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RELATIVE ENERGY GAP FOR HARMONIC MAPS OF RIEMANN SURFACES INTO REAL ANALYTIC RIEMANNIAN MANIFOLDS

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Abstract. We extend the well-known Sacks–Uhlenbeck energy gap result for harmonic maps from closed Riemann surfaces into closed Riemannian manifolds from the case of maps with small energy (thus near a constant map), to the case of harmonic maps with high absolute energy but small energy relative to a reference harmonic map.

1. Introduction

Let (M, g) and (N, h) be a pair of smooth Riemannian manifolds, with M orientable. One defines the *harmonic map energy function* by

$$\mathscr{E}_{g,h}(f) := \frac{1}{2}\int_M |df|^2_{g,h}\, d\,\mathrm{vol}_g, \tag{1.1}$$

for smooth maps, $f : M \to N$, where $df : TM \to TN$ is the differential map. A map $f \in C^\infty(M; N)$ is *(weakly) harmonic* if it is a *critical point* of $\mathscr{E}_{g,h}$, so $\mathscr{E}'_{g,h}(f)(u) = 0$ for all $u \in C^\infty(M; f^*TN)$, where

$$\mathscr{E}'_{g,h}(f)(u) = \int_M \langle df, u\rangle_{g,h}\, d\,\mathrm{vol}_g\,.$$

The purpose of this article is to prove

Theorem 1 (Relative energy gap for harmonic maps of Riemann surfaces into real analytic Riemannian manifolds)**.** *Let (M, g) be a closed Riemann surface and let (N, h) be a closed, real analytic Riemannian manifold equipped with a real analytic isometric embedding into a Euclidean space, $\mathbb{R}^n$. If $f_\infty \in C^\infty(M; N)$ is a harmonic map, then there is a constant $\epsilon = \epsilon(f_\infty, g, h) \in (0, 1]$ with the following significance. If $f \in C^\infty(M; N)$ is a harmonic map obeying*

$$\|d(f - f_\infty)\|_{L^2(M;\mathbb{R}^n)} + \|f - f_\infty\|_{L^2(M;\mathbb{R}^n)} < \epsilon, \tag{1.2}$$

then $\mathscr{E}_{g,h}(f) = \mathscr{E}_{g,h}(f_\infty)$.

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Remark 1.1 (Generalizations to the case of harmonic maps with potentials). In physics, harmonic maps arise in the context of non-linear sigma models and with such applications in mind, Theorem 1 should admit generalizations to allow, for example, the addition to $\mathscr{E}_{g,h}$ of a real analytic potential function, $V : C^\infty(M;N) \to \mathbb{R}$, in the definition (1.1) of the energy, as explored by Branding [6].

Naturally, Theorem 1 continues to hold if the condition (1.2) is replaced by the stronger (and conformally invariant) hypothesis,

$$\|d(f-f_\infty)\|_{L^2(M;\mathbb{R}^n)} + \|f-f_\infty\|_{L^\infty(M;\mathbb{R}^n)} < \epsilon.$$

Thus, if $\wp$ is any conformal diffeomorphism of (M,g), then the preceding condition on f, f_∞ holds if and only if the harmonic maps $f\circ\wp, f_\infty\circ\wp$ obey

$$\|d(f\circ\wp-f_\infty\circ\wp)\|_{L^2(M;\mathbb{R}^n)} + \|f\circ\wp-f_\infty\circ\wp\|_{L^\infty(M;\mathbb{R}^n)} < \epsilon.$$

Hence, the constants, Z, σ, θ in Theorem 1 are in this sense independent of the action of the conformal group of (M,g) on harmonic maps from M to N.

Theorem 1 may be viewed, in part, as a generalization of the following energy gap result due to Sacks and Uhlenbeck and who do not require that the target manifold be real analytic.

Theorem 1.2 (Energy gap near the constant map)**.** [34, Theorem 3.3] *Let (M,g) be a closed Riemann surface and (N,h) be a closed, smooth Riemannian manifold. Then there is a constant, $\epsilon > 0$, such that if $f \in C^\infty(M;N)$ is harmonic and $\mathscr{E}_{g,h}(f) < \epsilon$, then f is a constant map and $\mathscr{E}_{g,h}(f) = 0$.*

The Sacks–Uhlenbeck Energy Gap Theorem 1.2 has been generalized by Branding [5, Lemma 4.9] and by Jost and his collaborators [9, Proposition 4.2], [25, Proposition 5.2] to the case of Dirac-harmonic pairs. Theorem 1.2 ensures positivity of the constant $\hbar$ in the

Definition 1.3 (Dirac–Planck constant)**.** Let (N,h) be a closed, smooth Riemannian manifold. Then $\hbar$ denotes the *least energy* of a non-constant C^∞ map from $(S^2, g_{\mathrm{round}})$ into (N,h), where g_{round} is the standard round metric of radius one on S^2.

