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A NOTE ON CONSTRUCTING FAMILIES OF SHARP EXAMPLES FOR L^p GROWTH OF EIGENFUNCTIONS AND QUASIMODES

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ABSTRACT. In this note we analyse L^p estimates for Laplacian eigenfunctions and quasimodes and their associated sharp examples. In particular, we use previously determined estimates to produce a new set of estimates for restriction to thickened neighbourhoods of submanifolds. In addition, we produce a family of flat model quasimode examples that can be used to determine sharpness of estimates on Laplacian eigenfunctions restricted to subsets. For each quasimode in the family we show that there is a corresponding spherical harmonic that displays the same growth properties. Therefore it is enough to check L^p growth estimates against the simple flat model examples. Finally, we present a heuristic that for any subset determines which quasimode in the family is expected to produce sharp examples.

Let (M, g) be a Riemannian manifold and let $\Delta = \Delta_g$ be the (positive) Laplace–Beltrami operator defined by the metric. There has been much recent interest (for example [1], [3], [5], [7], [8], [9], [10]) in understanding how the L^p norms of Laplacian eigenfunctions

$$\Delta u = \lambda^2 u$$

grow for large λ. In particular in comparing the L^p estimates over the full manifold with that on subsets. The results in this area produce estimates of the form

$$\|u\|_{L^p(X)} \lesssim \lambda^{\delta(n,p,X)} \|u\|_{L^2(M)},$$

where X is a subset of M (not necessarily of full dimension). Is is often instructive to translate this to a semiclassical problem where $\lambda^{-1} = h$ and u is a solution to the semiclassical equation $(h^2\Delta - 1)u$. In fact, for a number of technical reasons, it is more customary to consider approximate solutions, that is, u, such that

$$\left\|(h^2\Delta - 1)u\right\|_{L^2(M)} \lesssim h \|u\|_{L^2(M)}.$$

The purpose of this note is twofold:

(1) To examine the known estimates and associated sharp examples and obtain new sharp estimates by "cheap" techniques (such as the application of Hölder's inequality or interpolation).
(2) To describe how to construct families of examples to examine questions of sharpness both for the flat model cases and for spherical harmonics.

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In particular, we will obtain L^p estimates where X is a thickened region of a submanifold. The examples we construct will show that these estimates are sharp (up to a possible log loss). The spherical harmonic examples have the advantage of being exact eigenfunctions, however, they are not so easy to write down explicitly. The flat model examples are in contrast very easy to explicitly produce. The flat model also has the advantage that for any given p (with knowledge of the semiclassical version of the L^p estimate proof) it is easy to determine which functions in the family will give rise to sharp examples. We will show that every flat model example has a matching spherical harmonic which shares all relevant features. Therefore any result that is sharp under the flat model is sharp under spherical harmonics. Finally we discuss how, given any particular p, one predicts which example will give rise to sharp estimates.

The whole and submanifold estimates are as follows:

$$\|u\|_{L^p(X)} \lesssim h^{-\delta(n,k,p)} \|u\|_{L^2(M)}$$

for X a k-dimensional smooth submanifold if $k < n$ and for $X = M$ if $k = n$. The function $\delta(n,k,p)$ is given by

$$\delta(n,n,p) = \begin{cases} \frac{n-1}{2} - \frac{n}{p} & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{4} - \frac{n-1}{2p} & 2 \leq p \leq \frac{2(n+1)}{n-1}, \end{cases}$$

$$\delta(n,n-1,p) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p} & \frac{2n}{n-1} \leq p \leq \infty, \\ \frac{n-1}{4} - \frac{n-2}{2p} & 2 \leq p \leq \frac{2n}{n-1}, \end{cases}$$

and for $k \leq n-2$,

$$\delta(n,k,p) = \frac{n-1}{2} + \frac{k}{p} \quad 2 < p < \infty.$$

In the case $k \leq n-3$ or $n = 3, k = 2$ the $p = 2$ estimate is included; otherwise, there is a logarithmic loss

$$\|u\|_{L^2(X)} \lesssim h^{-\frac{n-1+k}{2}} \log|h| \, \|u\|_{L^2}.$$

These estimates are due to

- Sogge [9] for L^p estimates over the full manifold and Koch-Tataru-Zworski [7] for the semiclassical problem.
- Burq-Gérard-Tzevtkov [1] for L^p estimates of eigenfunctions on submanifolds and Tacy [10] for the semiclassical problem.
- Chen-Sogge [3] for the endpoint estimate $(n,k,p) = (3,2,2)$.

It is well known that these estimates are saturated for high p by the zonal harmonics and for low p by the highest weight harmonics. The key features that saturate the estimates are a point concentration and a tube concentration (see Figures 1 and 2). Zonal harmonics have a point concentration at their north pole while highest weight harmonics are highly concentrated in an $h^{\frac{1}{2}}$ width tube around a great circle.

