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K_2 OF CERTAIN FAMILIES OF PLANE QUARTIC CURVES

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ABSTRACT. We construct three elements in the kernel of the tame symbol on families of quartic curves. We show that these elements are integral under certain conditions on the parameters. Moreover, we prove that these elements are in general linearly independent by calculating the limit of the regulator.

1. INTRODUCTION

In his pioneering work [2], Beilinson made very far-reaching conjectures on the relation between special values of L-functions and regulators of certain K-groups of smooth projective varieties defined over number fields. The conjecture for K_2 of curves is originally due to Bloch [5].

First we briefly review some notation and definitions concerning Beilinson's conjecture on K_2 of curves. For a smooth projective geometrically irreducible curve C defined over $\mathbb{Q}$ (similar definitions apply for curves over number fields), the localization sequence of K-theory gives us the exact sequence

$$\bigoplus_{x\in C^{(1)}} K_2(\mathbb{Q}(x)) \longrightarrow K_2C \longrightarrow K_2(\mathbb{Q}(C)) \xrightarrow{T} \bigoplus_{x\in C^{(1)}} \mathbb{Q}(x)^*,$$

where $C^{(1)}$ denotes the set of closed (codimension 1) points of C, and the x-component of the map T is the *tame symbol* at x, defined on generators by

$$T_x : \{a,b\} \mapsto (-1)^{\mathrm{ord}_x(a)\,\mathrm{ord}_x(b)} \frac{a^{\mathrm{ord}_x(b)}}{b^{\mathrm{ord}_x(a)}}(x). \tag{1.1}$$

The tame K_2 group $K_2^T(C)$ of C is defined by $\ker(T)$, which is isomorphic to $K_2(C)$ up to torsion since K_2 of a number field is torsion. For α in $K_2(\mathbb{Q}(C))$, we have the *product formula* [1, Theorem 8.2],

$$\prod_{x\in C^{(1)}} \mathrm{Nm}_{k(x)/k}(T_x(\alpha)) = 1. \tag{1.2}$$

Now we give the definition of the group $K_2^T(C)_{\mathrm{int}}$, which is a subgroup of $K_2^T(C)$. It plays a key role in Beilinson's conjecture on K_2 of curves. Fix a regular, proper model $\mathcal{C}/\mathbb{Z}$ of $C/\mathbb{Q}$. We define

$$K_2^T(C)_{\mathrm{int}} = \ker\left(K_2^T(C) \xrightarrow{T_{\mathcal{C}}} \bigoplus_{p,\mathcal{D}\subset\mathcal{C}_p} \mathbb{F}_p(\mathcal{D})^*\right),$$

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where the sum runs through all rational primes and the irreducible components of $\mathcal{C}_\mathfrak{p}$, and $\mathbb{F}_p(\mathcal{D})$ is the residue field at $\mathcal{D}$. The component of $T_\mathcal{C}$ for $\mathcal{D}$ is given by the tame symbol corresponding to $\mathcal{D}$ similar to (1.1),

$$\{a, b\} \mapsto (-1)^{v_\mathcal{D}(a)v_\mathcal{D}(b)} \frac{a^{v_\mathcal{D}(b)}}{b^{v_\mathcal{D}(a)}}(\mathcal{D}),$$

where $v_\mathcal{D}$ is the valuation on $\mathbb{Q}(C)$ corresponding to $\mathcal{D}$. The group $K_2^T(C)_{\mathrm{int}}$ is well-defined since it is independent of the choice of $\mathcal{C}$ (see Proposition 4.1 of [11]). We call it the integral tame K_2 group of C and call its elements *integral*. Note that $K_2^T(C)_{\mathrm{int}}$ agrees with $K_2(\mathcal{C})$ up to torsion by the localization sequence for $\mathcal{C}$.

Beilinson's conjecture relates K-theory of varieties to special values of their L-functions via the so-called *regulator*. There is a well-defined pairing between $K_2^T(C)$ and $H_1(C(\mathbb{C}), \mathbb{Z})$:

$$\langle \cdot, \cdot \rangle : H_1(C(\mathbb{C}); \mathbb{Z}) \times K_2^T(C)/\text{torsion} \to \mathbb{R}. \tag{1.3}$$

Let $H_1(C(\mathbb{C}), \mathbb{Z})^-$ be the part of $H_1(C(\mathbb{C}), \mathbb{Z})$ which is anti-invariant under the action of complex conjugation on $C(\mathbb{C})$. Suppose that C has genus g; then it is a free abelian group of rank g. We restrict the pairing to $H_1(C(\mathbb{C}), \mathbb{Z})^-$, giving us the *regulator pairing*

$$\begin{aligned} \langle \cdot, \cdot \rangle : H_1(X; \mathbb{Z})^- \times K_2^T(C)/\text{torsion} &\to \mathbb{R} \\ (\gamma, \alpha) &\mapsto \frac{1}{2\pi} \int_\gamma \eta(\alpha), \end{aligned} \tag{1.4}$$

with $\eta(\alpha)$ obtained by writing α as a sum of symbols $\{a, b\}$, and mapping $\{a, b\}$ to

$$\eta(a, b) = \log|a| \mathrm{d}\arg(b) - \log|b| \mathrm{d}\arg(a), \tag{1.5}$$

and γ is chosen such that $\eta(\alpha)$ is defined. If $\gamma_1, \dots, \gamma_g$ form a basis of $H_1(C(\mathbb{C}); \mathbb{Z})^-$ and $M_1, \dots, M_g$ are in $K_2^T(C)$, then the regulator $R(M_1, \dots, M_g)$ is defined by

$$R = |\det(\langle \gamma_i, M_j \rangle)|. \tag{1.6}$$

Beilinson's conjecture expects $K_2^T(C)_{\mathrm{int}} \otimes_\mathbb{Z} \mathbb{Q}$ to have $\mathbb{Q}$-dimension g and $R \neq 0$ if $M_1, \dots, M_g$ form a basis of it. Moreover, it relates R to the value of $L(C, s)$ at $s = 2$ (see, e.g., [8, Conjecture 3.11]).