The energy gap near the 'ground state' characterized by the constant maps from $(S^2, g_{\mathrm{round}})$ to (N,h) appears to be unusual in light of the following counter-example due to Li and Wang [27] when (N,h) is only C^∞ rather than real analytic.

Example 1.4 (Non-discreteness of the energy spectrum for harmonic maps from S^2 into a smooth Riemannian manifold with boundary)**.** (See [27, Section 4].) There exists a smooth Riemannian metric h on $N = S^2\times(-1,1)$ such that the energies of harmonic maps from $(S^2, g_{\mathrm{round}})$ to (N,h) have an accumulation point at the energy level 4π, where, g_{round} denotes the standard round metric of radius one.

Thus we would *not* expect Theorem 1 to hold when the hypothesis that (N,h) is real analytic is omitted, except for the case where f_∞ is a constant map. On the other hand, when (N,h) *is* real analytic, one has the following conjecture due to Lin [28].

Conjecture 1.5 (Discreteness for energies of harmonic maps from closed Riemann surfaces into analytic closed Riemannian manifolds)**.** (*Lin* [28, Conjecture 5.7].) *Assume the hypotheses of Theorem* 1 *and that (M,g) is the two-sphere, S^2, with its*

standard, round metric. Then the subset of critical values of the energy function, $\mathscr{E}_{g,h} : C^\infty(S^2; N) \to [0, \infty)$*, is closed and discrete.*

One may therefore view Theorem 1 as supporting evidence of the validity of Conjecture 1.5. In the special case that N is the Lie group $\mathrm{U}(n)$ with $n \geq 2$ and its standard Riemannian metric, Valli [48, Corollary 8] has shown (using ideas of Uhlenbeck [46]) that the energies of harmonic maps from $(S^2, g_{\mathrm{round}})$ into $\mathrm{U}(n)$ are integral multiples of 8π. If (N, h) has non-positive curvature sectional curvature, then Adachi and Sunada [1, Theorem 1] have shown that Conjecture 1.5 holds when (M, g) is any closed Riemann surface.

1.1. **Outline of the article.** In Section 2, we review the Łojasiewicz–Simon gradient inequality for the harmonic map energy function based on results of the author and Maridakis [12] and Simon [38, 39]. In Section 3, we prove certain a priori estimates for the difference of two harmonic maps and, with the aid of the Łojasiewicz–Simon gradient inequality, complete the proof of Theorem 1.

2. Łojasiewicz–Simon gradient inequality for the harmonic map energy function

In this section, we closely follow our treatment of the Łojasiewicz–Simon gradient inequality for abstract and harmonic map energy functions provided by the author and Maridakis in [12, Sections 1.1, 1.2, and 1.4]. Useful references for harmonic maps include Eells and Lemaire [10, 11], Hamilton [16], Hélein [17], Hélein and Wood [18], Jost [20–24], Moser [30], Parker [33], Sacks and Uhlenbeck [34, 35], Schoen [36], Simon [40], Struwe [42], Urakawa [47], and Xin [49], and the citations contained therein.

When clear from the context, we omit explicit mention of the Riemannian metrics g on M and h on N and write $\mathscr{E} = \mathscr{E}_{g,h}$. Although initially defined for smooth maps, the energy function $\mathscr{E}$ in (1.1), extends to the case of Sobolev maps of class $W^{1,2}$. To define the gradient, $\mathscr{M} = \mathscr{M}_{g,h}$, of the energy function $\mathscr{E}$ in (1.1) with respect to the L^2 metric on $C^\infty(M; N)$, we first choose an isometric embedding, $(N, h) \subset \mathbb{R}^n$ for a sufficiently large n (courtesy of the isometric embedding theorem due to Nash [31]), and recall that[1] by [40, Equations (2.2)(i) and (ii)]

$$\begin{aligned}(u, \mathscr{M}(f))_{L^2(M,g)} &:= \mathscr{E}'(f)(u) = \frac{d}{dt}\mathscr{E}(\pi(f + tu))\bigg|_{t=0} \\ &= (u, \Delta_g f)_{L^2(M,g)} \\ &= (u, d\pi_h(f)\Delta_g f)_{L^2(M,g)}\end{aligned}$$

