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THE LINEAR REQUEST PROBLEM

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Abstract. We propose a simple approach to a problem introduced by Galatolo and Pollicott, which can be called a *linear request* problem; in its general formulation, it consists of finding a first-order perturbation of a dynamical system such that its physical measure changes in a prescribed direction. Our method needs the physical measure to be absolutely continuous with smooth positive density: instead of using transfer operators, we use the well-known fact that a change in the density of a smooth measure can be reproduced by pushing forward along a well-chosen vector field. This implies that restricting to perturbations by infinitesimal conjugacy already yields a solution to the linear request problem, allowing us to work in any dimension and to dispense from additional dynamical hypotheses. In particular, we don't need to assume hyperbolicity to obtain a solution, but if the map is Anosov, we obtain the existence of an infinite-dimensional space of solutions.

Let M be a compact n-manifold and let $T : M \to M$ be a smooth map, seen as a discrete-time dynamical system (by "smooth" we shall always mean C^∞, but see Remark 6). The study of invariant measures of T, and most particularly of any "best" invariant measure, is a research area with a long and rich history. Often, one considers "best" the *physical* invariant measures, i.e., those whose basin of attraction (points for which the average along the orbit of any continuous function converges to the integral of the function) has positive volume. Physical measures can take quite general forms, but in this work we will mainly consider the particular case of absolutely continuous invariant measures (acim) with smooth positive density (identified with volume forms).

The *linear response theory* (see e.g. [Rue09, BS12]) is typically concerned with the following question: assuming uniqueness of the physical measure, how does it change (to first order) when the map T is perturbed? In the particular case of a smooth acim, the question becomes: if $(T_t)_{t\in(-\varepsilon,\varepsilon)}$ is a family of maps with $T_0 = T$, differentiable at $t = 0$, such that each T_t preserves a smooth measure ω_t, can we differentiate ω_t with respect to t and express $\dot\omega_0 := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\omega_t$ in terms of $\dot T_0 := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} T_t$?

A recent article by Galatolo and Pollicott [GP17] proposes to study the opposite direction, which we propose to call the *linear request problem*:

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Question 1 (Linear request problem in the smooth setting)**.** Given a smooth function $\rho : M \to \mathbb{R}$ of vanishing integral,[1] can we find a perturbation $(T_t)_{t\in(-\varepsilon,\varepsilon)}$ of T, differentiable at $t = 0$ and preserving a family of smooth measure $(\omega_t)_{t\in(-\varepsilon,\varepsilon)}$ such that $\dot{\omega}_0 = \rho\omega$? Can we then express the possible values of $\dot{T}_0$ in terms of ρ?

The possible values of $\dot{T}_0$ will be called *solutions* of the linear request problem (with parameter ρ).

Question 2. When the previous question has a positive answer, can we find an "optimal" solution?

In [GP17] some answers to these questions are provided for expanding maps of the circle. The main goal of this note is to observe that Question 1 has a positive answer in a very general setting, without dynamical hypothesis and in every dimension.

Theorem 3. *Let $T : M \to M$ be a smooth map acting on a compact smooth Riemannian manifold, preserving a smooth volume form ω. Let $\rho : M \to \mathbb{R}$ be a smooth function such that $\int_M \rho\omega = 0$.*

There exists a deformation $(T_t)_{(-\varepsilon,\varepsilon)}$ of T that is differentiable at $t = 0$ and preserves smooth volume forms ω_t, such that $\dot{\omega}_0 = \rho\omega$. Moreover one can ask all T_t to be smoothly conjugate to T.

This follows from well-known facts in differential geometry, and the only faint bit of novelty is in the observation that one can restrict to conjugate deformations, of the form $T_t = \varphi^t \circ T \circ \varphi^{-t}$ where $(\varphi^t)_{t\in(-\varepsilon,\varepsilon)}$ is the flow of a vector field.

