi(S)|\widetilde{T}_\Sigma = \sum_{K\in\Gamma\backslash\Sigma} (\varphi(S)|X_K)|M_K = \sum_{K\in\Gamma\backslash\Sigma} \varphi(S_K)|M_K = \varphi|[\Sigma](S)\,,$$

where we used (2.2). □

Corollary 2.3. *For any element* $\widetilde{T}_\Sigma \in \mathcal{R}_\Sigma$ *satisfying* (A) *we have*

$$\mathrm{Tr}([\Sigma], H^1_P(\Gamma, \mathcal{V})) + \mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{C}) = \mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{W}). \tag{2.5}$$

Proof. This is immediate from (2.3), once we show that the coboundary subspace $\mathcal{C}$ is also preserved by $\widetilde{T}_\Sigma$. Indeed, if $P|1-T = 0$, by (A) we have

$$P|(1-S)|\widetilde{T}_\Sigma = P|T_\Sigma^\infty|(1-S), \tag{2.6}$$

and since $T_\Sigma^\infty(1-T) \in (1-T)\mathcal{R}_n$ we also have $P|T_\Sigma^\infty|1-T = 0$. □

Remark 2.4. The previous proof shows that the following exact sequence is Hecke-equivariant:

$$0 \longrightarrow \mathcal{V}^\Gamma \longrightarrow \mathcal{D} \xrightarrow{P\mapsto P|1-S} \mathcal{C} \longrightarrow 0\,,$$

where $\mathcal{D} := \{P \in \mathcal{V} \ :\ P|1-T = 0\} \simeq H^0(\Gamma_\infty, \mathcal{V})$, and the Hecke action is by $\widetilde{T}_\Sigma$ on $\mathcal{C}$ and by T_Σ^∞ on the first two terms. Therefore we have $\mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{C}) = \mathrm{Tr}(T_\Sigma^\infty, \mathcal{D}) - \mathrm{Tr}(T_\Sigma^\infty, \mathcal{V}^\Gamma)$, and one can compute explicitly the right hand side for the modules of interest.

By Corollary 2.3, computing the trace of $[\Sigma]$ on the parabolic cohomology reduces to computing the trace of $\widetilde{T}_\Sigma$ on the period subspace. The latter trace is computed using an operator $\widetilde{T}_\Sigma$ satisfying an extra property introduced in [15]:

$$\text{(B)} \qquad \begin{cases} \widetilde{T}_\Sigma(1+S) & \in \quad (1+U+U^2)\mathcal{R}_\Sigma\,, \\ \widetilde{T}_\Sigma(1+U+U^2) & \in \quad (1+S)\mathcal{R}_\Sigma\,. \end{cases}$$

It is easy to show that there exist operators satisfying both (A) and (B), and the main difficulty in this approach to the trace formula is contained in the next theorem, proved in [10]. Recall that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in this section.

Theorem 2.5 ([10]). *Let* $\widetilde{T}_\Sigma = \sum_M c(M)M \in \mathcal{R}_\Sigma$ *be any element satisfying* (A) *and* (B).

(a) *For each* right coset $K \in \overline{\Sigma}/\overline{\Gamma}$ *we have* $\sum_{M\in K} c(M) = -1$.

(b) *For each conjugacy class* $X \subset \overline{\Sigma}$ *we have*

$$\sum_{M\in X} c(M) = \varepsilon(X)\ , \tag{C}$$

where $\varepsilon(X)$ *is defined in* (1.10).

Part (a) is easy to prove, and the main difficulty is to show that any element satisfying (A) and (B) also satisfies (C). Note that it is enough to produce such an operator for the double coset $\mathcal{M}_n$ of integral matrices of determinant n, because any double coset $\Sigma = \Gamma\sigma\Gamma$ can be scaled such that $\Sigma \subset \mathcal{M}_n$ for some n, and then the relations (A) − (C) for the operator $\widetilde{T}_n$ associated to the double coset $\mathcal{M}_n$ imply the corresponding relations for $\widetilde{T}_\Sigma$.

Before stating the main theorem of this section, we define $T_\Sigma \in \mathcal{R}_\Sigma$ to be the sum of a complete system of coset representatives for $\Gamma\backslash\Sigma$. The operator T_Σ acts (on the right) on the invariant space $\mathcal{V}^\Gamma$, and the action is clearly independent of the coset representatives chosen.