for all $u \in C^\infty(M; f^*TN)$, where π_h is the nearest point projection onto N from a normal tubular neighborhood and $d\pi_h(y) : \mathbb{R}^n \to T_yN$ is an orthogonal projection for all $y \in N$. By [17, Lemma 1.2.4], we have

$$\mathscr{M}(f) = d\pi_h(f)\Delta_g f = \Delta_g f - A_h(f)(df, df), \tag{2.1}$$

as in [40, Equations (2.2)(iii) and (iv)]. Here, A_h denotes the second fundamental form of the isometric embedding, $(N, h) \subset \mathbb{R}^n$, and

$$\Delta_g := -\operatorname{div}_g \operatorname{grad}_g = d^{*,g}d = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\det g}\,\frac{\partial f}{\partial x^\alpha}\right) \tag{2.2}$$

[1] Compare [24, Equations (8.1.10) and (8.1.13)], where Jost uses variations of f of the form $\exp_f(tu)$.

denotes the Laplace–Beltrami operator for (M, g) (with the opposite sign convention to that of [8, Equations (1.14) and (1.33)]) acting on the scalar components f^i of $f = (f^1, \ldots, f^n)$ and the $\{x^\alpha\}$ denote local coordinates on M. As usual, the gradient vector field, $\operatorname{grad}_g f^i \in C^\infty(TM)$, is defined by $\langle \operatorname{grad}_g f^i, \xi\rangle_g := df^i(\xi)$ for all $\xi \in C^\infty(TM)$ and $1 \leq i \leq n$ and the divergence function, $\operatorname{div}_g \xi \in C^\infty(M;\mathbb{R})$, by the pointwise trace, $\operatorname{div}_g \xi := \operatorname{tr}(\eta \mapsto \nabla^g_\xi \eta)$, for all $\eta \in C^\infty(TM)$.

Given a smooth map, $f : M \to N$, an isometric embedding, $(N, h) \subset \mathbb{R}^n$, a non-negative integer k, and constant $p \in [1, \infty)$, we define the Sobolev norms,

$$\|f\|_{W^{k,p}(M;\mathbb{R}^n)} := \left(\sum_{i=1}^n \|f^i\|^p_{W^{k,p}(M;\mathbb{R}^n)}\right)^{1/p},$$

with

$$\|f^i\|_{W^{k,p}(M;\mathbb{R}^n)} := \left(\sum_{j=0}^k \int_M |(\nabla^g)^j f^i|^p \, d\operatorname{vol}_g\right)^{1/p},$$

where ∇^g denotes the Levi–Civita connection on TM and all associated bundles (that is, T^*M and their tensor products), and if $p = \infty$, we define

$$\|f\|_{W^{k,\infty}(M;\mathbb{R}^n)} := \sum_{i=1}^n \sum_{j=0}^k \operatorname{ess\,sup}_M |(\nabla^g)^j f^i|.$$

If $k = 0$, then we denote $\|f\|_{W^{0,p}(M;\mathbb{R}^n)} = \|f\|_{L^p(M;\mathbb{R}^n)}$. For $p \in [1, \infty)$ and non-negative integers k, we use [2, Theorem 3.12] (applied to $W^{k,p}(M;\mathbb{R}^n)$ and noting that M is a closed manifold) and Banach space duality to define

$$W^{-k,p'}(M;\mathbb{R}^n) := \left(W^{k,p}(M;\mathbb{R}^n)\right)^*,$$

where $p' \in (1, \infty]$ is the dual exponent defined by $1/p + 1/p' = 1$. Elements of the continuous Banach dual space, $(W^{k,p}(M;\mathbb{R}^n))^*$, may be characterized via [2, Section 3.10] as distributions in the Schwartz space, $\mathscr{D}'(M;\mathbb{R}^n)$ [2, Section 1.57].

Spaces of Hölder continuous maps, $C^{k,\lambda}(M;N)$ for $\lambda \in (0,1)$ and integers $k \geq 0$, and norms,

$$\|f\|_{C^{k,\lambda}(M;\mathbb{R}^n)},$$

may be defined as in [2, Section 1.29].

We note that if (N, h) is real analytic, then the isometric embedding, $(N, h) \subset \mathbb{R}^n$, may also be chosen to be analytic by the analytic isometric embedding theorem due to Nash [32], with a simplified proof due to Greene and Jacobowitz [15]).