For any elliptic curve E over $\mathbb{Q}$, there is always a non-torsion element in $K_2^T(E)_{\mathrm{int}}$ such that the regulator of the element and $L(E, 2)$ have the relation predicted by Beilinson's conjecture. This was proved by Bloch [5] for E with complex multiplication and follows from Belinson's work on modular curves [3] and the modularity of E due to Wiles et al. for the non-CM case. On the other hand, not much is known about $K_2^T(C)_{\mathrm{int}}$ for curves of genus greater than 1 except Belinson's work on modular curves in [3], but see also [8, 9, 11, 13, 14] for some ad hoc constructions. Kühn and Müller in [10] considered the problem of constructing curves with elements in the tame K_2 group from their special intersection properties with other curves. In particular, they constructed a family of quartic curves with three elements in the integral tame K_2 group.

In this article, we study the K_2 group of certain families of quartic curves over $\mathbb{Q}$. In Section 2, following the ideas in [10], we construct a family of quartic curves with three elements in $K_2^T(C)$. The advantage of our construction is that the configuration is more symmetric, which makes the integrality of the elements almost automatic under certain conditions on the parameters. In Section 3, we prove that the elements are in general linearly independent by calculating the limit of the regulator. Hence these elements generate $K_2^T(C)_{\text{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$ provided that one accepts Beilinson's conjecture. By applying the same method, we can also show the linear independence of the elements on the quartic curves constructed in [10].

2. Elements in K_2 of a family of plane quartic curves

Suppose that C is a plane quartic curve. Since smooth quartic curves have genus 3, we want to construct quartic curves with three linearly independent elements in $K_2^T(C)_{\text{int}}$ as predicted by Beilinson's conjecture. But first we need three elements in $K_2^T(C)$. By Proposition 4.3 of [8], we only need to construct curves with four points whose pairwise differences are torsion divisors. To construct points with such property, we use the following simple observation.

Lemma 2.1. *Let P_1, P_2, P_3, P_4 be four distinct flex points on a smooth quartic curve C such that the tangent line at P_i passes through $P_{i+1}, i \in \mathbb{Z}/4\mathbb{Z}$. Then the pairwise differences of $P_i, i = 1, 2, 3, 4$, are torsion divisors.*

Proof. Suppose the tangent line T_{P_i} at P_i is defined by the equation $L_{P_i} = 0$. Since the curve is quartic and P_i are flex points, T_{P_i} intersect the curve with multiplicity 3 at P_i and with multiplicity 1 at P_{i+1}. Let $f_1 = \frac{L_{P_1}^7 L_{P_3}^3}{L_{P_2}^9 L_{P_4}}$. Then f_1 is a rational function on C and

$$\begin{aligned} \operatorname{div}(f_1) &= 7((3P_1) + (P_2)) + 3((3P_3) + (P_4)) - 9((3P_2) + (P_3)) - ((3P_4) + (P_1)) \\ &= 20(P_1) - 20(P_2). \end{aligned}$$

Similarly, let $f_2 = \frac{L_{P_2}^7 L_{P_4}^3}{L_{P_3}^9 L_{L_1}}$, $f_3 = \frac{L_{P_3}^7 L_{P_1}^3}{L_{P_4}^9 L_{P_2}}$, and $f_4 = \frac{L_{P_4}^7 L_{P_2}^3}{L_{P_1}^9 L_{P_3}}$. Then

$$\operatorname{div}(f_i) = 20(P_i) - 20(P_{i+1}), \qquad i \in \mathbb{Z}/4\mathbb{Z}.$$

Hence $(P_i) - (P_{i+1}), i \in \mathbb{Z}/4\mathbb{Z}$, are torsion divisors. Since any difference of two points can be expressed as a sum of differences of P_i and P_{i+1}, the pairwise differences of these four points are torsion divisors. □

Now we want to find quartic curves which satisfy the conditions of Lemma 2.1. Since C is quartic, the projective equation of C can be written as

$$\sum_{i+j+k=4, i,j,k \geqslant 0} a_{i,j,k} X^i Y^j Z^k = 0. \tag{2.2}$$

The equation has 14 degrees of freedom up to scale. Since any four points in $\mathbb{P}^2$ in the general position can be transformed to the four special points $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$ by an automorphism of $\mathbb{P}^2$, without loss of generality, we can assume $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, to be these four points. This imposes 4 conditions on the equation of the curve. To fulfill the requirements of Lemma 2.1, P_i must be flex points, which imposes 1 condition at each P_i, and the tangent line at P_i passes through P_{i+1}, which imposes another condition. So in total we have 12 conditions on the equation of the curve. Subtracting these 12 conditions from 14 degrees of freedom, we should have a family of quartic curves with 2 parameters. This is indeed what we get in the following lemma.