Theorem 2.6. *Let* $\mathcal{V}$ *be a* Γ*-module with period subspace* $\mathcal{W}$*, and let* Σ *be a double coset acting on* $\mathcal{V}$ *as in* (2.1). *Assume that* $\mathcal{V}$ *admits a nondegenerate,* Γ*-invariant pairing. If* $\widetilde{T}_\Sigma \in \mathcal{R}_\Sigma$ *is any element satisfying* (A)*, we have*

$$\mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{W}) = \mathrm{Tr}(T_\Sigma, \mathcal{V}^\Gamma) + \sum_{X\subset\overline{\Sigma}} \mathrm{Tr}(M_X, \mathcal{V})\, \varepsilon(X)\ ,$$

where the sum is over $\overline{\Gamma}$*-conjugacy classes* X *in* $\overline{\Sigma}$ *with representatives* $M_X \in \Sigma$.

Proof. We use Theorem 2.5. Any two elements satisfying (A) act in the same way on $\mathcal{W}$,[2] and we choose $\widetilde{T}_\Sigma$ satisfying (B) and (C) as well. Property (B) implies that $\widetilde{T}_\Sigma$ maps the spaces $A = \mathrm{Ker}(1+S)$ and $B = \mathrm{Ker}(1+U+U^2)$ into each other, and basic linear algebra shows that

$$\mathrm{Tr}(\widetilde{T}_\Sigma, A\cap B) = \mathrm{Tr}(\widetilde{T}_\Sigma, A+B). \tag{2.7}$$

We have $\mathcal{W} = A\cap B$, and denote $\mathcal{V}' = A+B$.

From the Γ-invariance of the pairing, we have $\mathrm{Ker}(1-S) \subseteq A^\perp$. Since $\mathrm{Ker}(1-S) = \mathrm{Im}(1+S)$, the nondegeneracy of the pairing implies that $\mathrm{Ker}(1-S)$ and $A^\perp$ have the same dimension, hence they are equal. Similary $B^\perp = \mathrm{Ker}(1-U)$, so $(A+B)^\perp = A^\perp \cap B^\perp = \mathcal{V}^\Gamma$ as Γ is generated by S and U. Therefore we have a direct sum decomposition

$$\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}^\Gamma\ . \tag{2.8}$$

Write $\widetilde{T}_\Sigma = \sum_{C\in\Sigma/\Gamma} R_C X_C$, where $R_C \in \Sigma$ is any representative for the coset C and $X_C \in \mathbb{Q}[\Gamma]$. We chose the representatives $\{R_C\}$ so that they also form a system of representatives for the left cosets $\Gamma\backslash\Sigma$ [13, Lemma 3.5], so that we can choose $T_\Sigma = \sum_C R_C$ in the statement.

[2] It was shown in [2] that the difference of such elements belongs to $(1+S)\mathcal{R}_\Sigma + (1+U+U^2)\mathcal{R}_\Sigma$, which annihilates $\mathcal{W}$.

The element $\widetilde{T}_\Sigma$ does not preserve $\mathcal{V}^\Gamma$, so we decompose for $P \in \mathcal{V}^\Gamma$ and a coset $C \in \Sigma/\Gamma$:

$$P|R_C = P_C + P'_C, \text{ with } P_C \in \mathcal{V}^\Gamma,\ P'_C \in \mathcal{V}'.$$

As $P_C \in \mathcal{V}^\Gamma$, it follows by Theorem 2.5(a) that $P_C|X_C = -P_C$, and we obtain $P|R_C X_C = -P_C + P''_C$ with $P''_C = P'_C|X_C \in \mathcal{V}'$ (as $\mathcal{V}' = \mathrm{Im}(1-S) + \mathrm{Im}(1-U)$ is invariant under the action of $\mathbb{Q}[\Gamma]$). Therefore

$$P|\widetilde{T}_\Sigma = -\sum_C P_C + \sum_C P''_C, \quad P|T_\Sigma = \sum_C P_C + \sum_C P'_C\,.$$

Since T_Σ preserves $\mathcal{V}_\Gamma$, we conclude from the direct sum decomposition (2.8) that $P|T_\Sigma = \sum_C P_C$, and therefore (2.7) and the previous relation give

$$\mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{W}) = \mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{V}') = \mathrm{Tr}(T_\Sigma, \mathcal{V}^\Gamma) + \mathrm{Tr}(\widetilde{T}_\Sigma, \mathcal{V})\,.$$

By Theorem 2.5(b), the last term can be written as a sum over conjugacy classes as in the statement of the theorem, finishing the proof. □

3. A trace formula on the space of period polynomials

Let $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. We now specialize the Γ_1-module of the last section to be the induced module $\mathrm{Ind}_\Gamma^{\Gamma_1}(\mathrm{Sym}^w \mathbb{C}^2 \otimes \chi)$, where $\Gamma \subset \Gamma_1$ is a finite index subgroup (note the change in notation from the last section). Its period subspace is the space of period polynomials associated with the space of cusp forms $S_{w+2}(\Gamma, \chi)$, and we apply the Eichler–Shimura isomorphism and the Shapiro lemma to show that the trace of Hecke operators on the period subspace equals the trace on $M_{w+2}(\Gamma,\chi) + S^c_{w+2}(\Gamma, \chi)$ (Proposition 3.6). We then apply Theorem 2.6 to compute this trace in Section 4, thus proving Theorem 1.