Definition 2.1 (Harmonic map)**.** (See [17, Definition 1.4.9].) A map $f \in W^{1,2}(M;N)$ is called *weakly harmonic* if it is a critical point of the L^2-energy functional (1.1), that is,

$$\mathscr{E}'(f)(u) = 0 \quad \forall\, u \in C^\infty(M; f^*TN),$$

and a map $f \in W^{2,p}(M;N)$, for $p \in [1, \infty]$, is called *harmonic* if

$$\Delta_g f - A_h(df, df) = 0 \quad \text{a.e. on } M. \tag{2.3}$$

A well-known result due to Hélein [17, Theorem 4.1.1] tells us that if M has dimension $d = 2$, then $f \in C^\infty(M;N)$; for $d \geq 3$, regularity results are far more limited — see, for example, [17, Theorem 4.3.1] due to Bethuel. From [12], we recall the following theorem.

Theorem 2.2 (Łojasiewicz–Simon gradient inequality for the energy function for maps between pairs of Riemannian manifolds)**.** (*See* [12, Theorem 5].) *Let $d \geq 2$ and $k \geq 1$ be integers and $p \in (1, \infty)$ be such that $kp > d$. Let (M, g) and (N, h) be closed, smooth Riemannian manifolds, with M of dimension d. If (N, h) is real analytic (respectively, C^∞) and $f \in W^{k,p}(M; N)$, then the gradient map $\mathscr{M}$ in* (2.1) *for the energy function, $\mathscr{E} : W^{k,p}(M; N) \to \mathbb{R}$, in* (1.1)*,*

$$W^{k,p}(M; N) \ni f \mapsto \mathscr{M}(f) \in W^{k-2,p}(M; f^*TN) \subset W^{k-2,p}(M; \mathbb{R}^n),$$

is a real analytic (respectively, C^∞) map of Banach spaces. If (N, h) is real analytic and $f_\infty \in W^{k,p}(M; N)$ is a weakly harmonic map, then there are positive constants $Z \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on f_∞, g, h, k, p, with the following significance. If $f \in W^{k,p}(M; N)$ obeys

$$\|f - f_\infty\|_{W^{k,p}(M;\mathbb{R}^n)} < \sigma, \tag{2.4}$$

then the gradient $\mathscr{M}$ in (2.1) *of the harmonic map energy function $\mathscr{E}$ in* (1.1) *obeys*

$$\|\mathscr{M}(f)\|_{W^{k-2,p}(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta. \tag{2.5}$$

Remark 2.3 (On the hypotheses of Theorem 2.2). When $k = d$ and $p = 1$, then $W^{d,1}(M; \mathbb{R}) \subset C(M; \mathbb{R})$ is a continuous embedding by [2, Theorem 4.12] and $W^{d,1}(M; \mathbb{R})$ is a Banach algebra by [2, Theorem 4.39]. In particular, $W^{d,1}(M; N)$ is a real analytic Banach manifold by [12, Proposition 3.2] and the harmonic map energy function, $\mathscr{E} : W^{d,1}(M; N) \to \mathbb{R}$, is real analytic by [12, Proposition 3.5]. However, the operator $\mathscr{M}'(f_\infty) : W^{d,1}(M; f_\infty^*TN) \to W^{d-2,1}(M; f_\infty^*TN)$ may not be Fredholm.

Theorem 2.2 extends a version of the Łojasiewicz–Simon gradient inequality that is stated by Simon in [39, Equation (4.27)] and can be derived from his more general [38, Theorem 3].

Theorem 2.4 (Łojasiewicz–Simon gradient inequality for the energy function for maps between pairs of Riemannian manifolds)**.** (*See* [12, Corollary 6], [38, Theorem 3], [39, Equation (4.27)].) *Let $d \geq 2$ and $\lambda \in (0, 1)$ be constants, let (M, g) be a closed, smooth Riemannian manifold of dimension d, and (N, h) is a closed, real analytic Riemannian manifold. If $f_\infty \in C^{2,\lambda}(M; N)$ is a harmonic map, then there are positive constants $Z \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on f_∞, g, h, λ, with the following significance. If $f \in C^{2,\lambda}(M; N)$ obeys*

$$\|f - f_\infty\|_{C^{2,\lambda}(M;\mathbb{R}^n)} < \sigma, \tag{2.6}$$

then the gradient $\mathscr{M}$ in (2.1) *of the harmonic map energy function $\mathscr{E}$ in* (1.1) *obeys*

$$\|\mathscr{M}(f)\|_{L^2(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta. \tag{2.7}$$