Lemma 2.3. *Suppose an irreducible quartic curve C passes through $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, and the P_i are flex points with tangent lines passing through P_{i+1}. Then C forms* *a two-parameter family of curves $C_{a,b}$ defined by the following projective equation with parameters a and b:*

(2.4)
$$(2+b-a)XY^3+X^3Z-(b+3)X^2YZ+aXY^2Z+bX^2Z^2-bXYZ^2+XZ^3-YZ^3=0,$$

where $2+b-a\neq 0$.

Proof. Basically what we do is to write the conditions and solve the other coefficients with $a_{1,2,1}$ and $a_{2,0,2}$.

Since C passes through P_1, P_2, P_3, we have $a_{4,0,0}=a_{0,4,0}=a_{0,0,4}=0$. The tangent line at P_1 passes through P_2, so it is $Z=0$. Substituting it into (2.2), we get $X(a_{3,1,0}X^2Y+a_{2,2,0}XY^2+a_{1,3,0}Y^3)=0$. Since $Z=0$ intersects C with multiplicity 3 at P_1, we must have $a_{3,1,0}=a_{2,2,0}=0$. Similarly, at P_2 we have $a_{0,3,1}=a_{0,2,2}=0$, and at P_3 we have $a_{1,0,3}+a_{0,1,3}=a_{2,0,2}+a_{1,1,2}=0$. Now (2.2) can be simplified to

$$\begin{aligned}(2.5)\quad a_{1,3,0}XY^3+a_{3,0,1}X^3Z+a_{2,1,1}X^2YZ+a_{1,2,1}XY^2Z+a_{2,0,2}X^2Z^2\\ -a_{2,0,2}XYZ^2+a_{1,0,3}XZ^3-a_{1,0,3}YZ^3=0.\end{aligned}$$

The tangent line at P_4 is $Y=Z$. Substituting $\tilde{Y}=Y-X, \tilde{Z}=Z-X$ into (2.5), the new curve with coordinate $(X,\tilde{Y},\tilde{Z})$ passes through $[1,0,0]$ at which the tangent line is $\tilde{Y}=\tilde{Z}$. Substituting $\tilde{Y}=\tilde{Z}, X=1$ into the equation of the curve, it must have a root of multiplicity 3 at $\tilde{Y}=0$, which implies that

$$\begin{aligned}a_{1,3,0}+a_{3,0,1}+a_{2,1,1}+a_{1,2,1} &= 0,\\ 3a_{1,3,0}+a_{3,0,1}+2a_{2,1,1}+3a_{1,2,1}-a_{2,0,2}-a_{1,0,3} &= 0,\\ 3a_{1,3,0}+a_{2,1,1}+3a_{1,2,1}-2a_{2,0,2}-3a_{1,0,3} &= 0.\end{aligned}$$

Solving these linear equations, we have

$$\begin{aligned}a_{1,3,0} &= 2a_{1,0,3}+a_{2,0,2}-a_{1,2,1},\\ a_{3,0,1} &= a_{1,0,3},\\ a_{2,1,1} &= -a_{2,0,2}-3a_{1,0,3}.\end{aligned}$$

Plugging this into equation (2.5), we have

$$\begin{aligned}(2a_{1,0,3}+a_{2,0,2}-a_{1,2,1})XY^3+a_{1,0,3}X^3Z-(a_{2,0,2}+3a_{1,0,3})X^2YZ+a_{1,2,1}XY^2Z\\ +a_{2,0,2}X^2Z^2-a_{2,0,2}XYZ^2+a_{1,0,3}XZ^3-a_{1,0,3}YZ^3=0.\end{aligned}$$

Note that if $a_{1,0,3}=0$, the curve splits into two lines and a conic. So up to scale we can assume $a_{1,0,3}=1$. Substituting $a_{1,2,1}=a$ and $a_{2,0,2}=b$, we get (2.4). Finally, we have $2+b-a\neq 0$ since otherwise C is reducible. □

Denote $x=\frac{X}{Z}$ and $y=\frac{Y}{Z}$. The elements of the function field of C are rational functions of x and y. Now can we write three elements in $K_2^T(C_{a,b})$.

Proposition 2.6. *Let $C_{a,b}$ be as in Lemma* 2.3. *Then we have three elements M_1, M_2, M_3 in $K_2^T(C_{a,b})$:*

$$\left\{\frac{f_1}{(a-b-2)^3},(a-b-2)f_2\right\},\left\{(a-b-2)^3f_2,\frac{f_3}{a-b-2}\right\},\left\{\frac{f_3}{(a-b-2)^3},(a-b-2)f_4\right\},$$

where

$$f_1=\frac{(x-y)^3}{x^9(y-1)},f_2=\frac{x^7(y-1)^3}{(x-y)^9},f_3=\frac{(x-y)^7}{(y-1)^9x},f_4=\frac{(y-1)^7x^3}{x-y}.$$

Proof. By the proof of Lemma 2.1, $\mathrm{div}(f_i) = 20(P_i) - 20(P_{i+1}), i \in \mathbb{Z}/4\mathbb{Z}$. To calculate the tame symbols, we only need to know the values of $f_i, i \in \mathbb{Z}/4\mathbb{Z}$, at suitable points.