These two examples alone are sometimes enough to analyse sharp L^p behaviour. We demonstrate this for restriction of eigenfunctions to sets near submanifolds. For Σ a smooth k-dimensional submanifold of M, let Σ_β be the set

$$\Sigma_\beta = \{x \in M \mid d(x,\Sigma) \leq h^\beta\},$$

where d is the usual distance associated with the metric g. We want an estimate of the form

$$\|u\|_{L^p(\Sigma_\beta)} \lesssim h^{-\sigma(n,k,p,\beta)} \|u\|_{L^2(M)}$$

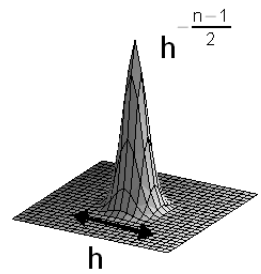

FIGURE 1. Concentration at a point

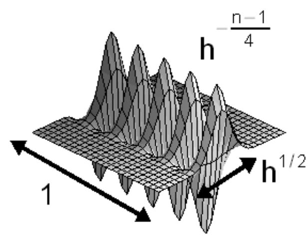

FIGURE 2. Concentration in a tube

for u a Laplacian eigenfunction or of a quasimode of $h^2\Delta_g - 1$. We first make some observations using prior results and the point/tube examples. These observations will enable us to determine $\sigma(n,k,p,\beta)$ in many cases.

Observation 1. Clearly the L^p norm of u on Σ_β must be bounded by the L^p norm of u on M. Therefore for all p we have

$$\|u\|_{L^p(\Sigma_\beta)} \lesssim h^{-\delta(n,n,p)} \|u\|_{L^2(M)}. \tag{1}$$

The question is then whether this can be improved. We must therefore first ask whether the known point and tube features fit inside Σ_β. If they do we can expect no better estimates that (1).

Observation 2. Since both the tube and point features can be placed inside Σ_β where $\beta \leq \frac{1}{2}$, we know immediately that there can be no better estimates in this case.

Observation 3. Writing $x \in M$ as $x = (y,z)$, where $\Sigma = \{(y,z) \in M \mid z = 0\}$, we see that

$$\begin{aligned}\int_{\Sigma_\beta} |u|^p dx &\leq \sup_{|z|\leq ch^\beta} \int |u|^p dy \times \int_{|z|\leq ch^\beta} dz \\ &\lesssim h^{-p\delta(n,k,p)} h^{\beta(n-k)}.\end{aligned}$$

So we may also say that

$$\|u\|_{L^p(\Sigma_\beta)} \lesssim h^{-\delta(n,k,p)+\frac{\beta(n-k)}{p}} \|u\|_{L^2(M)}. \tag{2}$$

If for all $|z| \leq h^\beta$ the submanifold estimate is sharp, we cannot expect to do better than (2).

Observation 4. Eigenfunctions and quasimodes have the property that they oscillate with frequency on the order h^{-1}; therefore, they cannot change much in a region of size h. This means that the estimate of (2) is the best we may expect for $\beta \geq 1$ and in fact we see that this is indeed the case for both the point and tube sharp examples.

From these four observations (along with interpolation from known results) we, in Section 1, generate a full set of $L^p(\Sigma_\beta)$. In Section 2 we construct a family of sharp quasimode examples in the flat model case that prove these L^p estimates to be sharp. In Section 3 we show that on the sphere we can construct exact eigenfunctions with the same properties as the sharp quasimode examples which means that, in any situation, we may check estimates against the flat model. In Section 4 we discuss how, given knowledge of the semiclassical techniques employed to prove the whole and submanifold estimates, one chooses the correct example to get a sharp quasimode.

1. L^p estimates on Σ_β

In this section we use our four observations along with known results to prove a full range of L^p estimates for Σ_β. Taken together, Observations 2 and 4 tell us that there are no nontrivial estimates outside $\frac{1}{2} \leq \beta \leq 1$, so we focus on this region. In the case that Σ is a hypersurface we obtain the following bounds.

Theorem 1.1. *Suppose u is an $O_{L^2}(h)$ quasimode of $h^2\Delta - 1$ on a Riemannian manifold (M, g). Further, for Σ a smooth embedded hypersurface in M and $\frac{1}{2} \leq \beta \leq 1$, let $\Sigma_\beta = \{x \in M \mid d(x, \Sigma) \leq h^\beta\}$. Then*

$$\|u\|_{L^p(\Sigma_\beta)} \lesssim h^{-\sigma(n,n-1,p)} \|u\|_{L^2(M)},$$

where

$$\sigma(n, n-1, p, \beta) = \begin{cases} \delta(n, n, p) & p \geq \frac{2(n+1)}{n-1}, \\ \frac{\beta(n-1)}{2} - \frac{\beta(n+1)}{p} + \frac{1}{p} & \frac{2n}{n-1} \leq p \leq \frac{2(n+1)}{n-1}, \\ \delta(n, n-1, p) - \frac{\beta}{p} & 2 \leq p \leq \frac{2n}{n-1}. \end{cases}$$

Proof. From Observation 1 we know that u must obey the full manifold estimates. Since $\beta \geq 1$ we may always fit the point type example into Σ_β, so we cannot expect better estimates than those arising from a point concentration. So we know that if $p \geq \frac{2(n+1)}{n-1}$ we cannot expect better estimates than those from over the full manifold. That is,

$$\sigma(n, n-1, p, \beta) = \delta(n, n, p), \quad p \geq \frac{2(n+1)}{n-1}.$$

The sharp example for the low p (that is, $2 \leq p \leq \frac{2n}{n-1}$) hypersurface estimates is the tube oriented with its long direction along the hypersurface. Since this example has relatively constant size in a $1 \times h^{\frac{n-1}{2}}$ region, the estimates of Observation 3 are the best we could expect in this range of p. That is,

$$\sigma(n, n-1, p, \beta) = \delta(n, n-1, p) - \frac{\beta}{p} \quad 2 \leq p \leq \frac{2n}{n-1}.$$