Remark 2.5 (Other versions of the Łojasiewicz–Simon gradient inequality for the harmonic map energy function). Topping [45, Lemma 1] proved a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to S^2$, with small energy, with the latter criterion replacing the usual small $C^{2,\lambda}(M; \mathbb{R}^n)$ norm criterion of Simon for the difference between a map and a critical point. Topping's result is generalized by Liu and Yang in [29, Lemma 3.3]. Kwon [26, Theorem 4.2] obtains a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to N$, that are $W^{2,p}(S^2; \mathbb{R}^n)$-close to a harmonic map, with $1 < p \leq 2$.

When $d = 2$ in the hypotheses of Theorem 2.2, the reader will note that the two cases that are most directly applicable to a proof of Theorem 1 are omitted, namely the cases $k = 2$ and $p = 1$ or $k = 1$ and $p = 2$, which are both critical since $kp = d$. We shall briefly comment on each of these two cases.

When $d = 2$, $k = 1$, and $p = 2$, it appears very difficult to verify the hypotheses of [12, Theorem 2]. The analytical difficulties are very much akin to those confronted by Hélein [17] in his celebrated proof of smoothness of weakly harmonic maps from Riemann surfaces. However, it is unclear that Hélein's methods could be used to extend Theorem 2.2 to the case $d = 2$, $k = 1$, and $p = 2$.

Similarly, when $d = 2$, $k = 2$, and $p = 1$, it is very difficult to verify the hypotheses of [12, Theorem 2]. One might speculate that a version of Theorem 2.2 could hold if the role of the pair of Sobolev spaces, $W^{2,1}(M; f_\infty^* TN)$ and $L^1(M; f_\infty^* TN)$, were replaced by suitably defined *local Hardy spaces*. We refer the reader to Semmes [37], Stein [41], and Taylor [44] for introductions to Hardy spaces of functions on Euclidean space and to Hélein [17] for their application to the problem of regularity for weakly harmonic maps from Riemann surfaces. Auscher, McIntosh, and Morris [3], Carbonaro, McIntosh, and Morris [7], and Taylor [43] provide definitions of local Hardy spaces on Riemannian manifolds. However, the analytical difficulties appear formidable in any such approach.

Fortunately, in our proof of Theorem 1, we can apply Theorem 2.2 with *non-critical* Sobolev exponents, namely $d = 2$, $k = 2$, and $p \in (1, \infty)$ by exploiting certain a priori estimates for harmonic maps similar to those used by Sacks and Uhlenbeck [34].

Theorem 2.2 is proved by the author and Maridakis in [12] as a consequence of a more general abstract Łojasiewicz–Simon gradient inequality for an analytic function on a Banach space, namely [12, Theorem 2], while Theorem 2.4 may be deduced as a consequence of [12, Theorem 2].

To state the abstract [12, Theorem 2], we let $\mathscr{X}$ be a Banach space and let $\mathscr{X}^*$ denote its continuous dual space. We call a bilinear form, $b : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$, *definite* if $b(x, x) \neq 0$ for all $x \in \mathscr{X} < \{0\}$. We say that a continuous *embedding* of a Banach space into its continuous dual space, $\jmath : \mathscr{X} \to \mathscr{X}^*$, is *definite* if the pullback of the canonical pairing, $\mathscr{X} \times \mathscr{X} \ni (x, y) \mapsto \langle x, \jmath(y)\rangle_{\mathscr{X}\times\mathscr{X}^*} \to \mathbb{R}$, is a definite bilinear form. (This hypothesis on the continuous embedding, $\mathscr{X} \subset \mathscr{X}^*$, is easily achieved given a continuous embedding of $\mathscr{X}$ into a Hilbert space $\mathscr{H}$ but the increased generality is often convenient.)