At $P_3 = [0, 0, 1]$, the affine equation of the curve is

$$(2 + b - a)xy^3 + x^3 - (b + 3)x^2y + axy^2 + bx^2 - bxy + x - y = 0.$$

Since $\mathrm{ord}_{P_3}(x - y) = 3$, by substituting $y = (2 + b - a)xy^3 + x^3 - (b + 3)x^2y + axy^2 + bx^2 - bxy + x$ into the equation, we get $x - y = (2 + b - a)x^3 + r(x, y)$ with $\mathrm{ord}_{P_3}(r(x, y)) \geqslant 4$. Hence $\frac{x-y}{x^3} = 2 + b - a$ at P_3, $f_1(P_3) = (a - b - 2)^3$.

Similarly we have

$$\begin{aligned} f_2(P_1) &= \frac{1}{a-b-2}, \\ f_2(P_4) &= \frac{1}{(a-b-2)^3}, \\ f_3(P_2) &= a - b - 2, \\ f_3(P_1) &= (a-b-2)^3, \\ f_4(P_3) &= \frac{1}{a-b-2}. \end{aligned}$$

Combined with the product formula (1.2), this implies that the tame symbols of M_1, M_2, M_3 are trivial. $\square$

Remark 2.7. The elements in Proposition 2.6 are $\{P_1, P_2, P_3\}, \{P_2, P_3, P_4\}, \{P_3, P_4, P_1\}$ under the notation of Proposition 4.3 in [8].

Since a lot of work has been done to find elements in K_2 of hyperelliptic curves [8, 9], we are interested in whether $C_{a,b}$ is hyperelliptic. If it is not, it means that we find a new class of curves which is geometrically more general.

Proposition 2.8. *The curve $C_{a,b}$ is non-hyperelliptic if it is smooth.*

Proof. If $C_{a,b}$ is smooth, then the genus of $C_{a,b}$ is 3. Denote the homogeneous polynomial defining $C_{a,b}$ by F. Then the space of holomorphic differentials on $C_{a,b}$ is generated by

$$\frac{dx}{\partial_y F(x,y,1)}, \qquad \frac{xdx}{\partial_y F(x,y,1)}, \qquad \frac{ydx}{\partial_y F(x,y,1)} \tag{2.9}$$

(see [4], page 99). By a theorem of Max Noether (see [12], [4], page 119), one can check when a plane algebraic curve is hyperelliptic from the knowledge of its holomorphic differentials: let

$$\{f_1(x, y)\mathrm{d}x, \ldots, f_g(x, y)\mathrm{d}x\}$$

denote the set of holomorphic differentials on the curve defined by $f(x, y) = 0$. Then the curve is hyperelliptic if and only if there are exactly $2g - 1$ linearly independent elements in the quadratic combinations:

$$\{f_i(x, y)f_j(x, y), i, j = 1, \ldots, g\}.$$

Obviously the quadratic combinations of (2.9) have dimension 6, which is greater than $2g - 1 = 5$, so by Noether's criteria $C_{a,b}$ is not hyperelliptic. $\square$

If $a, b \in \mathbb{Z}$, (2.4) defines an arithmetic surface $\mathcal{C}'$ over $\mathbb{Z}$. Now we want to understand the integrality of the elements given in Proposition 2.6.

Theorem 2.10. *Suppose $a, b \in \mathbb{Z}$ and $a-b-2=\pm 1$. The elements M_1, M_2, M_3 in Proposition 2.6 are integral.*

Proof. Assume that $a-b-2=1$; the case when $a-b-2=-1$ is similar. Using the notation in Proposition 2.6, the elements can simply be written as

$$M_1=\{f_1,f_2\}, \qquad M_2=\{f_2,f_3\}, \qquad M_3=\{f_3,f_4\}.$$

We follow the proof of Theorem 8.3 in [8]. Let $\mathcal{C}$ be a regular proper model of the arithmetic surface $\mathcal{C}'$ and let $\mathcal{C}_p$ be the fiber of $\mathcal{C}$ above p. To check the integrality, we only need to show that for every irreducible component $\mathcal{D}$ of $\mathcal{C}_p$, $v_{\mathcal{D}}(f_i)=0$. Since $\mathcal{C}$ is obtained by consecutively blowing up singularities, every such $\mathcal{D}$ maps onto an irreducible component D of the fiber above p of $\mathcal{C}'$ or a singular point. We only need to examine the functions $f_i, i \in \{1,2,3,4\}$, at D and singular points.

On any irreducible component D of the fiber above p of $\mathcal{C}'$, the functions $x, y-1$ and $x-y$ are rational functions which do not vanish identically on it since otherwise $C_{a,b}$ will have irreducible component defined by $X=0, Y=Z$, or $X=Y$, which is impossible. So if $\mathcal{D}$ maps to an irreducible component, $v_{\mathcal{D}}(f_i)=0, i\in\{1,2,3,4\}$.

From the equation of $C_{a,b}$, it is obvious that it is non-singular at $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, above any prime p. But from the definition of f_i, they only have zeroes or poles at these four points, so $f_i \neq 0$ is defined at any singular point of the fiber above p of $\mathcal{C}'$. Hence $f_i \neq 0$ is constant for any $\mathcal{D}$ that maps to a singular point of $C_{a,b}$, $v_{\mathcal{D}}(f_i)=0, i\in\{1,2,3,4\}$.