Therefore the only unknown estimates are those between $\frac{2n}{n-1}$ and $\frac{2(n+1)}{n-1}$. We interpolate between the estimate for $p\frac{2(n+1)}{n-1}$ and $p=\frac{2n}{n-1}$ to obtain

$$\sigma(n,n-1,p,\beta)=\begin{cases}\delta(n,n,p) & p\geq\frac{2(n+1)}{n-1},\\ \frac{\beta(n-1)}{2}-\frac{\beta(n+1)}{p}+\frac{1}{p} & \frac{2n}{n-1}\leq p\leq\frac{2(n+1)}{n-1},\\ \delta(n,n-1,p)-\frac{\beta}{p} & 2\leq p\leq\frac{2n}{n-1}.\end{cases}$$

Since Observations 1 and 3 (along with the known sharp examples for manifolds and hypersurfaces) tell us that we have sharp examples for $p\geq\frac{2(n+1)}{n-1}$ and $p\leq\frac{2n}{n-1}$, the only question remaining is whether the intermediate bounds obtained through interpolation are sharp. In Section 2 we will construct model quasimodes that demonstrate sharpness. The results of Section 3 guarantee that there are exact eigenfunctions on the sphere that are also sharp. $\square$

Where Σ is a lower-dimensional submanifold we obtain the following.

Theorem 1.2. *Suppose u is an $O_{L^2}(h)$ quasimode of $h^2\Delta-1$ on a Riemannian manifold (M,g). Further, for Σ a smooth embedded submanifold of dimension $k\leq n-3$ in M and $\frac{1}{2}\leq\beta\leq 1$, let $\Sigma_\beta=\{x\in M\mid d(x,\Sigma)\leq h^\beta\}$. Then*

$$\|u\|_{L^p(\Sigma_\beta)}\lesssim h^{-\sigma(n,k,p)}\|u\|_{L^2(M)},$$

where

$$\sigma(n,k,p,\beta)=\begin{cases}\delta(n,n,p) & p\geq\frac{2(n+1)}{n-1},\\ \frac{\beta(n-1)}{2}-\frac{\beta(n-1)}{p}+\frac{1}{p} & 2\leq p\leq\frac{2(n+1)}{n-1}.\end{cases}$$

If $k=n-2$ the same result holds for $p\geq\frac{2(n+1)}{n-1}$, and it holds with a log *loss when $p<\frac{2(n+1)}{n-1}$.*

Proof. Again Observation 1 along with the point sharp example tells us that if $p\geq\frac{2(n+1)}{n-1}$ we cannot expect better estimates than those from the full manifold. Therefore

$$\sigma(n,k,p,\beta)=\delta(n,n,p),\quad p\geq\frac{2(n+1)}{n-1}.$$

The sharp submanifold restriction examples, however, are the point type eigenfunctions. This feature only persists for an $O(h)$ region, so when $\beta\ll 1$ we cannot expect to get sharp examples for low p from Observation 3. However Burq and Zuily [2] have, in the case $k\leq n-3$, obtained

$$\|u\|_{L^2(\Sigma_\beta)}\lesssim h^{\beta-\frac{1}{2}}\|u\|_{L^2(M)}$$

and that when $k=n-2$ the same result holds with a log loss. so we may interpolate from this point to obtain

$$\sigma(n,k,p,\beta)=\begin{cases}\delta(n,n,p) & p\geq\frac{2(n+1)}{n-1},\\ \frac{\beta(n-1)}{2}-\frac{\beta(n-1)}{p}+\frac{1}{p} & 2\leq p\leq\frac{2(n+1)}{n-1}.\end{cases}$$

We know that the high p estimates are sharp. In Sections 2 and 3 we show that the low p estimates (modulo the log loss) are also sharp. $\square$

2. Flat model examples

We study the flat model, that is, localised quasimodes of the Laplacian in $\mathbb{R}^n$, to gain insight into sharp examples. Such quasimodes can be easily produced on the Fourier side. In keeping with the semiclassical theme we use the rescaled semiclassical Fourier transform,

$$\mathcal{F}_h[u](\xi) = \frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle} u(x)\,dx.$$

This operator has the property that

$$\mathcal{F}_h\left[hD_{x_i}\right] = \xi_i \mathcal{F}_h[u]$$

and

$$\|\mathcal{F}_h[u]\|_{L^2} = \|u\|_{L^2}.$$

The development of flat model examples was discussed in [4]. We include it here for the reader's convenience.

