PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 7, July 2018, Pages 2963–2971
http://dx.doi.org/10.1090/proc/13969
Article electronically published on February 8, 2018

# ON THE FUNDAMENTAL TONE OF IMMERSIONS AND SUBMERSIONS

MARCOS P. CAVALCANTE AND FERNANDO MANFIO

(Communicated by Guofang Wei)

ABSTRACT. In this paper we obtain lower bound estimates of the spectrum of the Laplace-Beltrami operator on complete submanifolds with bounded mean curvature, whose ambient space admits a Riemannian submersion over a Riemannian manifold with negative sectional curvature. Our main theorem generalizes many previously known estimates and applies for both immersions and submersions.

## 1. INTRODUCTION

Given a compact domain $\Omega$ on an $m$-dimensional Riemannian manifold $M^m$ let us denote by $Spec(\Omega) = \{\lambda_1(\Omega) < \lambda_2(\Omega) \leq ...\}$ the set of eigenvalues of the Laplace-Beltrami operator $-\Delta$ on $\Omega$ with Dirichlet boundary condition, repeated according to its multiplicity. That is, for each $\lambda_i = \lambda_i(\Omega)$, $i = 1, 2, \ldots$ there exists a nontrivial solution to the following problem:

$$\begin{cases} -\Delta\varphi = \lambda_i\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

The study of the relations between the eigenvalues and the geometry of the domain (or the manifold) is a very active topic in differential geometry and has attracted the attention of many pure and applied mathematicians for a long time.

In this paper, we are interested in obtaining lower bound estimates of the spectrum of Laplacian on a class of complete noncompact Riemannian manifolds in terms of its geometry. In order to state our results we need some notation.

We first recall that on compact domains, the set of eigenvalues is the whole spectrum of the Laplace-Beltrami operator for the Dirichlet problem. When we deal with noncompact domains some accumulation points or eigenvalues of infinite multiplicity may appear, composing the essential spectrum. In any case, the bottom of the spectrum is given by a limit of the first eigenvalues when we consider an exhaustion of the domain. More precisely, if $M$ is a Riemannian manifold and $\Omega_1 \subset \Omega_2 \subset \cdots$ is an exhaustion of $M$ the *fundamental tone* of $M$ is defined by $\lambda_1(M) = \lim_{k\to\infty} \lambda_1(\Omega_k)$.

Of course it does not depend on the choice of the exhaustion and coincides with the first eigenvalue when $M$ is compact. Moreover, $\lambda_1(M)$ can be characterized

Received by the editors May 15, 2017, and, in revised form, September 18, 2017.
2010 *Mathematics Subject Classification.* Primary 35P15, 53C20.
*Key words and phrases.* Eigenvalues, Mean Curvature, Isometric immersions, Riemannian submersions.

variationally as follows:

$$\lambda_1(M) = \inf\left\{ \frac{-\int_M \varphi\Delta\varphi}{\int_M \varphi^2} : \forall \varphi \in C_0^\infty(M) \right\}.$$

In particular, $\lambda_1(M) \geq 0$ and it is the bottom of the spectrum of $-\Delta$ on $M$.

Clearly it is much harder to give a lower bound for $\lambda_1(M)$ than an upper bound, and an important question that is proposed is to find conditions on $M$ which imply $\lambda_1(M) > 0$ (see [10, §III.4]).

In this direction, one important contribution was done by McKean [8], who proved that if $M$ is simply connected and its sectional curvature satisfies $K_M \leq -1$, then

$$\lambda_1(M) \geq \frac{(m-1)^2}{4} = \lambda_1(\mathbb{H}^m),$$

where $\mathbb{H}^m$ denotes the $m$-dimensional hyperbolic space of sectional curvature $-1$. This estimate was extended by Veeravalli [11] for a quite general class of manifolds.