In sum, on any irreducible component $\mathcal{D}$ of $\mathcal{C}_p$, $T_{\mathcal{D}}(M_j)=1, j\in\{1,2,3\}$. □

Combining Proposition 2.6, Proposition 2.8, and Theorem 2.10, we get two one-parameter families of plane quartics $C_t^{\pm}$ such that any smooth $C_t^{\pm}$ with $1/t \in \mathbb{Z}$ has at least 3 elements in $K_2^T(C_t^{\pm})_{\mathrm{int}}$.

Corollary 2.11. *Let $C_t^{\pm}$ be the families of quartic curves defined by*

(2.12)
$$t(\mp XY^3+X^3Z-3X^2YZ+(2\pm1)XY^2Z+XZ^3-YZ^3)-XZ(-Z+Y)(-Y+X)=0,$$

where $t\in\mathbb{Q}, t\neq 0$. Then:

(1) *The curve $C_t^{\pm}$ is smooth and non-hyperelliptic unless $1/t\in\{-6,-2,2\}$ for C_t^+ or $1/t\in\{-2,1\}$ for C_t^-.*
(2) *Let $1/t_1, 1/t_2 \in \mathbb{Z}$. Then $C_{t_1}^{\pm}$ and $C_{t_2}^{\pm}$ are isomorphic if and only if they have the same sign and $t_1=t_2$.*
(3) *The following elements are in $K_2^T(C_t^{\pm})$:*

$$M_1=\{\pm f_1,\pm f_2\}, \qquad M_2=\{\pm f_2,\pm f_3\}, \qquad M_3=\{\pm f_3,\pm f_4\}.$$

Moreover, if $1/t\in\mathbb{Z}$, then these elements are integral.

Proof. It is easy to write the conditions such that the curves are singular. Combining the conditions with the equation of the curve and using elimination theory with the help of Magma [6], we can solve t. This proves (1).

The first Dixmier-Ohno invariants (see [7]) of C_t^+ and C_t^- are $\frac{1}{72}(t^{-3}+6t^{-2}-10t^{-1}-36)$ and $\frac{1}{72}(t^{-3}+18t^{-1}+36)$ respectively. One can easily check they are equal for $1/t_1, 1/t_2\in\mathbb{Z}$ if and only if they are the invariants of the same family and $t_1=t_2$. This proves (2).

Let $a=b+2\pm1$ and $b=1/t$ in (2.4); we get the equation for $C_t^{\pm}$. Hence (3) is a direct corollary of Proposition 2.6 and Theorem 2.10. □

3. Linear independence of the elements

In this section, we prove the elements in Corollary 2.11 are linearly independent as $t \to 0$. The strategy of the proof is to show that the limit of the regulator is not zero as $t \to 0$. We also show the linear independence of the elements in K_2 on certain quartic curves constructed in [10] by applying the same method.

Let $X^{\pm}$ be the complex manifolds with points $\{C_t^{\pm}(\mathbb{C})\}_{t \in D}$ where D is a small enough disk. Then the fibers $X_t^{\pm} = C_t^{\pm}(\mathbb{C})$ for $t \neq 0$ are Riemann surfaces of genus 3 associated to $C_t^{\pm}(\mathbb{C})$. To calculate the limit of the regulator, we need a basis of $H_1(X_t^{\pm};\mathbb{Z})^-$ for $t \to 0$. But first we need a lemma.

Lemma 3.1. *Let Y be the fibered surface defined by*

$$F(x,y,t) = g(x,y)xy - h(x,y)t = 0 \tag{3.2}$$

with $g(x,y), h(x,y)$ holomorphic at $(0,0)$. Let γ_0 be a clockwise simple loop around 0 in the x-axis in Y_0. If $g(x,0) \not\equiv 0$, then y is a holomorphic function of x and t is in a neighbourhood of γ_0. In particular, for $x \in \gamma_0$ and t small enough, this gives a family of closed loops γ_t in the fibers Y_t.

Proof. We have $y, t \equiv 0$ on γ_0; hence $\frac{\partial F}{\partial y} = g(x,0)x$ on γ_0. Since $g(x,0)$ is holomorphic and $g(x,0) \not\equiv 0$, $g(x,0) \neq 0$ for $x \neq 0$ and $|x|$ small enough. Thus $g(x,0)x \neq 0$ if γ_0 is sufficiently small. Then the holomorphic implicit function theorem implies the lemma. $\square$

Obviously, around the points P_1, P_2, and P_3, $X^{\pm}$ is isomorphic to surfaces defined by (3.2) for certain holomorphic functions g, h with $g(x,0) \not\equiv 0$. By Lemma 3.1, this results in three families of closed loops $\gamma_1, \gamma_2, \gamma_3$ around P_1, P_2, P_3 in which we omit the parameter t since it will be clear from the context.

Lemma 3.3. *With notation and assumptions as above, for $|t| \to 0$:*

1. *the three loops $\gamma_i, i = 1,2,3$, can be complemented to a basis of $H_1(X_t^{\pm};\mathbb{Z})$;*
2. *if $t \in \mathbb{R}$, then $\gamma_i, i = 1,2,3$, give a basis of $H_1(X_t^{\pm};\mathbb{Z})^-$.*

Proof. By the construction, the loop γ_i is anti-invariant under the complex conjugation if $t \in \mathbb{R}$, namely in $H_1(X_t^{\pm};\mathbb{Z})^-$, since the equation of the curve has real coefficients. Hence we only need to prove part (1) as $H_1(X_t^{\pm};\mathbb{Z})^-$ has rank 3.