In Section 1 we give the (very short) proof of Theorem 3 and discuss the meaning of differentiability for families of maps (Section 1.1) and for families of measures (Section 1.3). Then in Section 2 we discuss uniqueness of the infinitesimal conjugacy used to prove Theorem 3 and prove in particular that for Anosov maps the method of Theorem 3 provides an infinite-dimensional set of solutions to the linear request problem. In Section 3 we discuss an optimality question, which leads us to consider a simple and classical PDE. In Section 4 we treat a simple example and compare to [GP17].

1. Solving the linear request problem by infinitesimal conjugacy

1.1. **Differentiating families of maps.**

Let us first define properly what it means for a family $(T_t)_{t\in(-\varepsilon,\varepsilon)}$ to be differentiable at $t = 0$. Galatolo and Pollicott work on the circle and implicitly use its parallelism (all tangent spaces can be identified), which is not possible on a general manifold. Pointwise, we want to ask that for each $x \in M$, the curve $(T_t(x))_{t\in(-\varepsilon,\varepsilon)}$ is differentiable at $t = 0$; moreover we want the derivative of this curve to depend smoothly on x. Let us stress that $\dot{T}_0(x) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} T_t(x)$ is an element of $T_{T(x)}M$, not of T_xM, and $\dot{T}_0$ is thus not a vector field.[2] We will thus consider the set

$$\Gamma_T(M) = \big\{Z : M \to TM \text{ smooth} \,\big|\, Z_x \in T_{T(x)}M \quad \forall x \in M\big\}$$

and say that a family $(T_t)_{t\in(-\varepsilon,\varepsilon)}$ of smooth maps $M \to M$ is *differentiable* at $t = 0$ if $\dot{T}_0(x)$ is defined for all x and $\dot{T}_0 \in \Gamma_T(M)$.

[1] The condition $\int_M \rho\omega = 0$ is needed whenever invariant measures are normalized to have fixed total mass.

[2] Beware not to confuse the letter 'T' in the tangent space TM with the map T.

Remark 4. Sometimes one considers perturbations of the form $\dot{T}_0(x) = X_{T(x)}$ for some vector field X. This is equivalent to considering $\dot{T}_0 \in \Gamma_T(M)$ if T is invertible, but is less general otherwise as the images of any $x, y \in M$ such that $T(x) = T(y)$ would be asked to be perturbed identically.

We will be concerned with perturbations by "infinitesimal conjugacy". If X is a smooth vector field on M, we can consider its flow $(\varphi^t)_{t\in\mathbb{R}}$ and the family $T_t := \varphi^t \circ T \circ \varphi^{-t}$. Then a direct computation shows that $(T_t)_{t\in\mathbb{R}}$ is differentiable at 0:

$$\begin{aligned}\varphi^t \circ T \circ \varphi^{-t}(x) &= \varphi^t \circ T\big(x - tX_x + o(t)\big)\\ &= \varphi^t\Big(T(x) - tD_xT(X_x) + o(t)\Big)\\ &= T(x) - tD_xT(X_x) + tX_{T(x)} + o(t)\end{aligned}$$

so that $\dot{T}_0(x) = -D_xT(X_x) + X_{T(x)}$, which we also write $\dot{T}_0 = -DT(X) + X_T$. This is naturally an element of $\Gamma_T(M)$, as it should be. The element $Z = -DT(X) + X_T$ of $\Gamma_T(M)$ is said to be the *infinitesimal conjugacy* induced by the vector field X.

1.2. **Proof of Theorem 3.** Observe that since T preserves ω, T_t then preserves $\omega_t := \varphi^t_*\omega$. But

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi^t_*\omega = -\mathscr{L}_X\omega, \tag{1}$$

where $\mathscr{L}$ denotes the Lie derivative. To prove Theorem 3, we thus only have to check that for all ρ such that $\int_M \rho\omega = 0$, there is a vector field X such that $\mathscr{L}_X\omega = -\rho\omega$. This is well-known (see e.g. the proof of Moser's theorem in [KH95, Theorem 5.1.27, page 195]), but let us recall the argument for the sake of completeness.