In the context of submanifolds, Cheung and Leung [5] gave lower bounds estimates when $M$ is complete and isometrically immersed in the hyperbolic space $\mathbb{H}^n$ with bounded mean curvature vector field $\|H\| \leq \alpha < m-1$. Namely they proved that

$$\lambda_1(M) \geq \frac{(m-1-\alpha)^2}{4}.$$

Later, Bessa and Montenegro (see [3, Corollary 4.4]) generalized Cheung-Leung's estimate for the case where $M^m$ is immersed in a complete simply connected Riemannian manifold $\overline{M}^n$ with bounded sectional curvature $K_{\overline{M}} \leq -b^2 < 0$ and bounded mean curvature vector $H$, with $\|H\| \leq \alpha < (m-1)b$. In this setting, they were able to prove that

$$\lambda_1(M) \geq \frac{[(m-1)b-\alpha]^2}{4}.$$

We point out that Castillon obtained a different lower bound estimate in the same situation (see Théorème 2.3 in [4]).

A few years ago, Bérard, Castillon, and the first author [1], using a different approach, obtained a sharp lower bound estimate for $\lambda_1(M)$, when $M$ is a hypersurface immersed into $\mathbb{H}^n \times \mathbb{R}$ with constant mean curvature.

Our first result is a dual estimate of Cheung and Leung's theorem in the context of Riemannian submersions. We obtain the following

**Theorem 1.1.** *Let $\pi : M^m \to \mathbb{H}^k$ be a Riemannian submersion of a complete Riemannian manifold $M^m$ onto the hyperbolic space. Let us denote by $H^{\mathcal{F}}$ the mean curvature of its fibers and assume that $\|H^{\mathcal{F}}\| \leq \beta < k-1$. Then*

$$\lambda_1(M) \geq \frac{(k-1-\beta)^2}{4}.$$

Notice that this estimate is sharp in the sense that it is archived by the canonical (totally geodesic) submersion of $\mathbb{H}^k \times \mathbb{R}^{m-k}$ over $\mathbb{H}^k$.

We have a similar estimate for submersions over a complete Riemannian manifold with sectional curvature bounded from above by a negative constant, and thus we also get the dual result of Bessa and Montenegro (it is a direct corollary of Theorem 5.1 below).

In fact, we found a general lower bound for $\lambda_1(M)$ for complete submanifolds with bounded mean curvature, whose ambient space admits a Riemannian submersion over a complete Riemannian manifold with bounded negative sectional curvature. In particular, when the base manifold of the submersion is the hyperbolic space $\mathbb{H}^k$, our main theorem reads as follows.

**Theorem 1.2.** *Let $f : M^m \to \widetilde{M}^n$ be an isometric immersion of a complete Riemannian manifold $M^m$ into a Riemannian manifold $\widetilde{M}^n$, which admits a Riemannian submersion $\pi : \widetilde{M} \to \mathbb{H}^k$. Let $H$ be the mean curvature of $M$, let $\alpha^{\mathcal{F}}$ be the second fundamental form of the fibers of $\widetilde{M}$, let $H^{\mathcal{F}}$ be its mean curvature, and let $A$ be the O'Neill tensor of $\widetilde{M}$. If*

$$c = \inf\{k - 1 - \|H\| - \|H^{\mathcal{F}}\| - (n-m)\big(2\|A\|_\infty + \|\alpha^{\mathcal{F}}\|_\infty + 1\big)\} > 0,$$

*then*

$$\lambda_1(M) \geq \frac{c^2}{4},$$

*where $\|A\|_\infty$ and $\|\alpha^{\mathcal{F}}\|_\infty$ denote the uniform norm of these tensors.*

Note that we get the Theorem 1.1 when the immersion $f$ is the identity, and we get (a new proof of) Cheung-Leung's theorem when the submersion $\pi$ is the identity. In fact, in the former case $n = m$ and $H = 0$ and in the latter case $\|H^{\mathcal{F}}\| = \|A\|_\infty = \|\alpha^{\mathcal{F}}\|_\infty = 0$.

