PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 3153–3166
http://dx.doi.org/10.1090/proc/13907
Article electronically published on March 20, 2018

A q-SERIES IDENTITY VIA THE $\mathfrak{sl}_3$ COLORED JONES POLYNOMIALS FOR THE $(2,2m)$-TORUS LINK

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(Communicated by David Futer)

Abstract. The colored Jones polynomial is a q-polynomial invariant of links colored by irreducible representations of a simple Lie algebra. A q-series called a tail is obtained as the limit of the $\mathfrak{sl}_2$ colored Jones polynomials $\{J_n(K;q)\}_n$ for some link K, for example, an alternating link. For the $\mathfrak{sl}_3$ colored Jones polynomials, the existence of a tail is unknown. We give two explicit formulas of the tail of the $\mathfrak{sl}_3$ colored Jones polynomials colored by $(n,0)$ for the $(2,2m)$-torus link. These two expressions of the tail provide an identity of q-series. This is a knot-theoretical generalization of the Andrews–Gordon identities for the Ramanujan false theta function.

1. Introduction

The Rogers–Ramanujan identities were first discovered and proved by Rogers [28]. After that, these identities appeared in Ramanujan's first letter to Hardy without proof. To this day, the Rogers–Ramanujan identities have been generalized by many people and have been proved in many ways, for example, partitions of integers [3], Bailey pairs, and the Bailey chain method [23, 30, 31]. One of these generalizations is the Andrews–Gordon identities.

Theorem 1.1 (The Andrews–Gordon identities for the Ramanujan theta function [1])**.**

$$f(-q^{2m},-q)=(q;q)_\infty \sum_{k_m\le\cdots\le k_2\le k_1}\frac{q^{\sum_{j=1}^{m-1}k_j(k_j+1)}}{\prod_{j=1}^{m-1}(q;q)_{k_j-k_{j+1}}},$$

where $m>0$ and $f(a,b)=\sum_{i=0}^{\infty}a^{\frac{i(i+1)}{2}}b^{\frac{i(i-1)}{2}}+\sum_{i=1}^{\infty}a^{\frac{i(i-1)}{2}}b^{\frac{i(i+1)}{2}}$ is Ramanujan's general theta function (see, for example, [2]).

In knot theory, the Andrews–Gordon identities appear through the ($\mathfrak{sl}_2$) colored Jones polynomial. The colored Jones polynomial is a q-polynomial invariant of knots and links. Dasbach and Lin showed a stability of coefficients of the colored Jones polynomials for alternating knots in [9, 10]. They conjectured that the kth coefficient of the $n+1$-dimensional colored Jones polynomial $J_n(K;q)$ for an alternating knot K coincides up to sign for $n\ge k$. This suggests the existence of q-series $\sum_{k=0}^{\infty}a_kq^k$ for an alternating link K called the *tail*. The coefficient a_k is given by the kth coefficient of $J_n(K;q)$ with $n\ge k$. Armond [4] proved the existence of tails

Received by the editors December 21, 2016 and, in revised form, July 16, 2017, and July 24, 2017.
2010 *Mathematics Subject Classification.* Primary 57M27; Secondary 11P84, 05A30.

for adequate links, in particular for alternating knots. Independently, for alternating knots, Garoufalidis and Lê [13] showed a more general stability of coefficients of $J_n(K;q)$.

In [6], Armond and Dasbach showed two explicit formulas for the tails of the colored Jones polynomial for the $(2,2m+1)$-torus knot. These two formulas conclude the left- and right-hand side of the Andrews–Gordon identities for the Ramanujan theta function. They used the formula for the colored Jones polynomial of the $(2,2m+1)$-torus knot in [25] and obtained by the method of Armond [5]. Hajij [16] also showed the Andrews–Gordon identities for the Ramanujan theta function and the Ramanujan false theta function using two expressions of the tail of the $(2,2m+1)$-torus knot and the $(2,2m)$-torus link, respectively. On the other hand, Bringmann and Milas showed the identity for Ramanujan false theta function from representation theory of the vertex operator algebra in Sect. 7 of [8].

Theorem 1.2 (The Andrews–Gordon identities for the Ramanujan false theta function [8, 16])**.**

$$\Psi(q^{2m-1},q)=(q;q)_\infty \sum_{k_{m-1}\leq\cdots\leq k_2\leq k_1}\frac{q^{\sum_{j=1}^{m-1}k_j(k_j+1)}}{(q;q)_{k_{m-1}}^2\prod_{j=1}^{m-2}(q;q)_{k_j-k_{j+1}}},$$

where $m>1$ and $\Psi(a,b)=\sum_{i=0}^{\infty}a^{\frac{i(i+1)}{2}}b^{\frac{i(i-1)}{2}}-\sum_{i=0}^{\infty}a^{\frac{i(i-1)}{2}}b^{\frac{i(i+1)}{2}}$ is Ramanujan's general false theta function (see, for example, [24]*).*

He used formulas for the colored Jones polynomial obtained by using a colored trivalent graph presentation of the Kauffman bracket skein elements (see, for example, [18]) and the Kauffman bracket bubble skein expansion formula in [15]. The Rogers–Ramanujan type identities for q-series are derived from other knots. For example, Elhamdadi and Hajij derived the q-series identities from Pretzel knots in [12] and singular knots in [11]. Keilthy and Osburn expressed tails of many knots as theta functions in [19]. These identities were conjectured in [13].