Since ω is an n-form, the Cartan formula reads $\mathscr{L}_X\omega = \mathrm{d}(i_X\omega)$ where $i_X\omega$ is the $(n-1)$-form $\omega(X;\cdot;\cdots;\cdot)$ obtained by contraction with X. Since M is compact, its top-dimensional cohomology is 1-dimensional, generated by the class of ω (or any volume form), and every n-form of vanishing total integral is exact. This means that there exists an $(n-1)$-form θ such that $\mathrm{d}\theta = -\rho\omega$. Now, ω being a volume form, it is non-degenerate, which means that any $(n-1)$-form can be obtained by contracting ω with a well-chosen vector field; in particular, there must exist a vector field X such that $i_X\omega = \theta$. Putting all this together, we get the desired conclusion:

$$\dot{\omega}_0 = -\mathscr{L}_X\omega = -\,\mathrm{d}(i_X\omega) = -\,\mathrm{d}\theta = \rho\omega.$$

Remark 5. Using Moser's theorem [Mos65] instead of its first-order version, one sees that for all volume forms ω', there is a conjugate of T that preserves ω'.

Remark 6. It could be asked what happens in lower regularity, e.g. C^k or $C^{k,\alpha}$. We do not enter into these details, since the strategy would be the very same, only keeping track of the available regularity for the various objects ω, θ, X, etc.

1.3. **Differentiating families of measures.** One could ask whether the same kind of method could be used for more general physical measures. While we used the smooth structure on the space of n-forms to define differentiability, when considering a family $(\mu_t)_{t\in(-\varepsilon,\varepsilon)}$ of more general measures the meaning of differentiability becomes central to the problem (be it linear request or linear response).

In the linear response literature it is common to define differentiability by using test functions as coordinates: one fixes a space $\mathcal{X}$ of functions $M \to \mathbb{R}$ and says that the family $(\mu_t)_{t\in(-\varepsilon,\varepsilon)}$ is differentiable at $t=0$ if $t \mapsto \int f \,\mathrm{d}\mu_t$ is differentiable at $t=0$ for all $f \in \mathcal{X}$. The larger $\mathcal{X}$ is, the stronger the definition. In the case of smooth measures, these definitions are satisfied when $\mu_t = \rho_t\omega$ with $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ a differentiable family of smooth functions, as soon as the regularity of test functions is sufficient to use the dominated convergence theorem. One can call these notions "vertical differentiability": they record how the amount of mass near each point changes.

Another notion, appearing notably in optimal transportation and which can be called "horizontal differentiability", calls differentiable a change of measure corresponding to the mass near each point moving in a certain direction: one asks for μ_t to be $o(t)$-close (in the so-called Wasserstein metric W_2) to the push-forward of μ_0 in the direction of some vector field X (which can be considered the derivative of $(\mu_t)_{t\in(-\varepsilon,\varepsilon)}$). It is in effect possible to make sense out of this definition for general probability measures, and the vector field X only needs to be L^2 with respect to μ_0 (and can even be multivalued; actually, the vector field should in general be replaced by a measure on the total space of the tangent bundle). While we will not enter into much detail, we want to stress that the question of what one means by a differentiable family of measures can have different answers.

The *continuity equation* generalizes equation (1) to horizontally differentiable families of general probability measures: if X is the "derivative" of $(\mu_t)_{t\in(-\varepsilon,\varepsilon)}$ for all smooth test functions $\varphi : M \to \mathbb{R}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \varphi \,\mathrm{d}\mu_t = \int d\varphi(X)\,\mathrm{d}\mu_0.$$

It translates horizontal differentiability into vertical differentiability (for regular enough test functions), which is thus in general weaker. What we did in the proof of Theorem 3 was to take advantage of the regularity of the measure to translate backwards, from vertical differentiability to horizontal differentiability.