For other uses of the Eicher–Shimura isomorphism together with the Shapiro lemma in the study of modular forms for congruence subgroups, see [5, 6].

3.1. **The Eichler–Shimura isomorphism.** Let Γ be a finite index subgroup of Γ_1, and let χ be a character of Γ whose kernel has finite index in Γ. For $w = k - 2 \geqslant 0$, we view the space V_w of complex polynomials of degree $\leqslant w$ as a Γ-module by

$$P|_\chi \gamma := \chi(\gamma^{-1}) P|_{-w}\gamma, \text{ for } P \in V_w, \gamma \in \Gamma\,, \tag{3.1}$$

and we denote it by V_w^χ to indicate this action.

Let Σ be a double coset as in Section 1.1. Using a function $\widetilde{\chi}$ as in (1.4), we define an operation of elements $M \in \Sigma$ on V_w^χ by

$$P|_\chi M = \widetilde{\chi}(M) P|_{-w} M,$$

and this operation is obviously compatible as in (2.1) with the action of Γ. Therefore we have an operator $[\Sigma]$ acting on $H^1_P(\Gamma, V_w^\chi)$ as in (2.2).

In order to state the Eichler–Shimura isomorphism, let $S^c_k(\Gamma, \chi)$ be the space of antiholomorphic cusp forms $\overline{S_k(\Gamma, \overline{\chi})}$, where the bar denotes complex conjugation. Functions $g \in S^c_k(\Gamma,\chi)$ are antiholomorphic on the upper half plane and satisfy $g|^c_k\gamma = \chi(\gamma) g$ for $\gamma \in \Gamma$, where

$$g|^c_k \gamma(z) := g(\gamma z) j(\gamma, \overline{z})^{-k}.$$

The operator $[\Sigma]$ acts on $S^c_k(\Gamma,\chi)$ as in (1.5), with the action $|_k$ replaced by $|^c_k$.

Theorem 3.1 (Eichler–Shimura). *We have a Hecke-equivariant isomorphism*

$$S_k(\Gamma,\chi)\oplus S_k^c(\Gamma,\chi)\longrightarrow H_P^1(\Gamma,V_w^\chi)$$

given by $(f,g)\mapsto[\varphi_f]+[\varphi_g^c]$ *with the parabolic cocycles* φ_f, φ_g^c *defined by*

$$\varphi_f(\gamma)(X)=\int_{\gamma^{-1}z_0}^{z_0} f(t)(t-X)^w dt,\quad \varphi_g^c(\gamma)(X)=\int_{\gamma^{-1}z_0}^{z_0} g(t)(\overline{t}-X)^w\overline{dt}\,,$$

where $z_0\in\mathcal{H}\cup\{\text{cusps}\}$.

Proof. The proof is given by Shimura in [13, Sec. 8], with the difference here being that Γ acts on the right on V_w^χ instead of on the left. Shimura defines the action of the Hecke operator $[\Sigma]$ using a character χ of the semigroup generated by Γ and $\Sigma^\vee$ inside $\widetilde{\Gamma}$, where $\Sigma^\vee$ is the adjoint of Σ. Our function $\widetilde{\chi}$ is related to such a χ by $\widetilde{\chi}(\sigma)=\chi(\sigma^\vee)$ for $\sigma\in\Sigma$. □

Remark 3.2. Assume that there exists $\eta\in\mathrm{GL}_2(\mathbb{R})$ with $\eta^2=1$ and $\det\eta=-1$ such that

$$\begin{cases}\quad\eta\Gamma\eta=\Gamma,\quad \eta\Sigma\eta=\Sigma,\\ \chi(\eta\gamma\eta)=\chi(\gamma),\quad \widetilde{\chi}(\eta\sigma\eta)=\widetilde{\chi}(\sigma)\quad \text{for }\gamma\in\Gamma,\ \sigma\in\Sigma.\end{cases}$$

For example, if $\Gamma=\Gamma_0(N)$ we can take $\eta=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$. Under this assumption, we have a Hecke-equivariant isomorphism

$$S_k(\Gamma,\chi)\longrightarrow S_k^c(\Gamma,\chi),\quad f\mapsto f^*(z)=f(\eta\overline{z})j(\eta,\overline{z})^{-k}\,,$$

with inverse $g\mapsto g^*$, $g^*(z)=g(\eta\overline{z})j(\eta,z)^{-k}$, so in this case all the trace formulas in this paper hold with the space $S_k^c(\Gamma,\chi)$ replaced by $S_k(\Gamma,\chi)$.