The main subject of this paper is the tail of the colored $\mathfrak{sl}_3$ Jones polynomials for the $(2,2m)$-torus link. We will give two explicit formulas of "the $\mathfrak{sl}_3$ tail" for the $(2,2m)$-torus link by using the A_2 bracket relations and colored trivalent graph presentations of some A_2 bracket skein elements. As the result, the following q-series identity holds.

Theorem 4.3.

$$\sum_{i=0}^{\infty}q^{-2i}q^{m(i^2+2i)}\frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q}=(q)_\infty\sum_{0\leq k_m\leq\cdots\leq k_2\leq k_1}\frac{q^{-2k_m}q^{\sum_{j=1}^{m}(k_i^2+2k_i)}}{(q)_{k_m}^2(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}}.$$

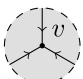
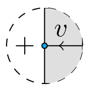
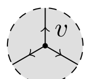
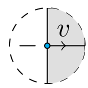
The right-hand side of the above identity is obtained from the full twist formula in [33]. The left-hand side is obtained by using of colored trivalent graph presentation of A_2 webs. We remark that the above theorem also claims the existence of the tail for the $\mathfrak{sl}_3$ colored Jones polynomials for the $(2,2m)$-torus link. Independently, Garoufalidis and Vuong formulated a conjecture about a stability of the coefficients of the $\mathfrak{g}$ colored Jones polynomials of a knot for general simple Lie algebra $\mathfrak{g}$ in [14]. They also proved the stability conjecture for all torus knots and all simple Lie algebras by using the Jones–Rosso formula [29]. For $\mathfrak{g}=\mathfrak{sl}_3$, they gave an explicit formula of the tail of the $(2,2m+1)$-torus knot. The paper is organized as follows. We first review definitions and formulas related to the A_2 web space in section 2. The A_2 web space is a generalization of the Kauffman bracket skein module.

In section 3, we introduce a method to represent some types of A_2 webs using colored trivalent graphs. Furthermore, we give values of some θ-graphs and quantum $6j$ symbols. As an application, we explicitly give the $\mathfrak{sl}_3$ colored Jones polynomial for a 2-bridge link. In section 4, we derive two q-series, that is "the $\mathfrak{sl}_3$ tails", by using explicit formulas for the $\mathfrak{sl}_3$ colored Jones polynomial for the $(2, 2m)$-torus link in section 3 and in [33].

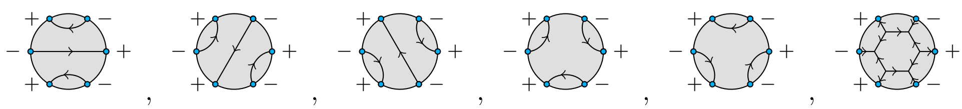
2. The A_2 web space and some formulas

The skein theory has been developed with a quantum representation of Lie algebra $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}_2$, the skein theory consists of the Kauffman bracket skein module, which is called the Temperley–Lieb algebra and the A_1 web space, and the Kauffman bracket. Kuperberg constructed the skein theory for Lie algebras of rank 2 in [21,22]. In this section, we review definitions of the A_2 web space, the A_2 bracket, and the A_2 clasp.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be an m-tuple of signs $+$ or $-$. Let D_ε denote the unit disk with signed marked points $\{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$ on its boundary. The sign of $\exp(2\pi\sqrt{-1}/m)^{j-1}$ is given by ε_j for $j = 1, 2, \dots, m$. A *bipartite uni-trivalent graph* G is a directed graph such that each vertex is either trivalent or univalent and the vertices are divided into the sinks and the sources. A sink (resp., source) is a vertex such that all edges adjoining to the vertex point into (resp., away from) it. A *bipartite trivalent graph* G in D_ε is an embedding of a uni-trivalent graph into D_ε such that any vertex v has the following neighborhoods:

 or  if v is a sink,  or  if v is a source.

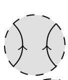
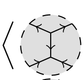
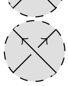
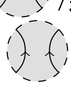
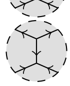
An A_2 *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph G in D_ε, where any internal face of $D_\varepsilon \setminus G$ has at least six sides. Let us denote B_ε as the set of A_2 basis webs in D_ε. For example, $B_{(+,-,+,-,+,-)}$ has the following A_2 basis webs:

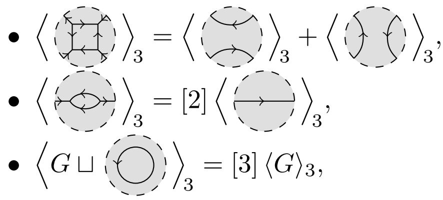


The A_2 *web space* W_ε is the $\mathbb{Q}(q^{\frac{1}{6}})$-vector space spanned by B_ε. A *tangled trivalent graph diagram* in D_ε is an immersed bipartite uni-trivalent graph in D_ε whose intersection points are only transverse double points of edges with crossing data  or . Tangled trivalent graph diagrams G and G' are regularly isotopic if G is obtained from G' by a finite sequence of boundary-fixing isotopies and Reidemeister moves (see Figure 1) with some direction of edges.