The paper is organized as follows. In Section 2, we recall some basic properties, in particular a useful condition on a Riemannian manifold which implies a positive lower bound estimate for the first eigenvalue. In Sections 3 and 4 we present some results on Riemannian submersions and on Busemann functions. A main step in our approach is to use a comparison theorem for the Hessian of Busemann functions. Finally, in Section 5, we state and prove our general theorem (Theorem 5.1), which generalizes Theorem 1.2 in two directions: when the base manifold has bounded negative sectional curvature and when the base manifold is a Riemannian warped-product of a complete manifold by the real line. We also describe some examples of submersions where the constant in the main theorem is positive.

## 2. Preliminaries

In this section we present two well-known results that will be used in the proofs of our results. The first result gives a general condition to get a positive lower bound to $\lambda_1(M)$ and its proof follows from integration by parts.

**Lemma 2.1.** *Let $M^m$ be a complete Riemannian manifold that carries a smooth function $F : M \to \mathbb{R}$ satisfying*

$$\|\text{grad } F\| \leq 1 \quad \textit{and} \quad |\Delta F| \geq c$$

*for some constant $c > 0$. Then, for any smooth and relatively compact domain $\Omega \subset M$ we have*

$$\lambda_1(\Omega) \geq \frac{c^2}{4},$$

*where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace-Beltrami operator $-\Delta$ in $\Omega$, with Dirichlet boundary condition.*

Now, given an isometric immersion $f : M^m \to \widetilde{M}^n$ between Riemannian manifolds $M$ and $\widetilde{M}$, let $\alpha$ denote its second fundamental form. Then, the *mean curvature vector* (not normalized) $H$ of $M$ is defined by $H = \operatorname{tr}\alpha$.

The second lemma relates the Laplacian of a function on $\widetilde{M}$ and its restriction to $M$ (see, for example, [6, Lemma 2]).

**Lemma 2.2.** *Let $f : M^m \to \widetilde{M}^n$ be an isometric immersion with mean curvature vector $H$. Let $\widetilde{F} : \widetilde{M} \to \mathbb{R}$ be a smooth function and let $F = \widetilde{F}|_M$ be its restriction to $M$. Then, on $M$, we have:*

$$\widetilde{\Delta}\widetilde{F} = \Delta F + \sum_{i=1}^{n-m} \operatorname{Hess}\widetilde{F}(N_i, N_i) - H(\widetilde{F}),$$

*where $\{N_1, \dots, N_{n-m}\}$ is an orthonormal frame of $TM^\perp$.*

## 3. Riemannian submersions

Let $\pi : \widetilde{M}^n \to B^k$ be a Riemannian submersion of Riemannian manifolds. As usual in the literature, given a vector field $X \in \mathfrak{X}(B)$ we will denote by $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$ its unique horizontal lifting. In general we use a tilde to denote the lifting to $\widetilde{M}$ of geometric objects in the base $B$. We also denote by $\widetilde{X}$ the *basics vectors* fields in $\widetilde{M}$, that is, the vectors fields that are $\pi$-related to some vector field $X \in \mathfrak{X}(B)$.

For $x \in B$, $\mathcal{F}_x = \pi^{-1}(x)$ denotes the fiber over $x$. Given $p \in \mathcal{F}_x$, the differential map $\mathrm{d}\pi$ restricted to the orthogonal subspace $T_p\mathcal{F}_x^\perp$ is an isometry onto $T_xB$. A vector field on $\widetilde{M}$ is called *vertical* if it is always tangent to fibers, and it is called *horizontal* if it is always orthogonal to fibers. Let $\mathcal{V}$ denote the *vertical distribution* consisting of vertical vectors and $\mathcal{H}$ denote the *horizontal distribution* consisting of horizontal vectors on $M$. The corresponding projections from $T\widetilde{M}$ to $\mathcal{V}$ and $\mathcal{H}$ are denoted by the same symbols.