This points to the fact that one should not expect the method of proof of Theorem 3 to generalize to physical measures, at least not if the perturbation of the measure is only asked to be vertically differentiable. Let us go further and provide an example showing how a generalization of Theorem 3 can actually fail (of course, Theorem 3 generalizes in a straightforward way if the perturbation of the measure is by pushing it forward by a smooth vector field).

Example 7. Let $T : [-1,1] \to [-1,1]$ be a contraction, say $T(x) = x/2$. It has a unique physical measure $\mu_0 = \delta_0$, whose basin of attraction is the whole phase space. Consider the perturbation $\mu_t = (1-t)\delta_0 + t\delta_{1/2}$ (for $t \in [0,1)$) of the physical measure. For any bounded $f : [-1,1] \to \mathbb{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int f \,\mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big((1-t)f(0) + tf(1/2)\big) = f(1/2) - f(0).$$

Therefore, $(\mu_t)_{t\in[0,1)}$ is differentiable against any space of test functions, however irregular. But for $t > 0$, μ_t cannot be a physical measure for a perturbation of T, since such a perturbation sends $1/2$ near $1/4$. There is no solution to the linear request problem in this case.

While this example might be only half convincing (the perturbation cannot be extended to $t < 0$), it points out the natural question to study linear response for a stronger definition of differentiability for families of measure: a completely satisfying theory should have both linear request and linear response use the same notion of differentiability.

2. Non-uniqueness of solutions

We now turn back to the smooth acim case and consider the size of the set of first-order perturbations $\dot{T}_0$ that solve the linear request problem $\dot{\omega}_0 = \rho\omega$. Our main result in this section shows that there can be very many solutions that are infinitesimal conjugacies. Our assumption relies on two notions: a vector field X is said to be T-*invariant* if $D_x T(X_x) = X_{T(x)}$ for all $x \in M$, and X is said to *preserve* the volume form ω if $\mathscr{L}_X \omega = 0$ (i.e., the flow of X preserves ω).

Theorem 8. *Assume that $n > 1$ and that the null vector field $X = 0$ is the only T-invariant vector field which preserves ω.*

Then for each smooth function $\rho : M \to \mathbb{R}$ such that $\int_M \rho\omega = 0$, there is an infinite-dimensional affine space $D \subset \Gamma_T(M)$ such that for each $Z \in D$, there is a differentiable family of maps $(T_t)_{(-\varepsilon,\varepsilon)}$ preserving smooth volume forms ω_t such that

$$T_0 = T, \quad \dot{T}_0 = Z, \quad \dot{\omega}_0 = \rho\omega.$$

Moreover one can ask all T_t to be smoothly conjugate to T.

In the vocabulary of Section 1.1, the above Z are infinitesimal conjugacies that solve the linear request problem. We will see that Theorem 8 applies in particular to Anosov maps, and thus in particular to expanding maps.

Proof. First, we investigate the size of the set of vector fields X that can be used in the proof of Theorem 3, i.e., such that $\mathscr{L}_X \omega = -\rho\omega$.

This reduces to solving $\mathrm{d}(i_X\omega) = -\rho\omega$ in the variable X and is decomposed into two steps: choose an $(n-1)$-form θ such that $\mathrm{d}\theta = -\rho\omega$, and then find X such that $i_X\omega = \theta$. This second step leaves no room, since $Z \mapsto i_Z\omega$ is an isomorphism from the space of vector fields to the space of $(n-1)$-forms.

On the contrary, θ is not unique: one can add to it any closed $(n-1)$-form and obtain a suitable $(n-1)$-form, and two suitable choices differ by a closed $(n-1)$-form. In other words, *the space of solutions to $\mathscr{L}_X\omega = -\rho\omega$ identifies (through the choice of a specific θ) to the space $Z^{n-1}(M)$ of closed $(n-1)$ forms.* Observe that since $n \geq 2$, $Z^{n-1}(M)$ is infinite-dimensional (it contains all exact forms $\mathrm{d}\alpha$ where α is an $(n-2)$-form).