Suppose that u is an L^2 normalised $O_{L^2}(h)$ quasimode of $\Delta_{\mathbb{R}^n}$. We must have

$$\left\|(|\xi|^2-1)\mathcal{F}_h[u]\right\|_{L^2(\mathbb{R}^n)} \lesssim h,$$

thus $\mathcal{F}_h[u]$ must be located near the sphere of radius 1 in the ξ-variables. We create a family of quaismodes indexed by α which controls the degree of angular dispersion of ξ. Write $\xi = (r,\omega)$, where $\omega \in S^{n-1}$ and set the coordinate system so that ω_0 corresponds with the unit vector in the ξ_1 direction. Let

$$\chi^h_\alpha(r,\omega) = \begin{cases} 1 & \text{if } |r-1|<h, |\omega-\omega_0|<h^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then set

$$f^h_\alpha(\xi) = f^h_\alpha(r,\omega) = h^{-1/2-\alpha(n-1)/2}\chi(r,\omega).$$

Note that f^h_α is L^2 normalised. Now set

$$T^h_\alpha(x) = \mathcal{F}_h^{-1}[f^h_\alpha](x) = \frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} f_\alpha(\xi)\,d\xi.$$

T^h_α is an L^2 normalised $O(h)$ quasimode of $\Delta_{\mathbb{R}^n}$. We may write

$$T^h_\alpha(x) = \frac{h^{-1/2-\alpha(n-1)/2-n/2} e^{\frac{i}{h}x_1}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x_1(\xi_1-1)+\langle x',\xi'\rangle)} \chi_\alpha(\xi)\,d\xi.$$

Note that if $|x_1| < \epsilon h^{1-2\alpha}$ and $|x'| < \epsilon h^{1-\alpha}$ for sufficiently small $\epsilon > 0$, the factor

$$e^{\frac{i}{h}(x_1(\xi_1-1)+\langle x',\xi'\rangle)}$$

does not oscillate, so in this region (shown in Figure 3)

$$|T^h_\alpha(x)| > ch^{-(n-1)/2+\alpha(n-1)/2}.$$

We claim that when $\alpha = 1-\beta$ the function T_α saturates the L^p estimates for Σ_β in the case where Σ is a hypersurface and $\frac{2n}{n-1} \le p \le \frac{2(n+1)}{n-1}$, as well as the case where Σ is a lower-dimensional submanifold and $p \le \frac{2(n+1)}{n-1}$.

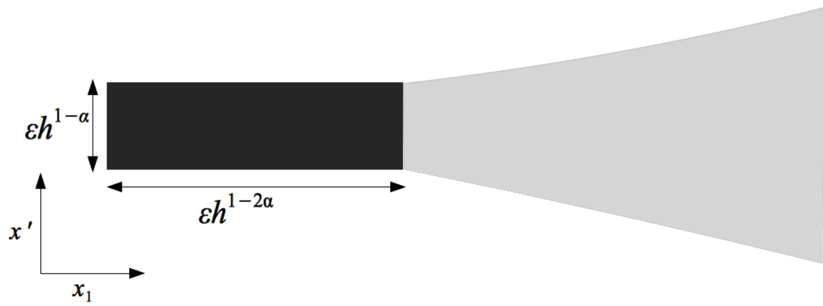

FIGURE 3. T_α^h is localised so that it is large in an $h^{1-2\alpha} \times (h^{1-\alpha})^{n-1}$ tube.

Example 2.1. We choose coordinates such that when we write $x \in M$ as $x = (y, z)$, $\Sigma = \{(y, z) \in M \mid z = 0\}$. By setting $\alpha = 1 - \beta$ we produce a function that has a $h^{2\beta-1} \times h^{\beta}$ tube where

$$|T_{1-\beta}| \geq ch^{-\frac{\beta(n-1)}{2}}.$$

Rotations and translations of T_α are still quasimodes, so we may align it so that the long direction lies in the submanifold. Therefore we have

$$\begin{aligned}\|T_\alpha\|_{L^p(\Sigma_\beta)} &> ch^{-\frac{\beta(n-1)}{2}} h^{\frac{\beta(n-1)}{p} + \frac{2\beta-1}{p}} \\ &= ch^{-\frac{\beta(n-1)}{2} + \frac{\beta(n+1)}{p} - \frac{1}{p}},\end{aligned}$$

as required.

One could obtain this example by calculating lower bounds for the L^p norm for every α and then maximising. However, by understanding the heuristics of the semiclassical proof one can immediately select the correct scale to find sharp examples in any situation. We discuss this heuristic in Section 4.

3. From quasimodes to exact eigenfunction

While they are easy to work with, the quasimodes T_α only show us that estimates are sharp for quasimodes of the flat Laplacian. However, we can construct exact L^2 normalised eigenfunctions ϕ_α on the sphere that have all the relevant properties of T_α. That is, they have an $h^{1-2\alpha} \times h^{(1-\alpha)(n-1)}$ region where $|\phi_\alpha| \geq ch^{-\frac{n-1}{2} + \frac{\alpha(n-1)}{2}}$. Since this is the only property of T_α used to prove sharp examples, this construction shows that any sharp examples from T_α give rise to sharp examples of exact eigenfunctions on the sphere. So any quasimode estimates that are sharp for the family of flat quasimode examples T_α are also sharp (with exact eigenfunctions) on the sphere.

To understand which spherical harmonics to pick we first re-express T_α as a sum of quasimodes, each of which has a Fourier transform localised in the angular variables on the scale of $h^{1/2}$. This is the localisation scale of $T_{1/2}$. Note that $T_{1/2}$ is localised about the point $(1, 0, \ldots, 0)$. We can produce a function $T_{1/2}^j$ with Fourier support in an $h \times h^{\frac{n-1}{2}}$ region of any $\xi_j \in S^{n-1}$ by a rotation applied to $\mathcal{F}_h(T_{1/2})$. The quasimode produced by this rotation is simply the standard $T_{1/2}$ quasimode rotated so that the long axis lies along ξ_j. Now $f_\alpha^h(\xi)$ is supported in an

$h^{\alpha(n-1)}$ angular region so we can cover this support with $h^{\left(\alpha-\frac{1}{2}\right)(n-1)}$ rotations of $\mathcal{F}_h[T_{1/2}]$. Therefore T_α can be thought of as a sum of $h^{\left(\alpha-\frac{1}{2}\right)(n-1)}$ functions each of which is a rotation of $T_{1/2}$.