Now we construct another set of loops on $X_t^{\pm}$. On $X_0^{\pm}$, we take the loops

$$\begin{aligned}
&P_1 \to P_4 \to [1,1,0] \to P_1,\\
&P_2 \to [0,1,1] \to P_4 \to [1,1,0] \to P_2,\\
&P_3 \to [0,1,1] \to P_4 \to P_3,
\end{aligned}$$

where the $\to$ means a line segment between two points such that it does not meet $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, $[0,1,1]$, and $[1,1,0]$ except at the endpoints. The first and third are triangles and the second is a quadrilateral.

We can make a linear coordinate transformation of x, y on $X^{\pm}$ such that the loops in the above paragraph are in the affine part of the variety after the transformation, and the lines passing through two of these points are not parallel to the y-axis. By an abuse of notation, we use the same symbol $X^{\pm}$ for the complex manifold after the transformation and the same symbol for the points after the transformation. Consider the projection $\pi : (x,y,t) \to (x,t)$ from the affine part of $X^{\pm}$ to $\mathbb{C} \times D$. The points $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, $[0,1,1]$, and $[1,1,0]$ split into two ramification points under this projection, and we parametrize one of the two ramification points for t in a suitable circle sector of D.[1]

We can connect the image of the ramification points under π by line segments to get loops in $\mathbb{C} \times D$ such that they continuously extend the image of the above loops in $X_0^{\pm}$ under π. Choosing small neighbourhoods of the image of $P_i, i \in \mathbb{Z}/4\mathbb{Z}$, $[0,1,1]$, and $[1,1,0]$ under π, we can lift part of the line segments in $\mathbb{C} \times D$ outside the neighbourhoods to get paths continuously extending the line segments in the above loops in $X_0^{\pm}$.[2] Shrinking the neighbourhoods if necessary, we can also lift part of the line segments inside the neighbourhoods such that they do not intersect γ_i. Connecting these lifted segments, we construct three families of loops $\delta_j = \delta_{j,t}, j = 1,2,3$, in $X_t^{\pm}$.

We see that γ_i and $\delta_j, i, j = 1,2,3$, meet exactly once in $X_0^{\pm}$ for $i = j$ and have intersection number 0 for $i \neq j$, hence also in $X_t^{\pm}$ by construction. Changing the orientation of γ_i if necessary, we can assume that $\gamma_i \cap \delta_j$ equals 1 if $i = j$ and 0 otherwise. Since different γ_i do not intersect, the intersection matrix of $\{\gamma_i\}_{i=1,2,3} \cap \{\delta_j\}_{j=1,2,3}$ on X_t with $t \neq 0$ and small enough is of the form

$$\begin{pmatrix} 0 & I_3 \\ -I_3 & * \end{pmatrix},$$

which has determinant 1. Hence $\gamma_i, \delta_j, i, j = 1,2,3$ give a basis of $H_1(X_t^{\pm};\mathbb{Z})$ for $t \to 0$. $\square$

Let Y be as defined by (3.2) and $g(x,0), h(x,0) \not\equiv 0$. Assume that there is a family of loops γ_t in the fibers Y_t, e.g., the family of loops constructed in Lemma 3.1, with γ_0 a clockwise simple loop around 0 in the x-axis in Y_0. Furthermore, let u and v be holomorphic functions in x and y around $(0,0)$ that do not vanish at $(0,0)$. The following lemma generalizes Lemma 6.4 in [11], which assumes $g(0,0) \neq 0$ and $h(x,y) \equiv 1$.

Lemma 3.4. *Let Y, u, v, and the γ_t be as above and assume γ_0 is sufficiently small. For integers a and b, let $\psi(ux^a, vy^b)$ be the 1-form $\log|ux^a| \mathrm{d}\arg(vy^b) - \log|vy^b| \mathrm{d}\arg(ux^a)$ on an open part of $Y \setminus Y_0$, and let $F(t) = \int_{\gamma_t} \psi(ux^a, vy^b)$ for $t \neq 0$ sufficiently small. Then $F(t) = 2\pi ab \log|t| + \mathrm{Re}(H(t))$ for a holomorphic*

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[1]Suppose $X^{\pm}$ is defined by $G(x,y,t) = l_1 l_2 l_3 l_4 - th(x,y) = 0$, where $l_i, i = 1,2,3,4$, define lines which are not parallel to the y-axis and $P_{i,j}$ is the intersection point of the lines defined by $l_i, l_j, i \neq j$, on $X_0^{\pm}$. Note that different pairs of lines have different intersection points. Combining the condition satisfied by the ramification points $\frac{\partial G}{\partial y} = 0$ with $G(x,y,t) = 0$, one can show that y is a holomorphic function of x, t and x is a double-valued function of t around $P_{i,j}$. In a suitable circle sector of D, we can choose a branch of the function $x(t)$ to give a parametrization of one of the two ramification points. (This is a local problem at each intersection point of the lines. To fix ideas, it might be helpful to look at a simplified version of this problem. Let $G(x,y,t) = x^2 - y^2 - t = 0$. The point $(0,0,0)$ splits into two ramification points $(\sqrt{t}, 0, t)$ in the fiber above t, so we can choose a branch of $\sqrt{t}$ in a circle sector to give a parametrization of one of the two ramification points.)