Then we have to determine to what extent different vector fields can lead to the same solution Z for the value of $\dot{T}_0$, using the formula $\dot{T}_0 = -DT(X) + X_T$.

Consider X and Y two smooth vector fields on M such that $\mathscr{L}_X\omega = \mathscr{L}_Y\omega = -\rho\omega$ and $-DT(X) + X_T = -DT(Y) + Y_T$. The first condition implies that $X - Y$ preserves ω, while the second one rewrites as $X_T - Y_T = DT(X-Y)$; i.e., $X - Y$ is T-invariant. By hypothesis then $X = Y$, and any two different elements of $Z^{n-1}(M)$ must induce different solutions to the linear request problem. □

In the case of a general smooth map T, the above proof shows that the space $D \subset \Gamma_T(M)$ of infinitesimal conjugacies that solve the linear request problem is determined by:

- $Z^{n-1}(M)$, whose size mostly depends on n: it is infinite-dimensional when $n \geq 2$, but when $n = 1$, i.e., $M = \mathbb{S}^1$, there are no $(n-2)$-forms, and $Z^{n-1}(M) = Z^0(\mathbb{S}^1)$ is the space of constant functions and is 1-dimensional;
- the space of volume-preserving, T-invariant vector fields (which strongly depend on the map T).

Let us now consider a few examples of maps.

Example 9. If T is the identity map, then all vectors are T-invariant and any two X, Y such that $\mathscr{L}_X\omega = \mathscr{L}_Y\omega = -\rho\omega$ will yield the same derivative $\dot{T}_0$. But of course this case is very degenerate, since for all X we have $-DT(X) + X_T = -X + X = 0$, which corresponds to $\varphi^t \circ T \circ \varphi^{-t} = \varphi^t \circ \varphi^{-t} = \mathrm{Id}$: the map is unchanged by conjugacy and preserves all measures anyway.

Example 10. If T is an irrational rotation of the torus $\mathbb{R}^n/\mathbb{Z}^n$, then ω must be the Lebesgue measure (which is the only invariant measure), and the only T-invariant vector fields are constant. As soon as $n > 1$ the linear request problem thus has an infinite-dimensional space of solutions. If $n = 1$, writing $X = v\,\partial/\partial x$ where v is a function and x the standard coordinate on the circle, the equation $\mathscr{L}_X\omega = -\rho\omega$ translates into $v' = -\rho$, which has solutions since $\int \rho(x)\,\mathrm{d}x = 0$. Any two solutions differ by a constant, which is T-invariant, and thus define the same first-order perturbation $\dot{T}_0$. In this case, there is a unique infinitesimal conjugacy solving the linear request problem.

Example 11. Assume now that $n > 1$ and T is Anosov; i.e., the tangent bundle of M admits an invariant decomposition $TM = E^s \oplus E^u$ and DT is uniformly exponentially expanding on E^u and uniformly exponentially contracting on E^s. Then there is no non-zero invariant vector field. Indeed, let X be a T-invariant vector field. Since M is compact, there is a point x at which the E^u-component of X has maximal norm, but invariance of X implies that this component blows up exponentially along the orbit of x. Thus the maximum is 0, and $X \in E^s$. But then the same argument, applied along any backward orbit, shows that the E^s component of X must also vanish, and $X = 0$.

Theorems 3 and 8 thus show that if T is an Anosov map admitting a smooth acim, then for all ρ of vanishing integral the linear request problem admits an infinite-dimensional space of solutions (which can moreover be chosen among infinitesimal conjugacies).

3. Optimality

We now take a look at Question 2, finding an "optimal" solution to the linear request problem. Assume that in addition to the usual data M, T, ω, ρ, a Riemannian metric has been fixed on M, and let S be the set of solutions Z to the linear request problem and let D be the subset of infinitesimal conjugacies (i.e., of the form $Z = -DT(X) + X_T$ for some X). In [GP17] Galatolo and Pollicott propose to find the solution $Z = \dot{T}_0 \in S$ that minimizes some norm, for example $\|Z\|_{L^2(\omega)}$. It would make little sense to minimize this over $Z \in D$: as we will see in an example, the optimal solution over S need not be in D.