The flat Laplacian quasimodes $T_{1/2}$ resemble the tubular concentrations we see in highest weight spherical harmonics. This leads us to the idea that we can create a suitable ϕ_α by considering a sum of rotated highest weight spherical harmonics. We write S^n as the subset of $\mathbb{R}^{n+1}$ where $|x| = 1$. It is well known that the function

$$u(x) = j^{\frac{n-1}{4}}(x_1 + ix_2)^j$$

is a solution to the spherical Laplacian eigenfunction equation with $j(j+n-1) = \lambda^2 = h^{-2}$. Further, if $x = (x_1, x_2, \bar{x})$, then

$$|u(x)|^2 = j^{\frac{n-1}{2}}(1 - |\bar{x}|^2)^j = j^{\frac{n-1}{2}} e^{j\log(1-|\bar{x}|^2)},$$

so $u(x)$ is highly concentrated on the equation $\bar{x} = 0$ with exponential decay when $|\bar{x}| \gg h^{1/2}$. The prefactor of $j^{\frac{n-1}{4}} \approx h^{-\frac{n-1}{4}}$ ensures that $\|u\|_{L^2} \approx 1$. We produce an example by summing rotations of $u(x)$.

Proposition 3.1. *For any $\epsilon > 0$ and $0 \leq \alpha \leq 1/2$, there exists a ϕ_α such that $\Delta_{S^n}\phi_\alpha = j(j+n-1)\phi_\alpha$ and ϕ_α is given by*

$$\phi_\alpha(x) = h^{-\frac{\alpha(n-1)}{2}} \sum_{k=1}^{N_\alpha} (x_1 + iP_k(x_2, \ldots, x_n))^j, \quad h^{-2} = j(j+n-1), \tag{3}$$

where $N_\alpha = \tilde{\epsilon} h^{(\alpha-1/2)(n-1)}$ for some small but fixed $\tilde{\epsilon}$ dependent on ϵ and P_k is a linear polynomial whose coefficients α_k^m obey

1. $|1 - \alpha_k^2| \leq \epsilon h^{2\alpha}$,
2. $|\alpha_k^m| \leq \epsilon h^\alpha, \quad m \neq 2.$

Further, there are constants c_1 and c_2 so that

$$c_1 \leq \|\phi_\alpha\|_{L^2} \leq c_2. \tag{4}$$

Proof. We construct ϕ_α by taking rotations of the standard highest weight harmonic

$$u(x) = (ix_1 + x_2)^j.$$

For $j = 3, \ldots, n+1$ we allow the rotation numbers s_j to take values in the set $\{h^{1/2} l \mid l = 1, 2, \ldots, \lfloor \tilde{\epsilon} h^{\alpha-1/2} \rfloor\}$ (where $\tilde{\epsilon}$ is some small but fixed number). For each s_j we define the associated rotation R_{s_j} by

$$(R_{s_j}(x))_2 = \sqrt{1 - s_j^2}\, x_2 + s_j x_j,$$
$$(R_{s_j}(x))_j = -s_j x_2 + \sqrt{1 - s_j^2}\, x_j,$$
$$(R_{s_j}(x))_m = x_m, \quad m \neq 2, j.$$

Let

$$\phi_\alpha = h^{-\frac{\alpha(n-1)}{2}} \sum_{[s_3, \ldots, s_{n+1}]} u \circ R_{s_{n+1}}(x) \circ R_{s_n} \circ \cdots \circ R_{s_3}.$$

We claim that ϕ_α has the necessary properties. Each individual term in the summand is an eigenfunction, so clearly ϕ_α is also an eigenfunction. Under the action of each rotation $R_{s_j}(x)$, x_1 is fixed, so it remains fixed under composition. Writing the $(n-1)$-tuple $S = (s_3, \ldots, s_{n+1})$ and denoting

$$R_S = R_{s_{n+1}}(x) \circ R_{s_n} \circ \cdots \circ R_{s_3},$$

we see that

$$(R_S)_2 = x_2 \left(\prod_{j=3}^{n+1} \sqrt{1-s_j^2} \right) + \sum_{k=3}^{n+1} x_k s_k \left(\prod_{j=k+1}^{n+1} \sqrt{1-s_j^2} \right). \tag{5}$$

Since each s_j obeys $|s_j| \leq \tilde{\epsilon} h^\alpha$ by making $\tilde{\epsilon}$ suitably small, we obtain the coefficient bounds

$$|1-\alpha_k^2| \leq \epsilon h^{2\alpha},$$
$$|\alpha_k^m| \leq \epsilon h^\alpha, \quad m \neq 2.$$

Therefore it remains only to prove the L^2 estimate. Note that there are $h^{(\alpha-1/2)(n-1)}$ terms in the summand each with L^2 norm of $h^{\frac{n-1}{4}}$, so (4) holds if for $S \neq S'$, $u \circ R_S$ and $u \circ R_{S'}$ are suitably orthogonal. We define

$$|S-S'| = \sup_j |s_j - s_j'|$$

and claim that for any $N>0$

$$\langle u \circ R_S, u \circ R_{S'} \rangle \leq h^{-\frac{n-1}{2}} \left(1 + \frac{|S-S'|}{h^{1/2}} \right)^{-N}.$$