Definition 2.6 (Gradient map)**.** (See [4, Section 2.5], [19, Definition 2.1.1].) Let $\mathscr{U} \subset \mathscr{X}$ be an open subset of a Banach space, $\mathscr{X}$, and let $\tilde{\mathscr{X}}$ be a Banach space with continuous embedding, $\tilde{\mathscr{X}} \subseteqq \mathscr{X}^*$. A continuous map, $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$, is called a *gradient map* if there exists a C^1 function, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$, such that

$$\mathscr{E}'(x)v = \langle v, \mathscr{M}(x)\rangle_{\mathscr{X}\times\mathscr{X}^*} \quad \forall\, x \in \mathscr{U}, \quad v \in \mathscr{X}, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_{\mathscr{X}\times\mathscr{X}^*}$ is the canonical bilinear form on $\mathscr{X} \times \mathscr{X}^*$. The real-valued function, $\mathscr{E}$, is called a *potential* for the gradient map, $\mathscr{M}$.

Theorem 2.7 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)**.** (*See* [12, Corollary 3].) *Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces with continuous embeddings, $\mathscr{X} \subset \tilde{\mathscr{X}} \subset \mathscr{X}^*$, and such that the embedding, $\mathscr{X} \subset \mathscr{X}^*$, is definite. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, let $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function with real analytic gradient map, $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$, and let $x_\infty \in \mathscr{U}$ be a critical point of*

$\mathscr{E}$, that is, $\mathscr{M}(x_\infty) = 0$. If $\mathscr{M}'(x_\infty) : \mathscr{X} \to \tilde{\mathscr{X}}$ is a Fredholm operator with index zero, then there are constants, $Z \in (0,\infty)$, and $\sigma \in (0,1]$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{2.9}$$

then

$$\|\mathscr{M}(x)\|_{\tilde{\mathscr{X}}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{2.10}$$

Theorem 2.2 follows from Theorem 2.7 by choosing

$$\mathscr{X} = W^{k,p}(M; f_\infty^* TN) \quad \text{and} \quad \tilde{\mathscr{X}} = W^{k-2,p}(M; f_\infty^* TN).$$

Theorem 2.4 follows from Theorem 2.2 when one can choose $k \geq 1$ and $p \in (1,\infty)$ with $kp > d$ so that there are continuous Sobolev embeddings, $C^{2,\lambda}(M; f_\infty^* TN) \subset W^{k,p}(M; f_\infty^* TN)$, and $L^2(M; f_\infty^* TN) \subset W^{k-2,p}(M; f_\infty^* TN)$. For example, if $d = 2$, then $k = p = 2$ will do. For $d \geq 2$, we may choose $k = 1$ and $d < p < \infty$ provided $L^2(M;\mathbb{R}) \subset W^{-1,p}(M;\mathbb{R})$ is a continuous embedding or, equivalently, $W^{1,p'}(M;\mathbb{R}) \subset L^2(M;\mathbb{R})$ is a continuous embedding, where $p' = p/(p-1)$. According to [2, Theorem 4.12] when $1 \leq p' < d$, the latter embedding is continuous if $(p')^* = dp'/(d-p') = dp/(d(p-1)-p) \geq 2$. But we must choose $p > d$ when $k = 1$ in Theorem 2.2 and if $p = d$, then $dp/(d(p-1)-p) = d^2/(d(d-1)-d) = d/(d-2) \geq 2$ implies $d \geq 2d-4$ or $d \leq 4$. Hence, Theorem 2.2 implies Theorem 2.4 when $d = 2, 3$ (the case $d = 4$ is excluded since $p > d$ leads to $d < 4$ in the preceding inequalities). For arbitrary $d \geq 2$, another abstract Łojasiewicz–Simon gradient inequality [19, Theorem 2.4.2 (i)] due to Huang implies Theorem 2.4 with the choices

$$\mathscr{X} = C^{2,\lambda}(M; f_\infty^* TN), \quad \tilde{\mathscr{X}} = C^\lambda(M; f_\infty^* TN), \qquad \mathscr{H} = L^2(M; f_\infty^* TN),$$

and $\mathscr{H}_{\mathscr{A}} = W^{2,2}(M; f_\infty^* TN)$ with $\mathscr{A} = \Delta_g + 1$ in [19, Hypotheses (H1)–(H3), pages 34–35]. We refer the reader to [13] for an exposition of Huang's [19, Theorem 2.4.2 (i)].

Alternatively, our [12, Theorem 3] implies Theorem 2.4, as we show in the proof of [12, Corollary 6].