Tangled trivalent graphs in D_ε are regular isotopy classes of tangled trivalent graph diagrams in D_ε. We denote T_ε as the set of tangled trivalent graphs in D_ε.

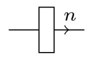
Definition 2.1 (The A_2 bracket [22])**.** We define a $\mathbb{Q}(q^{\frac{1}{6}})$-linear map $\langle \cdot \rangle_3 \colon \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \to W_\varepsilon$ by the following:

- $\Big\langle$  $\Big\rangle_3 = q^{\frac{1}{3}} \Big\langle$  $\Big\rangle_3 - q^{-\frac{1}{6}} \Big\langle$  $\Big\rangle_3$,
- $\Big\langle$  $\Big\rangle_3 = q^{-\frac{1}{3}} \Big\langle$  $\Big\rangle_3 - q^{\frac{1}{6}} \Big\langle$  $\Big\rangle_3$,

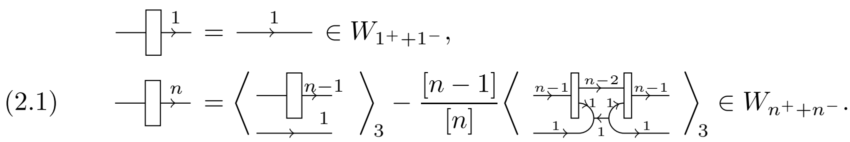


where $[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ is a quantum integer.

We remark that this map is invariant under the Reidemeister moves for tangled trivalent graphs.

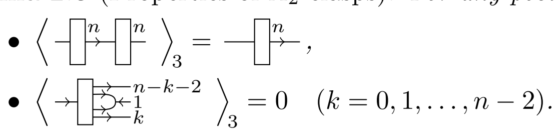
We next consider the A_2 web space $W_{n^+ + n^-} = W_{(+,+,\dots,+,-,-,\dots,-)}$ whose first n marked points are decorated with $+$ and where the next n marked points are decorated with $-$. We define A_2 clasps  $\in W_{n^+ + n^-}$ inductively by the following.

Definition 2.2 (The A_2 clasps)**.**



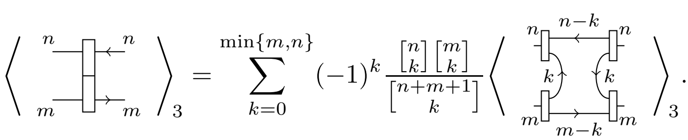
A_2 clasps have the following properties.

Lemma 2.3 (Properties of A_2 clasps)**.** *For any positive integer n,*



We also define the A_2 clasp of type (n, m) according to Ohtsuki and Yamada [26].

Definition 2.4 (The A_2 clasp of type (n, m))**.**



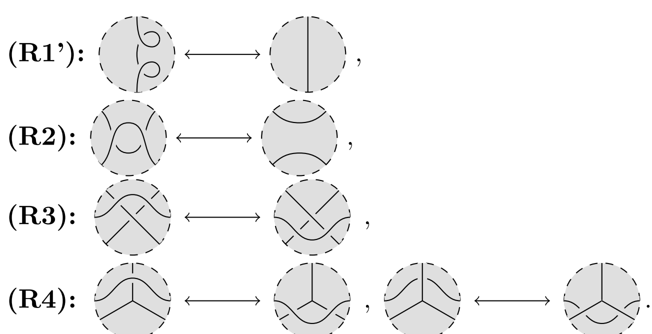
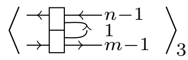


FIGURE 1. The Reidemeister moves for tangled trivalent graph diagrams

Lemma 2.5 (Property of A_2 clasps of type (m, n))**.**



$$= 0.$$

We review some formulas for clasped A_2 web spaces in [33]. We define a q-Pochhammer symbol as

$$(q;q)_k = \prod_{l=1}^{k}(1-q^l).$$

We abbreviate it as $(q)_k$. A q-binomial coefficient is defined by

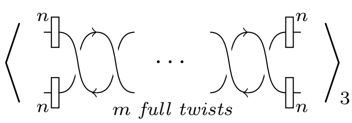
$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

for $k \le n$. If $k > n$, we define it by 0. We also define a q-multinomial coefficient as

$$\binom{n}{n_1, n_2, \dots, n_m}_q = \frac{(q)_n}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_m}},$$

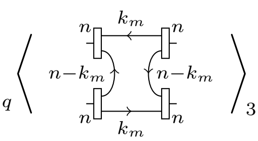
where $n_1, n_2, \dots, n_m$ are nonnegative integers such that $n_1 + n_2 + \cdots + n_m = n$.