Under a change of variables

$$x \to R_{S'}^{-1} = R_{s_3',3}^{-1} \circ \cdots \circ R_{s_{n+1}',n+1}^{-1},$$

this reduces to showing that

$$\left| \int \left(u \circ R_S \circ R_{S'}^{-1}(x) \right) u(x) d\mu(x) \right| \leq h^{-\frac{n-1}{2}} \left(1 - \frac{|S-S'|}{h^{1/2}} \right)^{-N}. \tag{6}$$

From the arguments leading to (5) we can say that

$$u \circ R_S \circ R_S^{-1} = (x_1 + iP_{S,S'}(x_2, \ldots, x_{n+1}))^j,$$

where $P_{S,S}$ is a linear polynomial in $x_2, \ldots, x_{n+1}$. Let k be such that $|s_k - s_k'| = |S-S'|$ and suppose that we have a lower bound on the x_k coefficient, $\alpha^k(S,S')$, of

$$|\alpha^k(S,S')| > c|S-S'| \quad \text{for some } c>0, \tag{7}$$

We will first assume (7) and use this to integrate by parts to show that (7) $\Rightarrow$ (6); we then prove (7). Let $\theta = (\theta_1, \ldots, \theta_n)$ be a spherical coordinate system so that $\theta_n \in [0,\pi]$ and the other $\theta_i \in [0, 2\pi]$ and

$$x_k = \cos(\theta_n),$$
$$x_{k+1} = \sin(\theta_n)$$
$$\vdots$$
$$x_2 = \sin(\theta_n) \cdots \sin(\theta_2)\sin(\theta_1),$$
$$x_1 = \sin(\theta_n) \cdots \sin(\theta_2)\cos(\theta_1).$$

Then

$$\frac{\partial(u \circ R_S \circ R_{S'}^{-1})}{\partial \theta_n} = j(x_1 + iP_{S,S'}(x_2, \ldots, x_n))^{j-1} \left(F(\theta_1, \theta_{n-1})\cos(\theta_n) + i\alpha^k(S,S')\sin(\theta_n) \right).$$

If we are suitably close to the region $\theta_n = \pi/2$ and (7) holds, we have the lower bound

$$\left|F(\theta_1, \theta_{n-1})\cos(\theta_n) + i\alpha^k(S,S')\sin(\theta_n)\right| \geq \frac{c}{2}|S-S'|$$

and can use this factor to integrate by parts. On the other hand, away from the region $\theta_n = \pi/2$ we know that $u(\theta)$ decays exponentially, so this contribution to the integral must be small. We complete the argument then by cutting the integral over S^n into two pieces, one where we may integrate by parts and the other where exponential decay dominates. Let χ be a smooth cut off function supported in $|\tau| \leq 2$ and equal to one in $|\tau| \leq 1$. Consider first

$$\int_{S^n} \left(u \circ R_S \circ R_{S'}^{-1}(\theta)\right) u(\theta) \chi\left(\frac{\cos(\theta_n)}{h^{1/4}|S-S'|^{1/2}}\right) d\mu(\theta).$$

On the support of χ we can write

$$\begin{aligned}(\sin(\theta_n \cdots \sin(\theta_2)\cos(\theta_1) + iP_{S,S'}(\theta))^j \\ = \frac{1}{j|S-S'|}\frac{\partial}{\partial\theta_n}(\sin(\theta_n \cdots \sin(\theta_2)\cos(\theta_1) + iP_{S,S'}(\theta))^{j+1}G(\theta),\end{aligned}$$

where $|G(\theta)| \leq 1$. Therefore we can integrate by parts. Any time a derivative hits the cut off function or $u(\theta)$, we lose at worst a factor of $\max(h^{-1/2}, h^{-1/4}|S-S'|^{1/2})$, so by repeating the argument $2N$ times we get

$$\begin{aligned}&\left|\int_{S^n} \left(u \circ R_S \circ R_{S'}^{-1}(\theta)\right) u(\theta) \chi\left(\frac{|\cos(\theta_n)|}{h^{1/2}|S-S'|^{1/2}}\right) d\mu(\theta)\right| \\ &\leq \left(1 + \frac{|S-S'|}{h^{1/2}}\right)^{-N} \int_{S^n} |u(x)| d\mu(x) \\ &= h^{-\frac{n-1}{2}}\left(1 + \frac{|S-S|}{h^{1/2}}\right)^{-N}.\end{aligned}$$

Now consider

$$\int_{S^n} \left(u \circ R_S \circ R_{S'}^{-1}(\theta)\right) u(\theta) \left(1 - \chi\left(\frac{|\cos(\theta_n)|}{h^{1/2}|S-S'|^{1/2}}\right)\right) d\mu(\theta).$$

On the support of $1-\chi$ we have $\cos(\theta_n) > h^{1/4}|S-S'|^{1/2}$, so $x_n^2 > h^{1/2}|S-S'|$ and

$$|u(x)| = e^{j\log(1-|\bar{x}|^2)} \leq e^{-h^{-1/2}|S-S'|}.$$

So

$$\begin{aligned}&\left|\int_{S^n} \left(u \circ R_S \circ R_{S'}^{-1}(\theta)\right) u(\theta)(1 - \chi\left(\frac{|\cos(\theta_n)|}{h^{1/2}|S-S'|^{1/2}}\right)) d\mu(\theta)\right| \\ &\qquad \leq e^{-h^{-1/2}|S-S'|}\int_{S^n} |u \circ R_S \circ R_{S'}^{-1}| d\mu(x) = h^{-\frac{n-1}{2}} e^{-h^{-1/2}|S-S'|},\end{aligned}$$

which is a much better estimate than we need.