However, it makes sense to minimize $\|X\|_{L^2(\omega)}$ over all X such that $\mathscr{L}_X\omega = -\rho\omega$, as it corresponds to the "shortest" vector field whose induced infinitesimal conjugacy solves the linear request problem. This is a classical and well-understood problem, which we present briefly for the sake of self-completeness.

First, one translates the differential geometric notation into their Riemannian version: denoting by η the density of ω with respect to ω_0, we have

$$\mathscr{L}_X\omega = \mathrm{d}(i_X(\eta\omega_0)) = \nabla\cdot(\eta X)\omega_0,$$

where $\nabla\cdot$ is the divergence operator (later, ∇u will denote the gradient of a function u). The equation for the infinitesimal conjugacy thus becomes $\nabla\cdot(\eta X) = \eta\rho$, where the product $\eta\rho$ is the density of the perturbation $\rho\omega$ with respect to ω_0.

We then rely on the following classical result to find a solution of a particular type: a gradient vector field.

Proposition 12. *Let η, g be smooth functions on a compact Riemannian manifold. If η is positive and $\int g\omega_0 = 0$, then there exists a smooth solution, unique up to constants, to the partial differential equation*

$$\nabla\cdot(\eta\nabla u) = g. \tag{2}$$

While it is not as easy as expected to find a proof of this precise result, it can be proved using the methods used for the "Poisson equation" $\Delta u = g$; see e.g. [Aub98]. For the sake of completeness, we provide a classical proof at the end of the section.

Taking $g = -\eta\rho$, we get a solution u to the above equation, and $X := \nabla u$ is then a solution to $\mathscr{L}_X\omega = -\rho\omega$. The first version of this note actually used this to obtain Theorem 3, but not insisting on finding a *gradient* vector field can be done even more easily, as above. I wish to thank Anatole Katok for interesting criticism on that first version, pointing to the simplified proof of Theorem 3 above.

The gradient solution is optimal in the above sense, as shown by the following (again well-known) lemma.

Lemma 13. *Among all vector fields which are solutions of $\mathscr{L}_X\omega = -\rho\omega$, the unique one which is the gradient of a function minimizes $\|X\|_{L^2(\omega)}$ (where the underlying norm is induced by the given Riemannian metric).*

Note that if one changes the Riemannian metric, then both the notion of gradient and the functional being optimized change.

Proof. Let $X_0 = \nabla u$ be the unique gradient solution provided by Proposition 12. Then any other solution writes as $X = X_0 + \frac{1}{\eta}F$ where F is a divergence-free vector field, and

$$\begin{aligned}
\int \big\|X_0 + \frac{1}{\eta}F\big\|^2\,\omega &= \int \|X_0\|^2\,\omega + 2\int (\nabla u)\cdot F\,\omega_0 + \int \big\|\frac{1}{\eta}F\big\|^2\,\omega \\
&= \int \|X_0\|^2\,\omega - 2\int u\,(\nabla\cdot F)\,\omega_0 + \int \big\|\frac{1}{\eta}F\big\|^2\,\omega \\
&= \int \|X_0\|^2\,\omega + \int \big\|\frac{1}{\eta}F\big\|^2\,\omega
\end{aligned}$$

so that X_0 is indeed uniquely minimizing among solutions. $\square$

Proof of Proposition 12. One first observes that by using test functions and then integration by parts, (2) is equivalent to

$$\int \varphi \nabla \cdot (\eta \nabla u)\, \omega_0 = -\int \varphi g \omega_0 \qquad \forall \varphi \in C^\infty(M),$$