[2]Let $X^{\pm}$ be defined as in the above footnote. Obviously, we have $\frac{\partial G}{\partial y} = 0$ on $X_0^{\pm}$ only at the intersection points of the lines. By the construction of the loops in $X_0^{\pm}$, the line segments do not meet the intersection points of the lines except at the endpoints. Hence by the implicit function theorem y is a function of x, t in a neighbourhood of the line segments outside a neighbourhood of the endpoints which gives a family of paths continuously extending the line segments in $X_0^{\pm}$.

function $H(t)$ around $t=0$. In particular,

$$\lim_{|t|\to 0}\frac{F(t)}{\log|t|}=2\pi ab\,.$$

Proof. The proof proceeds as the proof of Lemma 6.4 in [11]. Write $\mathrm{d}\eta(\{ux^a,vy^b\})=\omega_1\wedge\mathrm{d}t+\omega_2\wedge\mathrm{d}\bar{t}$ on a suitable smooth part of $Y\setminus Y_0$, with 1-forms ω_1 and ω_2. Applying Lemma 6.3 of [11] to a parametrization of $\{\gamma_t\}_{t\in D}$ where D is a small disk containing 0, we have $\frac{\partial F(t)}{\partial t}=-\int_{\gamma_t}\omega_1$.

Now we calculate ω_1. On $Y\setminus Y_0$ we have the identity

$$\frac{g_x\mathrm{d}x+g_y\mathrm{d}y}{g}+\frac{\mathrm{d}x}{x}+\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}t}{t}+\frac{h_x\mathrm{d}x+h_y\mathrm{d}y}{h}.$$

Since $g(x,0),h(x,0)\not\equiv 0$ and γ_0 is sufficiently small, we have $g(x,0),h(x,0)\neq 0$ on γ_0; therefore $g(x,y),h(x,y)\neq 0$ on $\{\gamma_t\}_{t\in D}$ for t sufficiently small. Thus we have $\frac{1+xh_1}{x}\mathrm{d}x+\frac{1+yh_2}{y}\mathrm{d}y=\frac{\mathrm{d}t}{t}$ for h_1 and h_2 holomorphic in a neighbourhood of $\{\gamma_t\}_{t\in D}$ and

$$\mathrm{d}x\wedge\mathrm{d}y=\frac{xy}{1+yh_2}\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}t}{t}=xy(1+yh_3)\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}t}{t},$$

with $h_3=\frac{-h_2}{1+yh_2}$ holomorphic in a neighbourhood of $\{\gamma_t\}_{t\in D}$. Therefore

$$\begin{aligned}\mathrm{d}\log(ux^a)\wedge\mathrm{d}\log(vy^b) &= (\mathrm{d}\log u+a\mathrm{d}\log x)\wedge(\mathrm{d}\log v+b\mathrm{d}\log y)\\ &= (ab+xh_4+yh_5)\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}t}{t}\end{aligned}$$

with $h_4(x,y)$ holomorphic around $(0,0)$ and $h_5(x,y)$ holomorphic in a neighbourhood of $\{\gamma_t\}_{t\in D}$.

Denote $(ab+xh_4+yh_5)\frac{\mathrm{d}x}{x}$ by ω. Clearly,

$$\mathrm{d}\eta(f_1,f_2)=\mathrm{Im}(\mathrm{d}\log(f_1)\wedge\mathrm{d}\log(f_2))=\frac{1}{2i}(\omega\wedge\frac{\mathrm{d}t}{t}-\overline{\omega}\wedge\frac{\mathrm{d}\bar{t}}{\bar{t}}),$$

so we have $\omega_1=\frac{1}{2it}\omega$. Viewing y around γ_t as a holomorphic function of x and t, we have

$$\frac{\partial F(t)}{\partial t}=-\int_{\gamma_t}\omega_1=-\int_{\gamma_t}\frac{1}{2it}(ab+xh_4+yh_5)\frac{\mathrm{d}x}{x}=\frac{\pi ab}{t}+h_6(t),$$

where $h_6(t)$ is a holomorphic function around $t=0$: $\int_{\gamma_t}(xh_4+yh_5)\frac{\mathrm{d}x}{x}$ is holomorphic in t and vanishes for $t=0$ because h_4 is holomorphic around $(0,0)$ and $y\equiv 0$ on γ_0.

Since $\frac{\partial\log|t|}{\partial t}=\frac{1}{2t}$, we have $\frac{\partial(F(t)-2\pi ab\log|t|)}{\partial t}=h_6$ around $t=0$. Both $F(t)$ and $\log|t|$ are real-valued; hence also $\frac{\partial(F(t)-2\pi ab\log|t|)}{\partial\bar{t}}=\overline{h_6}$ around $t=0$. Therefore $F(t)-2\pi ab\log|t|=\mathrm{Re}(H(t))$ with $H(t)$ holomorphic and $H'(t)=2h_6(t)$. □

Theorem 3.5. *Let C be defined by* (2.12). *For $t\to 0$, $t\in\mathbb{R}$, the classes of the elements M_1,M_2,M_3 in Proposition* 2.6 *have regulator $R=R(t)$ satisfying*

$$\lim_{t\to 0}\frac{R(t)}{|\log|t||^3}=20^4.$$

Proof. We can let $x=\frac{Z}{X}$ and $y=\frac{-Z+Y}{X}$ in (2.12) so that $X^{\pm}$ is isomorphic to a surface defined by (3.2) with $g(x,0),h(x,0)\not\equiv 0$. Under this transformation, M_1

becomes $\{u\frac{x^7}{y}, v\frac{y^3}{x}\}$ with $u(0,0), v(0,0) \neq 0$. Note that the restriction of the 1-form ψ in Lemma 3.4 on the fiber is just the regulator 1-form as in (1.5). Hence by Lemma 3.4, we have $\lim_{t\to 0} \frac{\int_{\gamma_1} \eta(M_1)}{\log|t|} = \pm 40\pi$.