Let $\mathcal{D} \subset T\widetilde{M}$ denote the smooth distribution on $\widetilde{M}$ consisting of vertical vectors. The orthogonal distribution $\mathcal{D}^\perp$ is the smooth rank $k$ distribution on $\widetilde{M}$ consisting of horizontal vectors. The *second fundamental form* of the fibers is a symmetric tensor $\alpha^{\mathcal{F}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}^\perp$, defined by

$$\alpha^{\mathcal{F}}(v, w) = (\widetilde{\nabla}_v W)^{\mathcal{H}},$$

where $W$ is a vertical extension of $w$. The *mean curvature vector* of the fiber is the horizontal vector field $H^{\mathcal{F}}$ defined by $H^{\mathcal{F}} = \operatorname{tr}\alpha^{\mathcal{F}}$. In terms of an orthonormal frame, we have

$$H^{\mathcal{F}}(p) = \sum_{i=1}^{n-k} \alpha^{\mathcal{F}}(e_i, e_i) = \sum_{i=1}^{n-k} (\widetilde{\nabla}_{e_i} e_i)^{\mathcal{H}}, \tag{3.1}$$

where $\{e_1, \dots, e_{n-k}\}$ is a local orthonormal frame to the fiber at $p$. The fibers are *minimal* submanifolds of $\widetilde{M}$ when $H^{\mathcal{F}} \equiv 0$, and are *totally geodesic* when $\alpha^{\mathcal{F}} \equiv 0$.

We need some formulas relating the derivatives of $\pi$-related objects in $\widetilde{M}$ and $B$. Let us start with the divergence of vector fields.

**Lemma 3.1.** *Let $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$ be a basic vector field, $\pi$-related to $X \in \mathfrak{X}(B)$. The following relation holds between the divergence of $\widetilde{X}$ and $X$ at $x \in B$ and $p \in \mathcal{F}_x$:*

$$\operatorname{div}\widetilde{X}(p) = \operatorname{div}X(x) - \langle \widetilde{X}(p), H^{\mathcal{F}}(p) \rangle.$$

*Proof.* Let $\widetilde{X}_1, \ldots, \widetilde{X}_k, \widetilde{X}_{k+1}, \ldots, \widetilde{X}_n$ be a local orthonormal frame of $T\widetilde{M}$, where $\widetilde{X}_1, \ldots, \widetilde{X}_k$ are basic fields. The equality follows from assertions 1 and 3 in [9, Lemma 1], and formula (3.1) using this frame. $\square$

Given a smooth function $F : B \to \mathbb{R}$ it is easy to see that the gradient of $\widetilde{F}$ is the horizontal lifting of the gradient of $F$, i.e.,

$$\text{grad } \widetilde{F} = \widetilde{\text{grad } F}. \tag{3.2}$$

The Laplace operator in $B$ of a smooth function $F : B \to \mathbb{R}$ and the Laplace operator in $\widetilde{M}$ of its lifting $\widetilde{F} = F \circ \pi$ are related by the following formula.

**Lemma 3.2.** *Let $F : B \to \mathbb{R}$ be a smooth function and set $\widetilde{F} = F \circ \pi$. Then, for all $x \in B$ and all $p \in \mathcal{F}_x$:*

$$\widetilde{\Delta}\widetilde{F}(p) = \Delta F(x) + \langle \text{grad } \widetilde{F}(p), H^{\mathcal{F}}(p) \rangle.$$

*Proof.* It follows easily from (3.2) and Lemma 3.1 applied to the vector fields $\widetilde{X} = \text{grad } \widetilde{F}$ and $X = \text{grad } F$. $\square$