3. A priori estimate for the difference of two harmonic maps

In this section, we give two proofs of Theorem 1, based on Theorems 2.2 and 2.4, respectively. We begin with the

Lemma 3.1 (A priori $W^{2,p}$ estimate for the difference of two harmonic maps)**.** *Let (M, g) be a closed Riemann surface, let (N, h) be a closed, smooth Riemannian manifold, and let $p \in (1, 2]$ be a constant. Then there is a constant $C = C(g, h, p) \in [1,\infty)$ with the following significance. If $f, f_\infty \in C^\infty(M; N)$ are harmonic maps and $q = 2p/(2-p) \in (2,\infty]$, then*

$$\begin{aligned} &\|f - f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} \\ &\qquad \leq C\left(\|df\|_{L^q(M;\mathbb{R}^n)} + \|df_\infty\|_{L^q(M;\mathbb{R}^n)} + 1\right)\|f - f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)}. \end{aligned} \tag{3.1}$$

Proof. Because f and f_∞ are harmonic, equation (2.3) implies that

$$\begin{aligned} \Delta_g f - A_h(df, df) &= 0, \\ \Delta_g f_\infty - A_h(df_\infty, df_\infty) &= 0, \end{aligned}$$

and, therefore,

$$\Delta_g(f-f_\infty) - A_h(df, d(f-f_\infty)) - A_h(d(f-f_\infty), df_\infty) = 0. \tag{3.2}$$

Because $1/p = 1/2 + 1/q$ by hypothesis, the preceding equality yields the estimate,

$$\|\Delta_g(f-f_\infty)\|_{L^p(M;\mathbb{R}^n)} \leq C\left(\|df\|_{L^q(M;\mathbb{R}^n)} + \|df_\infty\|_{L^q(M;\mathbb{R}^n)}\right)\|d(f-f_\infty)\|_{L^2(M;\mathbb{R}^n)},$$

with $C = C(h) \in [1,\infty)$. The standard a priori $W^{2,p}$ estimate for an elliptic, linear, scalar, second-order partial differential operator over a bounded domain in Euclidean space [14, Theorem 9.13] yields the bound,

$$\|f-f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} \leq C\left(\|\Delta_g(f-f_\infty)\|_{L^p(M;\mathbb{R}^n)} + \|f-f_\infty\|_{L^p(M;\mathbb{R}^n)}\right),$$

for a constant $C = C(g,p) \in [1,\infty)$. Combining the preceding two inequalities gives

$$\begin{aligned}\|f-f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} \leq C\left(\|df\|_{L^q(M;\mathbb{R}^n)} + \|df_\infty\|_{L^q(M;\mathbb{R}^n)}\right)\|d(f-f_\infty)\|_{L^2(M;\mathbb{R}^n)} \\ + C\|f-f_\infty\|_{L^p(M;\mathbb{R}^n)},\end{aligned}$$

for $C = C(g,h,p) \in [1,\infty)$. Since $p \leq 2$, this yields the desired estimate. □

Lemma 3.2 (A priori $W^{1,q}$ estimate for a harmonic map)**.** *Let (M,g) be a closed Riemann surface, let (N,h) be a closed, smooth Riemannian manifold, and let $q \in (2,\infty)$ be a constant. Then there is a constant $\epsilon = \epsilon(g,h,q) \in (0,1]$ with the following significance. If $f, f_\infty \in C^\infty(M;N)$ are harmonic maps obeying* (1.2)*, then*

$$\|f\|_{W^{1,q}(M;\mathbb{R}^n)} \leq 1 + 3\|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)}. \tag{3.3}$$

Proof. For $p \in (1,2)$ defined by $p^* := 2p/(2-p) = q$, we observe that $W^{1,p}(M;\mathbb{R}) \subset L^q(M;\mathbb{R})$ is a continuous Sobolev embedding by [2, Theorem 4.12]. Hence, the estimate (3.1) yields

$$\begin{aligned}\|f-f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)} &\leq C\|f-f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} \\ &\leq C\left(\|df\|_{L^q(M;\mathbb{R}^n)} + \|df_\infty\|_{L^q(M;\mathbb{R}^n)} + 1\right)\|f-f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|_{W^{1,q}(M;\mathbb{R}^n)} &\leq \|f-f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)} + \|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)} \\ &\leq C\|df\|_{L^q(M;\mathbb{R}^n)}\|f-f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)} \\ &\quad + C\left(\|df_\infty\|_{L^q(M;\mathbb{R}^n)} + 1\right)\|f-f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)} + \|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)}.\end{aligned}$$