Similarly we can calculate $\lim_{t\to 0} \frac{\int_{\gamma_i} \eta(M_j)}{\log|t|}$ for $1 \leqslant i, j \leqslant 3$. Then we have the 3×3 matrix

$$M = \frac{1}{2\pi} \lim_{t\to 0} \frac{(\langle \gamma_i, M_j \rangle)_{1\leqslant i,j\leqslant 3}}{\log|t|} = \begin{pmatrix} \pm 20 & 0 & \mp 60 \\ \pm 40 & \mp 20 & 0 \\ \pm 60 & \pm 40 & \mp 20 \end{pmatrix}.$$

Since $\gamma_i, i = 1, 2, 3$, give a basis of $H_1(X; \mathbb{Z})^-$ by Lemma 3.3 for $t \in \mathbb{R}$, we have

$$\lim_{t\to 0} \frac{R(t)}{|\log|t||^3} = |\det(M)| = 20^4.$$

□

Combining Theorem 3.5 and Corollary 2.11(3), we have the following immediate corollary.

Corollary 3.6. *Let C be defined by* (2.12) *and let $b = 1/t \in \mathbb{Z}$, $|b| \gg 0$. There are three independent elements M_1, M_2, M_3 in $K_2^T(C)_{\mathrm{int}}$.*

Remark 3.7. By Proposition 2.8 and Theorem 3.5, we have three independent elements in $K_2^T(C)_{\mathrm{int}}$ on two families of non-hyperelliptic curves of genus 3. These elements give a basis of $K_2^T(C)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$ provided that one accepts Beilinson's conjecture. In [11], Rob de Jeu and the first author constructed families of non-hyperelliptic curves of arbitrary composite genus g with g independent elements in $K_2^T(C)_{\mathrm{int}}$. It would be interesting to construct families of non-hyperelliptic curves of arbitrary genus g with g linearly independent elements in $K_2^T(C)_{\mathrm{int}}$.

Now we consider the family of quartic curves in Corollary 5.9 of [10] defined by

$$(y^3 + 3/16y^2 - 1/4x^2y + 1/64y + x^4) + by(y-x)(x+1/8) = 0, \tag{3.8}$$

where b is the parameter. Let $\widetilde{C}$ be the projective closure of the curve defined by (3.8). The elements

$$\widetilde{M_1} = \left\{-16y, 8(x+\frac{1}{8})\right\}, \widetilde{M_2} = \left\{-\frac{1}{4y}, \frac{y-x}{y}\right\}, \widetilde{M_3} = \left\{-\frac{1}{8(x+\frac{1}{8})}, -\frac{2^{12}(y-x)^4}{y}\right\}$$

are in $K_2^T(\widetilde{C})_{\mathrm{int}}$ by Corollary 5.9(iii) of [10]. We have results similar to Theorem 3.5 and Corollary 3.6 for this family of curves.

Theorem 3.9. *Let $\widetilde{C}$ be as above. For $t = 1/b \to 0$, $t \in \mathbb{R}$, the classes of the elements $\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}$ in Proposition* 2.6 *have regulator $R = R(t)$ satisfying*

$$\lim_{t\to 0} \frac{R(t)}{|\log|t||^3} = 4.$$

Proof. Let $O = (0,0), P' = (-\frac{1}{8}, 0), Q' = (-\frac{1}{8}, -\frac{1}{8})$. We can construct loops $\gamma_1, \gamma_2, \gamma_3$ close to O, P', Q' respectively and show that they give a basis of $H_1(X_t; \mathbb{Z})^-$ for $t \in \mathbb{R}$ as in Lemma 3.3. Applying Lemma 3.4, we have the 3×3 matrix

$$M = \frac{1}{2\pi} \lim_{t\to 0} \frac{\left(\int_{\gamma_i} \eta(\widetilde{M_j})\right)}{\log|t|} = \begin{pmatrix} 0 & \pm 1 & 0 \\ \mp 1 & 0 & \pm 1 \\ 0 & 0 & \pm 4 \end{pmatrix}.$$

Hence we have

$$\lim_{t\to 0}\frac{R(t)}{|\log|t||^3}=|\det(M)|=4.$$

□

Corollary 3.10. *Let* $\widetilde{C}$ *be as above and let* $b\in\mathbb{Z}$, $|b|\gg 0$. *There are three independent elements* $\widetilde{M_1},\widetilde{M_2},\widetilde{M_3}$ *in* $K_2^T(\widetilde{C})_{\mathrm{int}}$.

Proof. This follows from Theorem 3.9 and Corollary 5.9(iii) of [10]. □

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