Associated with a Riemannian submersion $\pi : \widetilde{M} \to B$, there are two natural $(1, 2)-$tensors $T$ and $A$ on $\widetilde{M}$, introduced by O'Neill in [9], and defined as follows: for vector fields $X$, $Y$ tangent to $\widetilde{M}$, the tensor $T$ is defined by

$$T_X Y = \left(\widetilde{\nabla}_{X^{\mathcal{V}}} Y^{\mathcal{V}}\right)^{\mathcal{H}} + \left(\widetilde{\nabla}_{X^{\mathcal{V}}} Y^{\mathcal{H}}\right)^{\mathcal{V}}.$$

Note that $\pi : \widetilde{M} \to B$ has totally geodesic fibers if and only if $T$ vanishes identically. The tensor $A$, known as the *integrability tensor*, is defined by

$$A_X Y = \left(\widetilde{\nabla}_{X^{\mathcal{H}}} Y^{\mathcal{H}}\right)^{\mathcal{V}} + \left(\widetilde{\nabla}_{X^{\mathcal{H}}} Y^{\mathcal{V}}\right)^{\mathcal{H}}.$$

The tensor $A$ measures the obstruction to integrability of the horizontal distribution $\mathcal{H}$. In particular, for any horizontal vector field $X$ and any vertical vector field $V$, we have:

$$A_X V = \left(\widetilde{\nabla}_X V\right)^{\mathcal{H}}. \tag{3.3}$$

The following lemma gives useful expressions for the Hessian of the lifting $\widetilde{F} : \widetilde{M} \to \mathbb{R}$ of a smooth function $F : B \to \mathbb{R}$, when we consider horizontal and vertical vector fields.

**Lemma 3.3.** *If $X$ and $Y$ are basic, and $V$ and $W$ are vertical vector fields, we have the following expressions for the Hessian of the lifting $\widetilde{F} = F \circ \pi$ of $F$ to $\widetilde{M}$:*

(a) $\text{Hess}\,\widetilde{F}(X, Y) = \text{Hess}\,F\,(\pi_* X, \pi_* Y) \circ \pi$,
(b) $\text{Hess}\,\widetilde{F}(V, W) = -\left\langle \alpha^{\mathcal{F}}(V, W), \text{grad } \widetilde{F} \right\rangle$,
(c) $\text{Hess}\,\widetilde{F}(X, V) = -\left\langle A_X V, \text{grad } \widetilde{F} \right\rangle$.

*Proof.* The first assertion follows from (3.2) and assertion 3 in [9, Lemma 1]. The second one is a straightforward calculation, and the third assertion follows directly from (3.3). $\square$

## 4. Comparison theorems for Busemann functions

In this section we describe comparison results for the Hessian of Busemann functions on two classes of Riemannian manifolds; both are generalization of the hyperbolic space. These classes of manifolds will be used as the base space of the Riemannian submersions we will consider in our main theorem.

### 4.1. **Busemann functions on manifolds with bounded negative sectional curvature.**

Given $a > 0$, let $\mathbb{H}^k(-a^2)$ denote the $k$-dimensional hyperbolic space with constant sectional curvature $-a^2$. We consider the warped-product model, that is,

$$\mathbb{H}^k(-a^2) = (\mathbb{R}^{k-1} \times \mathbb{R}, h),$$

where

$$h = e^{-2as}\mathrm{d}x^2 + \mathrm{d}s^2.$$

In this model, the curve $\gamma : \mathbb{R} \to \mathbb{H}^k(-a^2)$, given by $\gamma(s) = (x_0, s)$, is a geodesic for any $x_0 \in \mathbb{R}^{k-1}$, and the function $\overline{F} : \mathbb{H}^k(-a^2) \to \mathbb{R}$, given by

$$\overline{F}(x, s) = s, \tag{4.1}$$

is its associated Busemann function. By a direct computation we get

$$\begin{cases} \mathrm{Hess}\overline{F} & = e^{-2as}\mathrm{d}x^2, \\ \Delta\overline{F} & = (k-1)a. \end{cases}$$

Now we will estimate the Hessian of the Busemann function $\overline{F}$ defined in a complete Riemannian manifold $B^k$ with sectional curvature between two negative constants. In order to obtain the Hessian of $\overline{F}$, one takes a point $p$ on a geodesic sphere of radius $r$, and lets the center of the sphere go to infinity. In this case, the sphere converges to a horosphere, and the Hessian of the distance function will converge to the Hessian of the Busemann function. Thus, a comparison theorem for the Hessian of a Busemann function follows from the comparison theorem for the Hessian of the distance function (see [2] for a proof).