$$\int \nabla\varphi \cdot \nabla u\, \eta\omega_0 = \int \varphi g \omega_0 \qquad \forall \varphi \in C^\infty(M), \tag{3}$$

and we are thus asking for u solving

$$Q(u,\varphi) = L(\varphi) \qquad \forall \varphi \in C^\infty(M),$$

where Q and L are the bilinear form, respectively the linear form, defined by each side of (3), on the domain $H_1(M)$ of Sobolev functions. Recall that $H_1(M)$ can be defined as the set of $L^2(\omega_0)$ functions whose gradient in the distribution sense is $L^2(\omega_0)$ or as the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{H_1} = \int \varphi^2\, \omega_0 + \int \|\nabla\varphi\|^2\, \omega_0,$$

where $\|\cdot\|$ is the norm in each tangent space induced by the Riemannian metric.

Then one invokes the Lax-Milgram theorem, which precisely gives us a weak solution $u \in H_1$ as soon as we prove that Q and L are continuous in the H_1 norm (which follows from the Cauchy-Schwarz inequality) and that Q is coercive (i.e., the seminorm it induces on $H_1(M)$ is equivalent to the H_1 norm). That last statement does not hold as constants are in the kernel of Q; we thus decompose

$$H_1(M) = H_1^\perp \overset{\perp}{\bigoplus} \{\text{constants}\},$$

where $H_1^\perp$ is the subspace of functions of vanishing ω_0-average. Now, on a compact manifold one has:

Proposition 14 (Poincaré inequality)**.** *There exist a constant C depending only on M and its metric such that for all $\varphi \in H_1(M)$, denoting by $\bar\varphi = \frac{1}{\mathrm{vol}(M)}\int \varphi\,\omega_0$ the average of φ, it holds that*

$$\int |\varphi - \bar\varphi|^2\, \omega_0 \le C \int \|\nabla\varphi\|^2\, \omega_0.$$

This inequality is very classical and can for example be derived as follows. Looking at the Rayleigh quotient, one sees it is equivalent to $\lambda_1(M) > 0$, where λ_1 is the first eigenvalue of $-\Delta$ (in $H_1^\perp$). This then follows from Cheeger's bound $\lambda_1 \ge h_C^2/4$, where Cheeger's constant $h_C(M)$ can be bounded below in terms of the diameter of M and a (possibly negative) lower bound on its Ricci curvature; see for example Chapter IV of [Bér86].

On $H_1^\perp$, the Poincaré inequality yields

$$\begin{aligned} Q(\varphi,\varphi) &= \int \|\nabla\varphi\|^2 \eta\omega_0 \\ &\ge \min(\eta) \int \|\nabla\varphi\|^2\, \omega_0 \\ &\ge \min(\eta)\Big(\frac{1}{2}\int \|\nabla\varphi\|^2\, \omega_0 + \frac{1}{2C}\int \varphi^2\, \omega_0\Big) \\ &\ge C' \|\varphi\|_{H_1}^2, \end{aligned}$$

which is precisely the coercivity of Q restricted to $H_1^\perp$. It follows from the Lax-Milgram theorem that there is a $u \in H_1^\perp$ such that

$$Q(u, \varphi) = L(\varphi) \qquad \forall \varphi \in H_1^\perp.$$

To get rid of the restriction that φ must have vanishing average, observe that given $\varphi \in H_1(M)$, its centered version $\varphi - \bar{\varphi}$ is in $H_1^\perp$. On the one hand, $\nabla\varphi = \nabla(\varphi - \bar{\varphi})$ so that $Q(u, \varphi) = Q(u, \varphi - \bar{\varphi})$, and on the other hand,

$$L(\varphi - \bar{\varphi}) = \int \varphi g \omega_0 - \bar{\varphi} \int g \omega_0 = L(\varphi)$$

since g has vanishing average. It follows that $Q(u, \varphi) = L(\varphi)$ for all $\varphi \in H_1(M)$; in particular for all smooth φ, $u \in H_1(M)$ is a weak solution of (2).