3.2. **The Shapiro isomorphism.** The Shapiro lemma gives a Hecke equivariant isomorphism between the cohomology groups $H_P^1(\Gamma_1,\mathrm{Ind}_\Gamma^{\Gamma_1}V_w^\chi)$ and $H_P^1(\Gamma,V_w^\chi)$, and we now describe it explicitly.

Since V_w is a Γ_1-module as well as a Γ-module, we will identify the induced module $\mathrm{Ind}_\Gamma^{\Gamma_1}V_w^\chi$ with the space of functions $P:\Gamma_1\to V_w^\chi$ such that

$$P(\gamma A)=\chi(\gamma)P(A)\,,\quad A\in\Gamma_1,\gamma\in\Gamma\,,$$

on which Γ_1 acts by $P|g(A)=P(Ag^{-1})|_{-w}g$. By Remark 2.1, the cohomology group $H_P^1(\Gamma_1,\widetilde{V}_w^{\Gamma,\chi})$ does not change upon replacing this module with its subspace $V_w^{\Gamma,\chi}$ on which -1 acts trivially:

$$V_w^{\Gamma,\chi}:=\left\{P:\Gamma_1\to V_w\;\middle|\;\begin{matrix}P(-A)=(-1)^wP(A)\\ P(\gamma A)=\chi(\gamma)P(A)\end{matrix}\;,\ \text{for }\gamma\in\Gamma,A\in\Gamma_1\right\}.$$

We make the following assumption on the double coset Σ.

Assumption 3.3. *The map*

$$\Gamma\backslash\Sigma\longrightarrow\Gamma_1\backslash\Gamma_1\Sigma,\quad \Gamma\sigma\mapsto\Gamma_1\sigma$$

is bijective, or equivalently $|\Gamma\backslash\Sigma|=|\Gamma_1\backslash\Gamma_1\Sigma|$.

Remark 3.4. This assumption is satisfied by the double cosets giving the usual Hecke and Atkin–Lehner operators for the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$; see [9, Lemma 4.3]. It is related to the notion of compatible Hecke pairs in [1].

We now assume without loss of generality that $\Sigma \subset \mathcal{M}_n$ for some $n \geqslant 1$. Under Assumption 3.3, we define an action of elements $M \in \mathcal{M}_n$ on $V_w^{\Gamma,\chi}$, which for $\chi = \mathbf{1}$ is the same as in [8, Sec. 5]. For $A \in \Gamma_1$ with $MA^{-1} \in \Gamma_1\Sigma$, let $A_M \in \Gamma_1$, $M_A \in \Sigma$ such that $MA^{-1} = A_M^{-1}M_A$, and define:

$$P|_\Sigma M(A) = \begin{cases} \widetilde{\chi}(M_A)P(A_M)|_{-w}M & \text{if } MA^{-1} \in \Gamma_1\Sigma, \\ 0 & \text{if } MA^{-1} \notin \Gamma_1\Sigma\ . \end{cases} \tag{3.2}$$

By Assumption 3.3 and (1.4), the definition does not depend on the decomposition $MA^{-1} = A_M^{-1}M_A$. This action is compatible with the action of Γ_1 as in (2.1).

Formula (3.2) defines an action of the Γ_1-double coset $\Gamma_1\Sigma\Gamma_1$ on $V_w^{\Gamma,\chi}$, which is compatible with the action of Γ_1 as in (2.1).

Theorem 3.5 (Shapiro isomorphism)**.** *For $\Sigma \subset \mathcal{M}_n$ a double coset satisfying Assumption* 3.3*, we have a Hecke-equivariant isomorphism*

$$H_P^1(\Gamma_1, V_w^{\Gamma,\chi}) \simeq H_P^1(\Gamma, V_w^\chi)\ , \tag{3.3}$$

with $[\Sigma]$ *acting on the right side by* (2.2) *and with* $[\Gamma_1\Sigma\Gamma_1]$ *acting on the left side by* (3.2)*.*

Proof. The isomorphism is given on cocycles by $[\varphi] \mapsto [\varphi']$, where $\varphi'(\gamma) = \varphi(\gamma)(1)$, and the Hecke-equivariance is easily verified. See also [1, Lemma 1.1.4]. □