**Lemma 4.1.** *Let $B^k$ be a complete Riemannian manifold with sectional curvature $K$ satisfying $-a^2 \le K \le -b^2$ for some constants $a, b > 0$. If $\overline{F} : B \to \mathbb{R}$ is a Busemann function, then*

$$b\|X\|^2 \le \mathrm{Hess}\,\overline{F}(X, X) \le a\|X\|^2$$

*for any vector $X$ orthogonal to* grad $\overline{F}$.

### 4.2. **Busemann functions on a class of warped-product.**

Let $(N^{k-1}, g)$ be a complete Riemannian manifold and let $w : \mathbb{R} \to \mathbb{R}$ be a smooth function. Inspired in the hyperbolic space, we consider the Riemannian warped-product manifold

$$B = (N \times \mathbb{R}, h), \tag{4.2}$$

where

$$h = e^{2w(s)}g + \mathrm{d}s^2.$$

Consider now the Busemann function $\overline{F} : B \to \mathbb{R}$ defined by $\overline{F}(x, s) = s$. As above, a direct computation gives

$$\begin{cases} \mathrm{Hess}\,\overline{F} & = w'(s)e^{2w(s)}g, \\ \Delta\overline{F} & = w'(s)(k-1). \end{cases}$$

In particular we have the following lemma:

**Lemma 4.2.** *Let $B^k$ be a Riemannian manifold as in* (4.2) *and assume that the function $w$ satisfies $b \leq w' \leq a$ for some constants $a, b > 0$. If $\overline{F} : B \to \mathbb{R}$ is the Busemann function defined as above, then*

$$b\|X\|^2 \leq \operatorname{Hess} \overline{F}(X, X) \leq a\|X\|^2$$

*for any vector $X$ orthogonal to* grad $\overline{F}$.

In particular the following consequence will be used in the main theorem.

**Corollary 4.3.** *Under the conditions of Lemma* 4.1 *or Lemma* 4.2 *we have*

$$\Delta \overline{F} \geq (k-1)b.$$

*Remark* 4.4. It is important to point out that Riemannian manifolds given by (4.2) form a wide class. In particular, we may choose the manifold $N$ in such a way that $B$ has positive sectional curvature in some directions (see [11]).

## 5. Main result and examples

In this section, we will apply the previous results in order to get a lower bound estimate for the first eigenvalue of the Laplace operator on submanifolds immersed on Riemannian manifolds, which carries a Riemannian submersion on the two classes of manifolds described as before. In particular, using Lemmas 4.1 and 4.2, and its corollary above, we are able to present a unified proof to both cases.

**Theorem 5.1.** *Let $B^k$ be a complete Riemannian manifold as in Lemma* 4.1 *or as in Lemma* 4.2, *and let $\pi : \widetilde{M}^n \to B^k$ be a Riemannian submersion. Let $M^m$ be a complete Riemannian manifold and let $f : M^m \to \widetilde{M}^n$ be an isometric immersion. Assume that $\overline{F} : B \to \mathbb{R}$ is a Busemann function and consider its lifting $\widetilde{F} : \widetilde{M} \to \mathbb{R}$. If $F = \widetilde{F}|_M$ is its restriction to $M$, then*

$$\Delta F \geq (k-1)b + H^{\mathcal{F}}(\widetilde{F}) - (n-m)\big(a + 2\|A\|_\infty + \|\alpha^{\mathcal{F}}\|_\infty\big) + H(\widetilde{F}).$$