Now it only remains to ascertain (7). Since $u \circ R_S \circ R_S^{-1} = (x+ix_2)^j$, $\alpha^k(S,S) = 0$. Therefore if we expand it as a series in S' about S,

$$\alpha^k(S,S') = \sum_{i=3}^{n+1} \frac{\partial^2 P_{S,S'}}{\partial s_i' \partial x_k}\bigg|_{S=S'} (s_i - s_i') + O(|S-S'|^2).$$

If we write each rotation as a matrix M_{s_j}, then $\frac{\partial P_{S,S'}}{\partial x_k}$ is given by the first element of

$$V(S,S') = M_{s_{n+1}} \times \cdots \times M_{s_3} \times M^{-1}_{s'_3} \times \cdots \times M^{-1}_{s'_{n+1}} e_k,$$

where e_k is the standard unit vector with 1 in the entry corresponding to x_k. Now if $\partial_{s'_i} V(S,S')$ is the vector with elements given by the partial derivative of the elements of $V(S,S')$ with respect to s'_i,

$$\partial_{s'_i} V(S,S') = M_{s_{n+1}} \times \cdots \times M_{s_3} \times M^{-1}_{s'_3} \times \cdots \times \partial_{s'_i} M^{-1}_{s'_i} \times \cdots \times M^{-1}_{s'_{n+1}} e_k,$$

where $\partial_{s'_i} M^{-1}_{s'_i}$ is the matrix with elements given by the partial derivative of the elements of $M^{-1}_{s'_i}$ with respect to s'_i. So if we evaluate at $S = S'$,

$$\partial_{s'_i} V(S,S')\Big|_{S=S'} = M_{s_{n+1}} \times \cdots \times M_{s_{i+1}} \times W_{s_i} \times M^{-1}_{s_{i+1}} \times \cdots \times M^{-1}_{s_{n+1}} e_k,$$

where $W_{s_i} = M_{s_i} \times \partial_{s_i} M_{s_i}$. First consider the case $i = k$. If $j \neq k$, $M^{-1}_{s_j} e_k = e_k$, so

$$\partial_{s_i} V(S,S')\Big|_{S=S'} = M_{s_{n+1}} \circ M_{s_{k+1}} W_{s_k} e_k.$$

Since for any α, $s_i^2 < \tilde{\epsilon}^2$ we can say that

$$\sqrt{1 - s_i^2} = 1 + O(\tilde{\epsilon}),$$

then

$$W_{s_i} = \left[\begin{array}{c|ccc|c|ccc}
O(\tilde{\epsilon}) & 0 & \cdots & 0 & 1 + O(\tilde{\epsilon}) & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
-1 + O(\tilde{\epsilon}) & 0 & \cdots & 0 & O(\tilde{\epsilon}) & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{array}\right].$$

So $W_{s_k} e_k = (1 + O(\tilde{\epsilon}), 0, \ldots, 0, O(\tilde{\epsilon}), \ldots, 0)$. From (5) we have seen that multiplication of the matrices M_{s_j} produces a matrix with upper left entry β, obeying $|1-\beta| \leq \epsilon h^\alpha$. So the first component of $\partial_{s'_i} V(S,S')\Big|_{S=S'}$ has a lower bound of $c > 0$. Now consider the case when $i \neq k$. We have $M_{s_k} e_k = (s_k, 0, \ldots, 0, \sqrt{1 - s_k^2}, \ldots)$. Now the vector $(s_k, 0, \ldots, 0)$ has norm bounded by $\tilde{\epsilon}$, and if $i \neq k$, $W_{s_i} e_k = 0$. Since each of the matrices M_{s_j} and W_{s_i} represent a bounded operator on $\mathbb{R}^{n-1}$, we can say that

$$\left| \frac{\partial^2 P_{S,S'}}{\partial s_i \partial x_k}\Big|_{S'=S} \right| \leq \tilde{\epsilon}.$$

So by choosing $\tilde{\epsilon}$ small enough we have

$$\left| \sum_{i=3}^{n+1} \frac{\partial^2 P_{S,S'}}{\partial s'_i \partial x_k}\Big|_{S=S'} (s_i - s'_i) \right| \geq c|s_k - s'_k| = c|S - S'|.$$

Thus

$$|\alpha^k(S,S')| > c|S - S'|.$$

Therefore $u \circ R_S$ and $u \circ R_{S'}$ are suitably orthogonal and the L^2 estimates (4) hold. □

Having obtained our combination, ϕ_α of highest weight harmonics, it only remains to prove that there is indeed an $h^{1-2\alpha} \times h^{(1-\alpha)(n-1)}$ region where ϕ_α is large enough.