3.3. **Trace on period polynomials.** For $\mathcal{V} = \mathcal{V}_w^{\Gamma,\chi}$, we denote by $W_w^{\Gamma,\chi}$ the period subspace defined in Section 2. Let $\Sigma_1 := \Gamma_1\Sigma\Gamma_1$, and assume that $\Sigma \subset \mathcal{M}_n$. An operator $\widetilde{T}_{\Sigma_1}$ satisfying (A) acts on $W_w^{\Gamma,\chi}$ via (3.2), and its action is the same as that of the "universal operator" $\widetilde{T}_n$ that satisfies (A) for the double coset $\mathcal{M}_n$, since matrices in $\mathcal{M}_n \smallsetminus \Sigma_1$ act trivially in (3.2). To emphasize that its action depends on the coset Σ, we denote by $\mathrm{Tr}(X|_\Sigma\widetilde{T}_n)$ the trace of $\widetilde{T}_n$ (that is, of $\widetilde{T}_{\Sigma_1}$) on any subspace $X \subset \mathcal{W}_w^{\Gamma,\chi}$ preserved by it.

Proposition 3.6. *Let $\Gamma \subset \Gamma_1$ be a finite index subgroup, let $k = w+2 \geqslant 2$ be an integer, let χ be a character of Γ with kernel of finite index in Γ, and let $\Sigma \subset \mathcal{M}_n$ be a double coset satisfying Assumption* 3.3*. For any $\widetilde{T}_n \in \mathcal{R}_n$ satisfying* (A)*, we have*

$$\mathrm{Tr}(W_w^{\Gamma,\chi}|_\Sigma\widetilde{T}_n) = \mathrm{Tr}([\Sigma], M_k(\Gamma,\chi) + S_k^c(\Gamma,\chi))\ .$$

Proof. By (2.5) and the Eichler–Shimura isomorphism combined with the Shapiro lemma we have

$$\mathrm{Tr}(W_w^{\Gamma,\chi}|_\Sigma\widetilde{T}_n) = \mathrm{Tr}([\Sigma], S_{w+2}(\Gamma,\chi) + S_{w+2}^c(\Gamma,\chi)) + \mathrm{Tr}(C_w^{\Gamma,\chi}|_\Sigma\widetilde{T}_n)\ , \tag{3.4}$$

where $C_w^{\Gamma,\chi}$ is the coboundary subspace defined by (2.3). Therefore it is enough to show that $\mathrm{Tr}(C_w^{\Gamma,\chi}|_\Sigma\widetilde{T}_n) = \mathrm{Tr}([\Sigma], E_k(\Gamma,\chi))$, where $E_k(\Gamma,\chi) \subset M_k(\Gamma,\chi)$ is the Eisenstein subspace. This can be shown by explicitly computing the left side, using Remark 2.4, and by comparing the result with the formula for the Eisenstein trace, which is easily computed. For brevity we omit this computation and refer to our arXiv preprint 1408.4998v2 for the details.

A more conceptual proof is provided by the theory of modular symbols of Ash and Stevens. The space $W_w^{\Gamma,\chi}$ is isomorphic with the space of modular symbols $\mathrm{Symb}_\Gamma(V_w^\chi)$ defined in [1, Sec. 4], and the isomorphism is compatible with the action of Hecke operators on both sides. By [1, Prop. 4.2], we have a Hecke equivariant isomorphism between $\mathrm{Symb}_\Gamma(V_w^\chi)$ and the cohomology with compact support $H^1_c(X_\Gamma, \widetilde{V}^\chi_w)$ of the local system $\widetilde{V}^\chi_w$ associated to V^χ_w on the modular surface $X_\Gamma = \Gamma\backslash\mathcal{H}$. Therefore we have a Hecke-equivariant isomorphism $W^{\Gamma,\chi}_w \simeq H^1_c(X_\Gamma, \widetilde{V}^\chi_w)$, and since the latter space is Hecke isomorphic with $M_k(\Gamma,\chi)+S^c_k(\Gamma,\chi)$ by a version of the Eichler–Shimura isomorphism, the conclusion follows. $\square$

4. Proof of Theorem 1

First we show that the second version of Theorem 1 is equivalent to the first. For Γ a finite index subgroup of Γ_1, let $[M]_\Gamma$ denote the Γ-conjugacy class in $\mathrm{PGL}_2^+(\mathbb{R})$ of the projection of a matrix $M \in \mathrm{GL}_2^+(\mathbb{R})$. For fixed $M \in \Sigma$, the subsum in (1.9) over Γ-conjugacy classes $X \subset [M]_{\Gamma_1}$ equals