*In particular, if*

$$c = \inf\{(k-1)b - \|H^{\mathcal{F}}\| - (n-m)\big(a + 2\|A\|_\infty + \|\alpha^{\mathcal{F}}\|_\infty\big) - \|H\|\} > 0,$$

*then*

$$\lambda_1(M) \geq \frac{c^2}{4}.$$

*Proof.* From Lemma 3.2 and Corollary 4.3 we have:

$$\widetilde{\Delta}\widetilde{F} = \overline{\Delta}\,\overline{F} + \langle \operatorname{grad} \widetilde{F}, H^{\mathcal{F}} \rangle \geq (k-1)b + H^{\mathcal{F}}(\widetilde{F}). \tag{5.1}$$

On the other hand, from Lemma 2.2 we obtain

$$\widetilde{\Delta}\widetilde{F} = \Delta F + \sum_{i=1}^{n-m} \operatorname{Hess} \widetilde{F}(N_i, N_i) - H(\widetilde{F}), \tag{5.2}$$

where $\{N_1, \ldots, N_{n-m}\}$ is an orthonormal frame of $TM^\perp$. For each $1 \leq i \leq n-m$, we write

$$N_i = N_i^{\mathcal{H}} + N_i^{\mathcal{V}},$$

where $N_i^{\mathcal{H}}$ and $N_i^{\mathcal{V}}$ denote the horizontal and vertical projection of $N_i$ onto $T\widetilde{M}$, respectively. Moreover, since (5.2) is a tensorial equation, we may assume that each $N_i^{\mathcal{H}}$ is basic. Thus, using Lemmas 3.3, 4.1 and 4.2 we get

$$\widetilde{\Delta}\widetilde{F} \leq \Delta F + (n-m)\big(a + 2\|A\|_\infty + \|\alpha^{\mathcal{F}}\|_\infty\big) - H(\widetilde{F}).$$

So, plugging this in (5.1) we obtain

$$\Delta F \geq (k-1)a + H^{\mathcal{F}}(\widetilde{F}) - (n-m)\big(b + 2\|A\|_\infty + \|\alpha^{\mathcal{F}}\|_\infty\big) + H(\widetilde{F}).$$

The result follows from Lemma 2.1. □

5.1. **Lower bounds in warped-products.** Suppose that the ambient space $\widetilde{M}^n = \mathbb{H}^k \times_\rho F^{n-k}$ admits a warped-product structure, where the warped function $\rho$ satisfies $\|\text{grad } \rho\|/\rho \leq 1$. By considering the projection on the first factor $\pi : \mathbb{H}^k \times_\rho F^{n-k} \to \mathbb{H}^k$ as a Riemannian submersion, we have that the tensor $A$ is identically zero, $\|\alpha^{\mathcal{F}}\|_\infty \leq 1$, and in particular $\|H^{\mathcal{F}}\| \leq n-k$.

Let $M^m$ be a complete Riemannian manifold and $f : M^m \to \widetilde{M}^n$ be an isometric immersion such that its mean curvature vector $H$ satisfies $\|H\| \leq \alpha$, where $\alpha$ is a positive constant to be determined . If $\overline{F} : \mathbb{H}^k \to \mathbb{R}$ is the Busemann function given in (4.1), a lower bound estimate for the infimum in (5.1) goes as follows:

$$\begin{aligned} c &= \inf\{k - 1 - \|H^{\mathcal{F}}\| - (n-m)(1 + \|\alpha^{\mathcal{F}}\|_\infty) - \|H\|\} \\ &\geq \inf\{k - 1 - n + k - 2(n-m) - \|H\|\} \\ &= 2(k+m) - 3n - 1 - \alpha. \end{aligned}$$

In particular, $\lambda_1(M) > 0$ if we take $0 < \alpha < 2(k+m) - 3n - 1$.