Theorem 2.6 (The m full twists formula [33])**.**



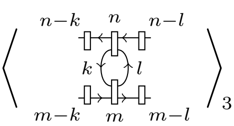
$$= q^{-\frac{2m}{3}(n^2+3n)} \sum_{0\le k_m \le \cdots \le k_1 \le n} q^{n-k_m} q^{\sum_{i=1}^{m}(k_i^2+2k_i)}$$

$$\times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q$$

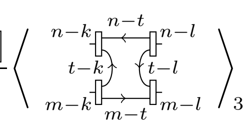


where k_i, k'_i are integers such that $k_0 = n$, $k'_{i+1} = k_i - k_{i+1}$ for $i = 0, 1, \dots, m-1$.

Theorem 2.7 (The A_2 bracket bubble skein expansion formula [33])**.**



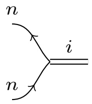
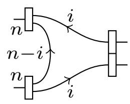
$$= \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \frac{\begin{bmatrix} n \\ t \end{bmatrix}\begin{bmatrix} m \\ t \end{bmatrix}\begin{bmatrix} t \\ k \end{bmatrix}\begin{bmatrix} t \\ l \end{bmatrix}\begin{bmatrix} n+m-t+2 \\ n+m-k-l+2 \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix}\begin{bmatrix} m \\ l \end{bmatrix}}$$



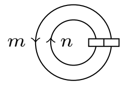
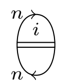
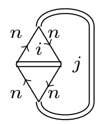
where $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}$ for $a \ge b$ and $[a]! = \prod_{i=1}^{a}[i]$.

3. Trivalent graphs for clasped A_2 web spaces

For the Kauffman bracket skein module, a method for representing its elements by colored trivalent graphs is well known. Formulas related to colored trivalent graphs play a very important role in defining the quantum SU(2) invariants for 3-manifolds and the quantum representations for mapping class groups of surfaces, etc. (See, for example, Kauffman and Lins [18], Turaev and Viro [32] and Roberts [27].) In this section, we represent a certain clasped A_2 web space by using colored trivalent graphs and give some formulas. In general, the diagrammatic expression of a trivalent vertex for A_2 is given by Kim [20].

Definition 3.1. Let n be a nonnegative integer. For $0 \le i \le n$, we define a colored trivalent graph  by  .

We use the following notation:

- $\Delta(m,n) = \Big\langle$  $\Big\rangle_3$,
- $\theta(n,n,(i,i)) = \Big\langle$  $\Big\rangle_3$,
- $\mathrm{Tet}\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} = \Big\langle$  $\Big\rangle_3$,
- $\begin{Bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{Bmatrix} = \dfrac{\mathrm{Tet}\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} \Delta(j,j)}{\theta(n,n,(j,j))^2}$,

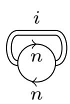
where m,n are any nonnegative integers and $0 \le i,j \le n$.

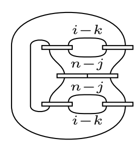
Lemma 3.2.

(1) $\Delta(i,j) = \frac{[i+1][j+1][i+j+2]}{[2]}$,

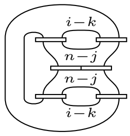
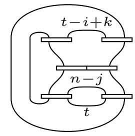
(2) $\theta(n,n,(i,i)) = \sum_{k=0}^{i} (-1)^k \frac{\begin{bmatrix} i \\ k \end{bmatrix}^2}{\begin{bmatrix} 2i+1 \\ k \end{bmatrix}} \frac{\Delta(n,0)^2}{\Delta(n-i+k,0)} = \frac{\begin{bmatrix} n+i+2 \\ 2i+2 \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}^2} \Delta(i,i)$,

(3) $\mathrm{Tet}\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} = \sum_{k=\max\{0,i+j-n\}}^{i} (-1)^k \frac{\begin{bmatrix} i \\ k \end{bmatrix}^2 \begin{bmatrix} n-j \\ i-k \end{bmatrix} \begin{bmatrix} n+j+2 \\ i-k \end{bmatrix}}{\begin{bmatrix} 2i+1 \\ k \end{bmatrix} \begin{bmatrix} n \\ i-k \end{bmatrix}^2} \theta(n,n,(j,j))$.

Proof. We only show (2) and (3). The first equation of (2) is obtained by expanding the double-lined edge using Definition 2.4. $\theta(n,n,(i,i))$ is also represented by a colored graph $\Big\langle$  $\Big\rangle_3$. This clasped A_2 web has a bubble skein element colored by $n-i$ and we use Theorem 2.7. Thus, we obtain the second equation of (2). The proof of (3) is obtained by using Definition 2.4 and Theorem 2.7. We describe $\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix}$ as an A_2 web and expand the A_2 clasp of type (i,i) by definition. Then, we obtain the following:

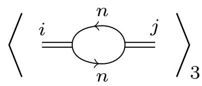
$$\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} = \sum_{k=0}^{i} (-1)^k \frac{\begin{bmatrix} i \\ k \end{bmatrix}^2}{\begin{bmatrix} 2i+1 \\ k \end{bmatrix}} \Big\langle$$  $$\Big\rangle_3.$$

Theorem 2.7 can be applied the upper bubble skein:

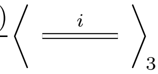
$$\Big\langle$$  $$\Big\rangle_3 = \sum_{t=\max\{i-k,n-j\}}^{\min\{n,n+i-k-j\}} \frac{\begin{bmatrix} n \\ t \end{bmatrix}^2 \begin{bmatrix} t \\ i-k \end{bmatrix} \begin{bmatrix} t \\ n-j \end{bmatrix} \begin{bmatrix} 2n-t+2 \\ n-i+k+j+2 \end{bmatrix}}{\begin{bmatrix} n \\ i-k \end{bmatrix}^2 \begin{bmatrix} n \\ n-j \end{bmatrix}^2} \Big\langle$$  $$\Big\rangle_3.$$

By Lemma 2.5, the above A_2 web vanishes if $i-k>n-j$. For $i-k\geq n-j$, the summand of $t=n-j$ only survives and the value is $\frac{\begin{bmatrix} n-j \\ i-k \end{bmatrix}\begin{bmatrix} n+j+2 \\ i-k \end{bmatrix}}{\begin{bmatrix} n \\ i-k \end{bmatrix}^2}\theta(n,n,(j,j))$. The summation index k runs from 0 to i, that is, $0\leq i-k\leq i$. If $i<n-j$, then summands from 0 to i remain. If $n-j<i$, then summands from $i+j-n$ to i remain. □

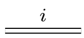
Lemma 3.3.

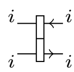


$$= \delta_{ij}\frac{\theta(n,n,(i,i))}{\Delta(i,i)}$$



,

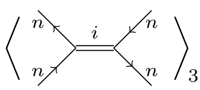
where 

denotes 

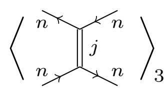
and δ_{ij} *is the Kronecker delta function.*

Proof. Theorem 2.7 and Lemma 2.5 show the value is zero if $i\neq j$. If $i=j$, then the coefficient is obtained by closing double-lined edges of both sides. □

Proposition 3.4 (Recoupling Theorem)**.**



$$=\sum_{j=0}^{n}\begin{Bmatrix} n & n & (j,j) \\ n & n & (i,i)\end{Bmatrix}$$



.

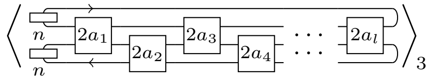
Proof. We can prove this proposition in the same way as the recoupling theory of Temperley–Lieb algebra (see [18, Chapter 7]) using Definition 2.4 and Lemma 3.3. □

We give a formula of the $\mathfrak{sl}_3$ colored Jones polynomial $J^{\mathfrak{sl}_3}_{(n,0)}([2a_1,2a_2,\ldots,2a_l];q)$ for a 2-bridge link $[2a_1,2a_2,\ldots,2a_l]$ using colored trivalent graphs. In this paper we define the $\mathfrak{sl}_3$ colored Jones polynomial for framed links obtained by a link diagram with the blackboard framing.

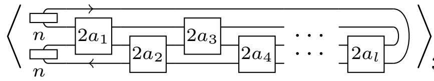
Definition 3.5.

$J^{\mathfrak{sl}_3}_{(n,0)}([2a_1,2a_2,\ldots,2a_l];q)$

$=$

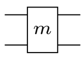


$/\Delta(n,0)$ if l is odd,



$/\Delta(n,0)$ if l is even,

where, $a_1,a_2,\ldots,a_l$ are nonzero integers and



$=$

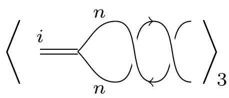
right-handed m half twists, if $m>0$,

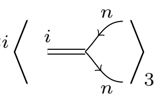
left-handed m half twists, if $m<0$.

Theorem 3.6.

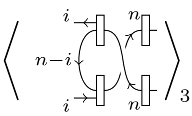
$$J^{\mathfrak{sl}_3}_{(n,0)}([2a_1, 2a_2, \dots, 2a_l]\,; q) = \sum_{0 \le i_1, i_2 \dots i_l \le n} \frac{\Delta(i_1, i_1)}{\Delta(n,0)} \frac{\theta(n,n,(i_l,i_l))}{\theta(n,n,(i_1,i_1))} q^{-\frac{2}{3}(n^2+3n)(\sum_{k=1}^{l} a_k)} q^{\sum_{k=1}^{l} a_k (i_k^2 + 2i_k)} \times \prod_{k=1}^{l-1} \begin{Bmatrix} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{Bmatrix}.$$

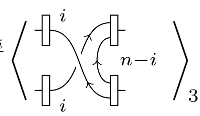
Lemma 3.7.



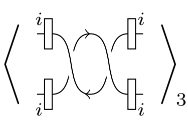
$$= q^{-\frac{2}{3}(n^2+3n)+i^2+2i}$$


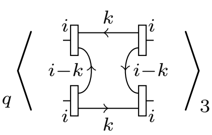
Proof. First, we describe the trivalent vertex as a web and slide the web labelled by $n-i$ by twice using the following formula [33, Lemma 3.16]:



$$= q^{-\frac{n^2+3n-i^2-3i}{3}}$$


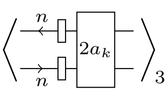
Next, we use Theorem 2.6 with $m = 1, n = i$:

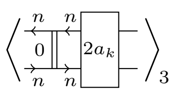


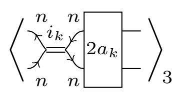
$$= q^{\frac{i^2}{3}} \sum_{k=0}^{i} q^{k^2-i^2+k-i} \frac{(q)_i}{(q)_k} \binom{i}{k}_q$$


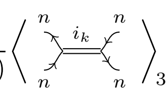
From the property of the A_2 clasp of Lemma 2.5, the terms except for $k = i$ are vanished. ☐

Proof of Theorem 3.6.