Choosing $\epsilon = \epsilon(g,h,q) \leq 1/(2C)$ in (1.2) and applying rearrangement in the preceding inequality yields

$$\|f\|_{W^{1,q}(M;\mathbb{R}^n)} \leq \|df_\infty\|_{L^q(M;\mathbb{R}^n)} + 1 + 2\|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)},$$

as desired. □

It remains to complete the

Proof of Theorem 1 *using Theorem* 2.2. Combining the inequalities (3.1) and (3.3) yields

$$\|f-f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} \leq C\left(1 + \|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)}\right)\|f-f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)}.$$

We now fix $p \in (1,2)$ and $q = 2p/(2-p)$ (say with $p = 3/2$) and choose $\epsilon = \epsilon(f_\infty, g, h) \in (0,1]$ in (1.2) small enough that

$$\epsilon C \left(1 + \|f_\infty\|_{W^{1,q}(M;\mathbb{R}^n)}\right) \leq \sigma,$$

where the constant $\sigma \in (0,1]$ is as in Theorem 2.2. Consequently,

$$\|f - f_\infty\|_{W^{2,p}(M;\mathbb{R}^n)} < \sigma,$$

and the hypothesis (2.4) is satisfied. The Łojasiewicz–Simon gradient inequality (2.5) (with $d = k = 2$) in Theorem 2.2 therefore yields

$$\|\mathscr{M}(f)\|_{L^p(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta.$$

But $\mathscr{M}(f) = 0$ since f is harmonic and thus $\mathscr{E}(f) = \mathscr{E}(f_\infty)$. □

It is possible to give an alternative proof of Theorem 1 using Theorem 2.4 with the aid of an elliptic bootstrapping argument to fulfill the stronger hypothesis (2.6). We first observe that Lemma 3.1 can be strengthened to give

Lemma 3.3 (A priori $W^{k,p}$ estimate for the difference of two harmonic maps)**.** *Let (M,g) be a closed Riemann surface, let (N,h) be a closed, smooth Riemannian manifold, let $p \in (1,\infty)$ be a constant, let $k \geq 2$ be an integer, and let $f_\infty \in C^\infty(M;N)$ be a harmonic map. Then there are constants $\epsilon = \epsilon(f_\infty, g, h, k, p) \in (0,1]$ and $C = C(f_\infty, g, h, k, p) \in [1,\infty)$ with the following significance. If $f \in C^\infty(M;N)$ is a harmonic map that obeys* (1.2)*, then*

$$\|f - f_\infty\|_{W^{k,p}(M;\mathbb{R}^n)} \leq C\|f - f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)}. \tag{3.4}$$

Proof. For $k = 2$ and $p \in (1,2]$, the conclusion follows by combining (3.1) and (3.3). For $k \geq 3$ and $p \in (1,\infty)$, the conclusion follows by taking derivatives of (3.2) and applying a standard elliptic bootstrapping argument. □

We can now give the

Proof of Theorem 1 *using Theorem* 2.4. For $p \in (1,\infty)$ and $\lambda \in (0,1)$ and large enough $k = k(g,p,\lambda) \geq 2$, there is a continuous Sobolev embedding, $W^{k,p}(M;\mathbb{R}^n) \subset C^{2,\lambda}(M;\mathbb{R}^n)$, and thus a constant $C = C(g,k,p,\lambda) \in [1,\infty)$ such that

$$\|f - f_\infty\|_{C^{2,\lambda}(M;\mathbb{R}^n)} \leq C\|f - f_\infty\|_{W^{k,p}(M;\mathbb{R}^n)}.$$

Combining the preceding inequality with (3.4) yields the bound

$$\|f - f_\infty\|_{C^{2,\lambda}(M;\mathbb{R}^n)} \leq C\|f - f_\infty\|_{W^{1,2}(M;\mathbb{R}^n)},$$

for a constant $C = C(f_\infty, g, h, k, p, \lambda) \in [1,\infty)$.

We now fix k, p, λ and choose $\epsilon = \epsilon(f_\infty, g, h) \in (0,1]$ in (1.2) small enough that $C\epsilon \leq \sigma$, where the constant $\sigma \in (0,1]$ is as in Theorem 2.4. Consequently,

$$\|f - f_\infty\|_{C^{2,\lambda}(M;\mathbb{R}^n)} < \sigma,$$

and the hypothesis (2.6) is satisfied. The Łojasiewicz–Simon gradient inequality (2.7) in Theorem 2.4 therefore yields

$$\|\mathscr{M}(f)\|_{L^2(M;f^*TN)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta.$$

Again, $\mathscr{M}(f) = 0$ since f is harmonic and thus $\mathscr{E}(f) = \mathscr{E}(f_\infty)$. □

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