5.2. **Lower bounds in submersions with totally geodesic fibers.** Let $\widetilde{M}^n$ be a Riemannian manifold with nonpositive sectional curvature and let $\pi : \widetilde{M}^n \to \mathbb{H}^k$ be a Riemannian submersion with totally geodesic fibers. This means that $\alpha^{\mathcal{F}} = 0$, and thus $H^{\mathcal{F}} = 0$. Furthermore, the submersion $\pi$ is integrable in the sense that the horizontal distribution is integrable (cf. [7, Proposition 3.1]). Thus, if $f : M^m \to \widetilde{M}^n$ is an isometric immersion, whose mean curvature vector $H$ satisfies $\|H\| \leq \alpha$ for some positive constant $\alpha < k + m - n - 1$, we have

$$\begin{aligned} c &\geq k - 1 - (n-m) - \|H\| \\ &\geq k + m - n - 1 - \alpha > 0, \end{aligned}$$

and thus $\lambda_1(M) > 0$.

*Remark* 5.2. As suggested by the referee, the complex hyperbolic space and bounded symmetric domains may be interesting examples which are fitted in Theorem 5.1.

## Acknowledgment

The authors are grateful to Professors P. Piccione, H. Rosenberg, and D. Zhou for helpful comments about this work. The authors also thank the referee for valuable comments and suggestions that have improved this article. The first author was partially supported by CNPq/Brazil and FAPEAL/Brazil. The second author was partially supported by Fapesp/Brazil.

## References

[1] Pierre Bérard, Philippe Castillon, and Marcos Cavalcante, *Eigenvalue estimates for hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$ and applications*, Pacific J. Math. **253** (2011), no. 1, 19–35, DOI 10.2140/pjm.2011.253.19. MR2869432

[2] G. Pacelli Bessa, Jorge H. de Lira, Stefano Pigola, and Alberto G. Setti, *Curvature estimates for submanifolds immersed into horoballs and horocylinders*, J. Math. Anal. Appl. **431** (2015), no. 2, 1000–1007, DOI 10.1016/j.jmaa.2015.06.010. MR3365851

[3] G. Pacelli Bessa and J. Fábio Montenegro, *Eigenvalue estimates for submanifolds with locally bounded mean curvature*, Ann. Global Anal. Geom. **24** (2003), no. 3, 279–290, DOI 10.1023/A:1024750713006. MR1996771

[4] Philippe Castillon, *Sur l'opérateur de stabilité des sous-variétés à courbure moyenne constante dans l'espace hyperbolique* (French), Manuscripta Math. **94** (1997), no. 3, 385–400, DOI 10.1007/BF02677861. MR1485444

[5] Leung-Fu Cheung and Pui-Fai Leung, *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*, Math. Z. **236** (2001), no. 3, 525–530, DOI 10.1007/PL00004840. MR1821303

[6] Jaigyoung Choe and Robert Gulliver, *Isoperimetric inequalities on minimal submanifolds of space forms*, Manuscripta Math. **77** (1992), no. 2-3, 169–189, DOI 10.1007/BF02567052. MR1188579

[7] Richard H. Escobales Jr., *Riemannian submersions with totally geodesic fibers*, J. Differential Geom. **10** (1975), 253–276. MR0370423

[8] H. P. McKean, *An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366. MR0266100

[9] Barrett O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469. MR0200865

[10] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; Translated from the Chinese by Ding and S. Y. Cheng; Preface translated from the Chinese by Kaising Tso. MR1333601

[11] Alain R. Veeravalli, *Une remarque sur l'inégalité de McKean* (French, with French summary), Comment. Math. Helv. **78** (2003), no. 4, 884–888, DOI 10.1007/s00014-003-0782-9. MR2016701

Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, CEP 57072-970, Brazil
*Email address*: marcos@pos.mat.ufal.br

ICMC, Universidade de São Paulo, São Carlos, SP, CEP 13561-060, Brazil
*Email address*: manfio@icmc.usp.br