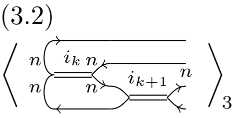
(3.1)


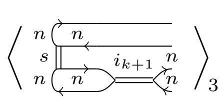
$$=$$


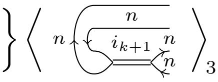
$$= \sum_{i_k=0}^{n} \begin{Bmatrix} n & n & (i_k, i_k) \\ n & n & (0,0) \end{Bmatrix}$$


$$= \sum_{i_k=0}^{n} q^{-\frac{2a_k}{3}(n^2+3n)+a_k(i_k^2+2i_k)} \frac{\Delta(i_k,i_k)}{\theta(n,n,(i_k,i_k))}$$


This equation is derived from Proposition 3.4, Lemma 3.7, and $\begin{Bmatrix} n & n & (i_k, i_k) \\ n & n & (0,0) \end{Bmatrix} = \frac{\Delta(i_k,i_k)}{\theta(n,n,(i_k,i_k))}$. By Proposition 3.4 and Lemma 3.3,

(3.2)


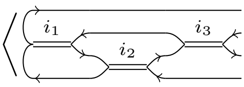
$$= \sum_{s=0}^{n} \begin{Bmatrix} n & n & (s,s) \\ n & n & (i_k, i_k) \end{Bmatrix}$$


$$= \frac{\theta(n,n,(i_{k+1},i_{k+1}))}{\Delta(i_{k+1},i_{k+1})} \begin{Bmatrix} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{Bmatrix}$$


First, we apply (3.1) to boxed $2a_k$ for all $0 \leq k \leq l$ of the 2-bridge link diagram. Then, we obtain

$$\sum_{0\leq i_1,i_2,\ldots,i_l\leq n} q^{-\frac{2}{3}(n^2+3n)(\sum_{k=1}^{l} a_k)} q^{\sum_{k=1}^{l} a_k(i_k^2+2i_k)}$$

$$\times \prod_{k=1}^{l} \frac{\Delta(i_k,i_k)}{\theta(n,n,(i_k,i_k))} \Big\langle \quad \cdots \Big\rangle_3 .$$



Next, we apply (3.2) to the doubled edge with label $i_1, i_2, \ldots, i_{l-1}$ in turn. Then, $\prod_{k=2}^{l} \frac{\Delta(i_k,i_k)}{\theta(n,n,(i_k,i_k))}$ is vanished and $\prod_{k=1}^{l-1} \begin{Bmatrix} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{Bmatrix}$ appears. Finally, the remaining A_2 web is $\theta(n,n,(i_l,i_l))$. □

4. The tail of the $\mathfrak{sl}_3$ colored Jones polynomial for $T(2,2m)$

In this section we discuss the stability of coefficients of the $\mathfrak{sl}_3$ colored Jones polynomial for the torus link $T(2,2m)$ for positive integer m. Two explicit formulas of $J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m))$ are obtained from Theorem 3.6 and [33, Theorem 5.7] because $T(2,2m)$ has a presentation $[2m]$ as a 2-bridge link. We write these two formulas below. Let m be a positive integer.

From Theorem 3.6,

$$\begin{aligned}\psi_n^{(m)}(q) = J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m)) &= q^{-\frac{2m}{3}(n^2+3n)} \sum_{i=0}^{n} \frac{\Delta(i,i)}{\Delta(n,0)} q^{m(i^2+2i)} \\ &= q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{i=0}^{n} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)(1-q^{n+1})(1-q^{n+2})}.\end{aligned}$$

From [33, Theorem 5.7],

$$\begin{aligned}g_n^{(m)}(q) &= J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m)) \\ &= q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{0\leq k_m\leq\cdots\leq k_2\leq k_1\leq n} q^{-2k_m} q^{\sum_{j=1}^{m}(k_i^2+2k_i)} \\ &\quad\times \frac{(q)_n^2}{(q)_{k_m}^2 (q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{n-k_m+1})(1-q^{n-k_m+2})}.\end{aligned}$$

Definition 4.1. Suppose $f(q), f_n(q) \in \mathbb{Z}[[q]]$ for $n \geq 1$. *The limit of* $\{f_n(q)\}_n$ *is* $f(q)$, denoted $\lim_{n\to\infty} f_n(q) = f(q)$, which means that $f_n(q) = f(q)$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$ for all n.