The last step we need is to improve this into a strong solution. This is well-known but subtle and is purely local: the manifold case is handled just as in the $\mathbb{R}^n$ case, using charts. We refer for example to [Aub98, Theorem 3.55, page 85] for a suitable statement, which itself refers to [LU68], where it is proved that $u \in C^{k+2,\alpha}(M)$ whenever the coefficients of (2) are $C^{k,\alpha}$. This happens as soon as $\eta \in C^{k+1,\alpha}(M)$ and $g \in C^{k,\alpha}(M)$ (in particular, if η and g are smooth, so is u).

4. A model case

Let us now spell out what happens in the model case when $M = \mathbb{R}/\mathbb{Z}$ and $T(x) = 2x$ (modulo 1, implicitly). The acim of T is then the Lebesgue measure, which coincides with the Riemannian volume ω_0, and we thus have $\eta \equiv 1$.

A smooth vector field can be written $X = v\,\partial/\partial x$, where x is the standard coordinate on $\mathbb{R}/\mathbb{Z}$ and v is a smooth function. We will identify X with v, and since the circle is naturally parallel we can also identify any element of $\Gamma_T(\mathbb{R}/\mathbb{Z})$ with a function whose value at a point x represents a vector at the point $2x$. Under this identification we have

$$\begin{aligned}\big[-DT(X) + X_T\big](x) &= -2v(x) + v(2x),\\ -\mathscr{L}_X\omega(x) &= -v'(x).\end{aligned}$$

Let ρ be a smooth function with vanishing average, which will be identified with its 1-periodic lift to $\mathbb{R}$. The gradient solution to $-\mathscr{L}_X\omega = \rho\omega$ is given by

$$v_0(x) = -\int_0^x \rho(t)\,\mathrm{d}t + \int_0^1 \int_0^y \rho(t)\,\mathrm{d}t\,\mathrm{d}y$$

(the only primitive of $-\rho$ which has vanishing average).

To compare with [GP17], let us consider the case $\rho(x) = \sin(2\pi x)$. Then

$$v_0(x) = \frac{1}{2\pi}\cos(2\pi x),$$

and the corresponding perturbation of T is given by

$$w_0(x) = -\frac{1}{\pi}\cos(2\pi x) + \frac{1}{2\pi}\cos(4\pi x).$$

Meanwhile, the L^2-norm minimizing perturbation found in [GP17] is given by $w_1(x) = \frac{1}{2\pi}\cos(4\pi x)$; note that this cannot be obtained from an infinitesimal conjugacy, as they all come from vector fields $v = v_0 + c$ where c is a constant (see Section 2) and thus are of the form $w(x) = -2v(x) + v(2x) = w_0(x) - c$.

Parceval identity easily shows that in fact, not only does v_0 minimize the L^2 norm among solutions of $\mathscr{L}_X\omega = -\rho\omega$, but w_0 also minimizes the L^2 norm among solutions of the linear request problem that are infinitesimal conjugacies. This means that the difference in the two optimization problems does not stem merely from the difference in the functionals to be minimized but also from the restriction to infinitesimal conjugacy. This restriction facilitates the answer to Question 1, but is not appropriate for Question 2 as meant in [GP17].

As a last remark, let us observe that, using the particular form of T and Fourier series, w_1 can be written as the image $-DT(X) + X_T$ of some non-smooth vector field X, given by

$$v_1(x) = \sum_{k\geq 1} \frac{-1}{2^{k+1}\pi} \cos(2\pi \cdot 2^k \cdot x),$$

and can thus be thought of as a *non-smooth* infinitesimal conjugacy. This is not surprising: by structural stability, we know that any deformation of T must be topologically conjugate to T. The equation $-DT(X) + X_T = Y$ in the unknown X has been studied under the name "twisted cohomological equation", and the regularity of its solutions when T is an expanding circle map has for example been finely analyzed by de Lima and Smania [dLS15].

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