(4.1)
$$\sum_{\substack{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1\\ \pm AMA^{-1}\in\Sigma}} p_{k-2}(\operatorname{Tr} M, \det M)(\pm1)^k\widetilde{\chi}(\pm AMA^{-1})\frac{\varepsilon_\Gamma([AMA^{-1}]_\Gamma)}{[\operatorname{Stab}_{\overline{\Gamma}_1} M : \operatorname{Stab}_{\overline{A^{-1}\Gamma A}} M]}\ ,$$

where the same sign is chosen in all three places, and if $-1 \notin \Gamma$ at most one choice of signs is possible for each A.[3] Indeed, the Γ-conjugacy classes contained in $[M]_{\Gamma_1}$ are $[AMA^{-1}]_\Gamma$, with A running through a set of representatives for $\overline{\Gamma}\backslash\overline{\Gamma}_1$ such that $\pm AMA^{-1} \in \Sigma$; for fixed such A and varying $h \in \operatorname{Stab}_{\Gamma_1} M$, elements in the cosets ΓAh give the same conjugacy class $[AMA^{-1}]_\Gamma$, and the number of such distinct cosets is easily seen to equal the index in the denominator above. When $\operatorname{Stab}_{\overline{\Gamma}_1} M$ is finite, the fraction in the sum above equals $\varepsilon([M]_{\Gamma_1})$, since $|\operatorname{Stab}_{\overline{A^{-1}\Gamma A}} M| = |\operatorname{Stab}_{\overline{\Gamma}} AMA^{-1}|$. Therefore formula (1.9) implies (1.11), and since the reasoning above is reversible, the two trace formulas are equivalent.

We now apply Theorem 2.6 to the Γ_1-module $V^{\Gamma,\chi}_w$ to prove the second version of Theorem 1. The module $V^{\Gamma,\chi}_w$ admits a Γ_1-equivariant pairing, given by

$$\langle\!\langle P, Q \rangle\!\rangle := \frac{1}{[\Gamma_1 : \Gamma]} \sum_{A\in\Gamma\backslash\Gamma_1} \langle\, P(A), \overline{Q(A)}\,\rangle\,,$$

where $\langle\, (ax+b)^w, (cx+d)^w\,\rangle = (ad-bc)^w$ is the well-known $\mathrm{SL}_2(\mathbb{R})$-invariant pairing on V_w; e.g., [1, Lemma 3.1]. It is clear that the definition is independent of the system of representatives chosen in the summation, and this pairing is nondegenerate and Γ_1-invariant [8, Sec. 3], so the hypothesis of Theorem 2.6 is satisfied.

The space of Γ_1-invariants $(V^{\Gamma,\chi}_w)^{\Gamma_1}$ is trivial if $(w,\chi) \neq (0,\mathbf{1})$, and if $w = 0$ and $\chi = \mathbf{1}$, it is one dimensional spanned by the constant polynomial P_0, with $P_0(A) = 1$ for $A \in \Gamma_1$. In the latter case we have

$$\operatorname{Tr}((V^{\Gamma,\mathbf{1}}_0)^{\Gamma_1}|_\Sigma T^\infty_n) = P_0|_\Sigma T^\infty_\Sigma(I) = \sum_{M\in\Gamma_1\backslash\Gamma_1\Sigma} \widetilde{\chi}(M_I), \tag{4.2}$$

where M runs through a system of representatives for $\Gamma_1\backslash\Gamma_1\Sigma$ and $M_I \in \Sigma$ is any element such that $M \in \Gamma_1 M_I$. By Assumption 3.3, M_I runs over a system of representatives for $\Gamma\backslash\Sigma$, so the last sum equals $\sum_{\sigma\in\Gamma\backslash\Sigma}\widetilde{\chi}(\sigma)$.

By Theorem 2.6 and Proposition 3.6, the second version of Theorem 1 is proved once we compute below the trace of $M \in \mathcal{M}_n$ on the module $V^{\Gamma,\chi}_w$. $\square$

[3]The assumption on Σ implies that if $-1 \notin \Gamma$, then $\Sigma \cap -\Sigma = \emptyset$. See the remark following Lemma 4.1.

Lemma 4.1. *Assume that $\chi(-1)=(-1)^k$ if $-1\in\Gamma$. For any $M\in\mathcal{M}_n$ we have*

$$\operatorname{Tr}(V_w^{\Gamma,\chi}|_\Sigma M)=p_w(\operatorname{Tr}M,\det M)\cdot\mathcal{C}^\chi_{\Gamma,\Sigma}(M)\ ,$$

where

$$\mathcal{C}^\chi_{\Gamma,\Sigma}(M)=\sum_{\substack{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1\\ \pm AMA^{-1}\in\Sigma}}(\pm1)^w\widetilde{\chi}(\pm AMA^{-1})\ .$$

Remark 4.2. If $-1\in\Gamma$ the signs can be chosen arbitrarily. If $-1\notin\Gamma$, Assumption 3.3 implies that M and $-M$ cannot both belong to Σ, so in each term at most one choice of signs is possible.