A trivial example is that $f_n(q) = (1-q^{n+1})$ and $f(q) = 1$. We remark that $(1-q^{n+1}) = (1-q)(1+q+q^2+\cdots+q^n)$. Thus, $(1-q)(\sum_{k=0}^{\infty} q^k) = 1$, that is, $\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$. Another simple example is that $f_n(q) = (q;q)_n$

and $f(q) = (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)$. In fact,

$$\begin{aligned}(q;q)_\infty &= \prod_{k=1}^{\infty}(1-q^k)\\ &= (1-q)(1-q^2)\cdots(1-q^n)-q^{n+1}(1-q)(1-q^2)\\ &\qquad \cdots(1-q^n)\prod_{k=1}^{\infty}(1-q^{n+1+k})\\ &= (q;q)_n \quad \text{in } \mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]].\end{aligned}$$

We can see that the minimum degree of $J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m))$ is $q^{-\frac{2m}{3}(n^2+3n)+n}$ and consider the limit of $\Psi^{(m)}_n(q) = q^{\frac{2m}{3}(n^2+3n)-n}\psi_n(q)$ and

$$G^{(m)}_n(q) = q^{\frac{2m}{3}(n^2+3n)-n}g_n(q).$$

The following lemma ensures the existence of the limit.

Lemma 4.2. $\Psi^{(m)}_n(q) = \Psi^{(m)}_{n+1}(q)$ *in* $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$.

Proof. From the the above explicit formula of $\psi^m_n(q)$,

$$\Psi^{(m)}_{n+1}(q) = \sum_{i=0}^{n+1} q^{-2i}q^{m(i^2+2i)}\frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)(1-q^{n+1})(1-q^{n+2})}.$$

As we confirmed in the first example of Definition 4.1, $\frac{1}{1-q^N} = 1$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$ if $N \geq n+1$. The lowest exponent of the summand $i = n+1$ is

$$-2(n+1)+m((n+1)^2+2(n+1)) = m(n+1)^2+2(n+1)(m-1) \geq m(n+1)^2 \geq n+1.$$

Consequently,

$$\begin{aligned}\Psi^{(m)}_{n+1}(q) &= \sum_{i=0}^{n+1} q^{-2i}q^{m(i^2+2i)}\frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)(1-q^{n+1})(1-q^{n+2})}\\ &= \sum_{i=0}^{n} q^{-2i}q^{m(i^2+2i)}\frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)} \in \mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]. \qquad (4.1)\end{aligned}$$

□

We explicitly give the limit of $\{\Psi^{(m)}_n(q)\}_n$ and $\{G^{(m)}_n\}_n$. First, we can obtain the following from (4.1):

$$\lim_{n\to\infty}\Psi^{(m)}_n(q) = \sum_{i=0}^{\infty} q^{-2i}q^{m(i^2+2i)}\frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q}. \qquad (4.2)$$

Next, we consider the limit of $\{G^{(m)}_n\}_n$. In $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$,

$$\begin{aligned}\frac{q^{k_m^2}}{1-q^{n-k_m+1}} &= q^{k_m}(1+q^{n-k_m+1}+q^{2(n-k_m+1)}+\dots)\\ &= q^{k_m^2}+q^{n+1+k_m^2-k_m}+(\text{higher-order tems})\\ &= q^{k_m^2}\end{aligned}$$

TABLE 1. Expansion of $\Psi^{(m)}(q)$ up to order 150 for $m = 1, 2, \ldots, 6$.