Proof. Let C_Γ be a system of representatives for $\Gamma\backslash\Gamma_1/\{\pm1\}$. We have a decomposition

$$V_w^{\Gamma,\chi}=\bigoplus_{A\in C_\Gamma}V_w^{(A)}\ ,$$

where $V_w^{(A)}\simeq V_w$ is the space of $P\in V_w^{\Gamma,\chi}$ with $P(B)=0$ if $A\neq B\in C_\Gamma$. If $MA^{-1}\notin\Gamma_1\Sigma$, then M maps $V_w^{(A)}$ into $\bigoplus_{B\neq A}V_w^{(B)}$; if $MA^{-1}\in\Gamma_1\Sigma$, there are unique $A_M\in C_\Gamma$, $M_A\in\Sigma$ such that $MA^{-1}=\pm A_M^{-1}M_A$ (the sign can be assumed $+1$ if $-1\in\Gamma$), and

$$P|_\Sigma M(A)=\widetilde{\chi}(M_A)(\pm1)^wP(A_M)|_{-w}M.$$

It follows that the space $V_w^{(A)}$ contributes to the trace only if $A_M=A$, that is, $\pm AMA^{-1}\in\Sigma$, and its contribution is $(\pm1)^w\widetilde{\chi}(\pm AMA^{-1})\operatorname{Tr}(V_w|_{-w}M)$. To conclude the proof, we leave as an exercise to the reader to check using (1.8) that

$$\operatorname{Tr}(V_w|_{-w}M)=p_w(\operatorname{Tr}M,\det M),$$

as both sides equal $\sum_{n=0}^w\alpha^n\beta^{w-n}$, where α,β are the eigenvalues of M. $\square$

5. Proof of Theorem 3

Since $(\ell,n)=1$ we easily check that $\Sigma=\Gamma_n\sigma\Gamma_n$ satisfies Assumption 3.3, and

$$|\Gamma_n\backslash\Sigma|=|\Gamma_0(\ell)\backslash\Gamma_1|=\varphi_1(\ell).$$

In the trace formula (1.11), the conjugacy classes $X\subset\overline{\Gamma_1\sigma\Gamma_1}$ with $\varepsilon(X)\neq0$ have representatives $M_X\in M_\ell$ such that $|\operatorname{Tr}M_X|\leqslant l+1$. The condition $\pm AM_XA^{-1}\in\Gamma_n\sigma\Gamma_n$ implies that $\pm\operatorname{Tr}M_X\equiv\operatorname{Tr}\sigma=\ell+1\pmod n$, and from $n>2\ell+2$ we conclude that the conjugacy classes that contribute to the formula have a representative with $\operatorname{Tr}M_X=\operatorname{Tr}\sigma=l+1$. Trace formula (1.11) becomes

$$\operatorname{Tr}([\Gamma_n\sigma\Gamma_n],M_k(\Gamma_n)+S_k(\Gamma_n))=p_{k-2}(\ell+1,\ell)\cdot\sum_{\substack{X\subset M_\ell\\ \operatorname{Tr}M_X=\ell+1}}C_{n,\sigma}(M_X)+\delta_{k,2}\varphi_1(\ell),$$

where $C_{n,\sigma}(M_X)=\#\{A\in\overline{\Gamma}_n\backslash\overline{\Gamma}_1:AM_XA^{-1}\equiv\sigma\pmod n\}$. Here we used the fact that $(\ell,n)=1$ to conclude $\Gamma_n\sigma\Gamma_n=\{M\in M_\ell:M\equiv\sigma\pmod n\}$ [13, Lemma 3.29].

As representatives for Γ_1-conjugacy classes in M_ℓ of trace $\ell+1$ we take the matrices $\left(\begin{smallmatrix}\ell&b\\0&1\end{smallmatrix}\right)$ with $0\leqslant b<\ell-1$, so we can assume $M_X=\left(\begin{smallmatrix}\ell&b\\0&1\end{smallmatrix}\right)$. The equation $AM_X\equiv\sigma A\pmod n$ for $A=\left(\begin{smallmatrix}x&y\\z&t\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ is equivalent to

$$bz\equiv0,\quad(\ell-1)z\equiv0,\quad(\ell-1)y\equiv-bt\pmod n.$$

Write $n = dn'$, $\ell - 1 = ds$ with $(n', s) = 1$. From $d|bz$, $d|bt$ it follows that $d|b$, so

$$z \equiv 0 \pmod{n'}, \quad sy \equiv -tb/d \pmod{n'}.$$

It follows that $xt \equiv 1 \pmod{n'}$, and for every such choice of x, t, there is a unique solution $A \in \mathrm{SL}_2(\mathbb{Z}/n'\mathbb{Z})$. Each such solution has $\#\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})/\#\mathrm{SL}_2(\mathbb{Z}/n'\mathbb{Z})$ lifts to $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, and since $\#\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\} = [\overline{\Gamma}_1 : \overline{\Gamma}_n] = \varphi_2(n)$ we obtain

$$C_{n,\sigma}(M_X) = \frac{\varphi(n')}{2} \cdot \frac{\varphi_2(n)}{\varphi_2(n')},$$

independent of b. Since $p_{k-2}(\ell+1, \ell) = \frac{\ell^{k-1}-1}{\ell-1}$ and there are $\ell - 1$ conjugacy classes in the sum over X above, the claim follows.

Acknowledgments

I am grateful to Don Zagier for introducing me to the period polynomial approach for proving the trace formula for $\mathrm{SL}_2(\mathbb{Z})$, and for generously sharing his insights. I also thank Vicenţiu Paşol for many illuminating discussions on the subject of period polynomials and modular symbols.

References

[1] Avner Ash and Glenn Stevens, *Modular forms in characteristic l and special values of their L-functions*, Duke Math. J. **53** (1986), no. 3, 849–868, DOI 10.1215/S0012-7094-86-05346-9. MR860675

[2] Youngju Choie and Don Zagier, *Rational period functions for* $\mathrm{PSL}(2, \mathbf{Z})$, A tribute to Emil Grosswald: number theory and related analysis, Contemp. Math., vol. 143, Amer. Math. Soc., Providence, RI, 1993, pp. 89–108, DOI 10.1090/conm/143/00992. MR1210514

[3] Sharon Frechette, Ken Ono, and Matthew Papanikolas, *Combinatorics of traces of Hecke operators*, Proc. Natl. Acad. Sci. USA **101** (2004), no. 49, 17016–17020, DOI 10.1073/pnas.0407223101. MR2114776

[4] Klaus Haberland, *Perioden von Modulformen einer Variabler and Gruppencohomologie. I, II, III* (German), Math. Nachr. **112** (1983), 245–282, 283–295, 297–315, DOI 10.1002/mana.19831120113. MR726861

[5] Kazuyuki Hatada, *On the rationality of periods of primitive forms*, Acta Arith. **40** (1981/82), no. 1, 9–40. MR654020

[6] Adriaan Herremans, *A combinatorial interpretation of Serre's conjecture on modular Galois representations* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **53** (2003), no. 5, 1287–1321. MR2032935

[7] Loïc Merel, *Universal Fourier expansions of modular forms*, On Artin's conjecture for odd 2-dimensional representations, Lecture Notes in Math., vol. 1585, Springer, Berlin, 1994, pp. 59–94, DOI 10.1007/BFb0074110. MR1322319

[8] Vicenţiu Paşol and Alexandru A. Popa, *Modular forms and period polynomials*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 4, 713–743, DOI 10.1112/plms/pdt003. MR3108829

[9] Alexandru A. Popa, *On the trace formula for Hecke operators on congruence subgroups, II.* Preprint, arXiv:1706.02691

[10] Alexandru A. Popa and Don Zagier, *An elementary proof of the Eichler-Selberg trace formula.* Preprint, arXiv:1711.00327

[11] Florin Rădulescu, *Endomorphisms of spaces of virtual vectors fixed by a discrete group*, Russian Math. Surveys **71** (2016), 291–343.

[12] Jean-Pierre Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke* T_p (French), J. Amer. Math. Soc. **10** (1997), no. 1, 75–102, DOI 10.1090/S0894-0347-97-00220-8. MR1396897

[13] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Kanô Memorial Lectures, No. 1, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. MR0314766

[14] Don Zagier, *Hecke operators and periods of modular forms*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 3, Weizmann, Jerusalem, 1990, pp. 321–336. MR1159123

[15] Don Zagier, *Periods of modular forms, traces of Hecke operators, and multiple zeta values*, Sūrikaisekikenkyūsho Kōkyūroku **843** (1993), 162–170. Research into automorphic forms and L functions (Japanese) (Kyoto, 1992). MR1296720

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

Email address: aapopa@gmail.com