m	$\Psi^{(m)}(q)$
1	$1 + O\left(q^{151}\right)$
2	$1 - q - q^{2} + q^{3} + q^{4} + q^{5} - q^{6} - q^{7} - q^{8} - q^{9} + q^{10} + q^{11} + q^{12} + q^{13} + q^{14} - q^{15} - q^{16} - q^{17} - q^{18} - q^{19} - q^{20} + q^{21} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} - q^{28} - q^{29} - q^{30} - q^{31} - q^{32} - q^{33} - q^{34} - q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} + q^{44} - q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} - q^{53} - q^{54} + q^{55} + q^{56} + q^{57} + q^{58} + q^{59} + q^{60} + q^{61} + q^{62} + q^{63} + q^{64} + q^{65} - q^{66} - q^{67} - q^{68} - q^{69} - q^{70} - q^{71} - q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} + q^{78} + q^{79} + q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{85} + q^{86} + q^{87} + q^{88} + q^{89} + q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} - q^{100} - q^{101} - q^{102} - q^{103} - q^{104} + q^{105} + q^{106} + q^{107} + q^{108} + q^{109} + q^{110} + q^{111} + q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} + q^{118} + q^{119} - q^{120} - q^{121} - q^{122} - q^{123} - q^{124} - q^{125} - q^{126} - q^{127} - q^{128} - q^{129} - q^{130} - q^{131} - q^{132} - q^{133} - q^{134} - q^{135} + q^{136} + q^{137} + q^{138} + q^{139} + q^{140} + q^{141} + q^{142} + q^{143} + q^{144} + q^{145} + q^{146} + q^{147} + q^{148} + q^{149} + q^{150} + O(q^{151})$
3	$1 - q - q^{2} + q^{3} + q^{7} + q^{8} - q^{9} - q^{10} - q^{11} - q^{12} + q^{13} + q^{14} + q^{20} + q^{21} + q^{22} - q^{23} - q^{24} - q^{25} - q^{26} - q^{27} - q^{28} + q^{29} + q^{30} + q^{31} + q^{39} + q^{40} + q^{41} + q^{42} - q^{43} - q^{44} - q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} + q^{51} + q^{52} + q^{53} + q^{54} + q^{64} + q^{65} + q^{66} + q^{67} + q^{68} - q^{69} - q^{70} - q^{71} - q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} - q^{78} + q^{79} + q^{80} + q^{81} + q^{82} + q^{83} + q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{100} - q^{101} - q^{102} - q^{103} - q^{104} - q^{105} - q^{106} - q^{107} - q^{108} - q^{109} - q^{110} - q^{111} - q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} + q^{118} + q^{132} + q^{133} + q^{134} + q^{135} + q^{136} + q^{137} + q^{138} - q^{139} - q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} - q^{150} + O\left(q^{151}\right)$
4	$1 - q - q^{2} + q^{3} + q^{10} + q^{11} - q^{12} - q^{13} - q^{14} - q^{15} + q^{16} + q^{17} + q^{28} + q^{29} + q^{30} - q^{31} - q^{32} - q^{33} - q^{34} - q^{35} - q^{36} + q^{37} + q^{38} + q^{39} + q^{54} + q^{55} + q^{56} + q^{57} - q^{58} - q^{59} - q^{60} - q^{61} - q^{62} - q^{63} - q^{64} - q^{65} + q^{66} + q^{67} + q^{68} + q^{69} + q^{88} + q^{89} + q^{90} + q^{91} + q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} - q^{100} - q^{101} - q^{102} + q^{103} + q^{104} + q^{105} + q^{106} + q^{107} + q^{130} + q^{131} + q^{132} + q^{133} + q^{134} + q^{135} - q^{136} - q^{137} - q^{138} - q^{139} - q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} + q^{148} + q^{149} + q^{150} + O\left(q^{151}\right)$
5	$1 - q - q^{2} + q^{3} + q^{13} + q^{14} - q^{15} - q^{16} - q^{17} - q^{18} + q^{19} + q^{20} + q^{36} + q^{37} + q^{38} - q^{39} - q^{40} - q^{41} - q^{42} - q^{43} - q^{44} + q^{45} + q^{46} + q^{47} + q^{69} + q^{70} + q^{71} + q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} - q^{78} - q^{79} - q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{112} + q^{113} + q^{114} + q^{115} + q^{116} - q^{117} - q^{118} - q^{119} - q^{120} - q^{121} - q^{122} - q^{123} - q^{124} - q^{125} - q^{126} + q^{127} + q^{128} + q^{129} + q^{130} + q^{131} + O\left(q^{151}\right)$
6	$1 - q - q^{2} + q^{3} + q^{16} + q^{17} - q^{18} - q^{19} - q^{20} - q^{21} + q^{22} + q^{23} + q^{44} + q^{45} + q^{46} - q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} + q^{53} + q^{54} + q^{55} + q^{84} + q^{85} + q^{86} + q^{87} - q^{88} - q^{89} - q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{136} + q^{137} + q^{138} + q^{139} + q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} - q^{150} + O\left(q^{151}\right)$

because $k_m^2 - k_m \geq 0$. In a similar way, we obtain $\frac{q^{k_m^2}}{1-q^{n-k_m+2}} = q^{k_m^2}$, $q^{k_1^2+2k_1}\frac{(q)_n}{(q)_{n-k_1}}$ $= q^{k_1^2+2k_1}$. From the above, we conclude that the limit of $\{G_n^{(m)}\}_n$ is

$$\lim_{n\to\infty} G_n^{(m)}(q) = (q)_\infty \sum_{0\leq k_m\leq\cdots\leq k_2\leq k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^m (k_i^2+2k_i)}}{(q)_{k_m}^2 (q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}}. \tag{4.3}$$

Consequently, we have the following identity of q-series.

Theorem 4.3.

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q} = (q)_\infty \sum_{0\leq k_m\leq\cdots\leq k_2\leq k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^m (k_i^2+2k_i)}}{(q)_{k_m}^2 (q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}}.$$

The above identity is a knot-theoretical generalization of the Andrews–Gordon identities for the Ramanujan false theta function.

Remark 4.4.

- It is shown that the limit (4.2) appears as a summand of the false theta function for $\mathfrak{sl}_3$ by Bringmann, Kaszian and Milas [7].
- Jeremy Lovejoy tells us that Theorem 4.3 can also be proven by using Bailey pair techniques.

We compute the expansion of q-series $\Psi^{(m)}(q) = \lim_{n\to\infty}\Psi_n^{(m)}(q)$ up to order 150 by Mathematica 11.0 [17] in Table 1. It is conjectured that the collection of coefficients of the expansion of $\Psi^{(m)}(q)$ except for zeros is

$$\{1,-1,-1,1,1,1,-1,-1,-1,-1,1,1,1,1,1,\dots\}$$

for $m > 0$. Sequences of nonzero coefficients and zero coefficients appear in the expansion of $\Psi^{(m)}(q)$ alternatively. The nth sequence of nonzero coefficients has length $4n$, and the nth sequence of zero coefficients has length $(2n+1)(m-2)$ for $m > 0$.

Acknowledgment

The author would like to express his gratitude to his advisor, Hisaaki Endo, for encouragement. He also would like to thank Antun Milas for pointing out relationships betweeen his works and his and Jeremy Lovejoy for letting him know another proof of our theorem.

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