Proposition 3.2. *Suppose ϕ_α is given by* (3) *. Then there is an $h^{1-2\alpha} \times h^{(1-\alpha)(n-1)}$ region in which $|\phi_\alpha| > ch^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}$.*

Proof. We prove this by expanding ϕ_α about the point $(\theta_1, \dots, \theta_n) = (0, \pi/2, \dots, \pi/2)$. This corresponds to the point $(1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ which is fixed by all the rotations, so all terms in the sum are equal to 1 at this point. This point lies on the equator where the original harmonic is equal to $(x_1 + ix_2)^j = e^{ij\theta_1}$, and at $(0, \pi/2, \dots, \pi/2)$, $|e^{-ij\theta_1}\phi_\alpha| = h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}$. When $|\theta_m - \pi/2| \leq \epsilon h^{1/2}, m \neq 1$, the conditions on the coefficients of Proposition 3.1 tell us that for each term in the sum defining ϕ_α,

$$\left| \frac{\partial}{\partial \theta_1} e^{-ij\theta_1} (\sin(\theta_n) \cdots \sin(\theta_2)\cos(\theta_1) + iP_k(\theta))^j \right| \leq jh^{2\alpha} \leq h^{2\alpha-1},$$

and for $m \neq 1$,

$$\left| \frac{\partial}{\partial \theta_m} e^{-ij\theta_1} (\sin(\theta_n) \cdots \sin(\theta_2)\cos(\theta_1) + iP_k(\theta))^j \right| \leq jh^{\alpha} \leq h^{\alpha-1}.$$

So

$$\left| \frac{\partial}{\partial \theta_1} (e^{-ij\theta_1}\phi_\alpha) \right| \leq h^{2\alpha-1} \cdot h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}},$$

and when $m \neq 1$,

$$\left| \frac{\partial}{\partial \theta_m} (e^{-ij\theta}\phi_\alpha) \right| \leq h^{\alpha-1} \cdot h^{-\frac{n-1}{2}+\frac{\alpha(n-1}{2}}.$$

So if we take a $h^{1-2\alpha}$ in θ_1 by $h^{1-\alpha}$ in the other θ_m region about $(0, \pi/2, \dots, \pi/2)$ we will still have

$$|\phi_\alpha| = |e^{-ij\theta_1}\phi_\alpha| > h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}.$$

□

4. Predicting the correct scale

In this section we discuss the heuristics of the semiclassical proof. The details of the proof can be found in [7] and [10] and thus we will not address them here. The semiclassical approach to eigenfunction estimates is to study quasimodes, that is, functions such that

$$\left\| (h^2\Delta - 1)u \right\|_{L^2(M)} \lesssim h \left\| u \right\|_{L^2(M)},$$

or more generally,

$$\left\| p(x, hD)u \right\|_{L^2(M)} \lesssim h \left\| u \right\|_{L^2(M)},$$

where $p(x, hD)$ is a semiclassical pseudodifferential operator,

$$p(x, hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y, \xi \rangle} p(x, \xi) u(y) d\xi dy,$$

whose symbol, $p(x, \xi)$ satisfies the admissibility criteria:

1) If $p(x_0, \xi_0) = 0$, then $\nabla_\xi p(x_0, \xi_0) \neq 0$.

2) The characteristic set $\{\xi \mid p(x_0, \xi) = 0\}$ has positive definite second fundamental form.

For Laplacian eigenfunctions the semiclassical symbol $p(x, \xi) = |\xi||_g^2 - 1$, so clearly this is admissible. In fact, in the flat case the characteristic set is the $n-1$ sphere (the canonical example of a hypersurface with positive definite second fundamental form). Quasimodes, as distinct from eigenfunctions, have the nice property that they remain quasimodes under localisation so we may work locally. It is relatively easy to show that contributions localised away from the characteristic set are small. Therefore we may work locally around some point (x_0, ξ_0) such that $p(x_0, \xi_0) = 0$. To prove L^p estimates we perform the following steps.

Step 1. Factorise the symbol. Since the characteristic set is nondegenerate (by admissibility condition 1) we can always find some ξ_i such that

$$|\partial_{\xi_1} p(x_0, \xi_0)| > c > 0,$$

so by the implicit function theorem, locally

$$p(x, \xi) = e(x, \xi)(\xi_i - a(x, \xi')),$$

where $|e(x, \xi)| > c > 0$. The semiclassical calculus then tells us we may invert $e(x, hD)$ to obtain

$$(hD_{x_i} - a(x, hD_{x'}))u = hf,$$

where $\|f\|_{L^2(M)} \lesssim \|u\|_{L^2(M)}$.

Step 2. By setting $x_i = t$ we find that u is an approximate solution to the semiclassical evolution equation

$$(hD_t - a(t, x', hD_{x'}))v(t, x) = 0.$$

Therefore by Duhammel's principle we may write

$$u = U(t, 0)u(0, x') + \int_0^t U(t - \tau, \tau)f(\tau)d\tau,$$

where $U(t, \tau)$ satisfies

$$\begin{cases} (hD_t - a_1(\tau + t, x', hD_{x'}))U_h(t, \tau) = 0, \\ U_h(0, \tau) = \mathrm{Id}\,. \end{cases}$$

The problem then reduces to finding (uniform in τ) $L^2 \to L^p$ mapping norms of $U(t, \tau)$ or the restriction of $U(t, \tau)$ to a submanifold.

Step 3. We estimate the $L^2(M) \to L^p(X)$ norms through a $TT^\star$ method. The key point is to obtain estimates of the form

$$\|U(t, \tau)U(s, \tau)^\star\|_{L^1(X) \to L^\infty(X)} \lesssim h^{-\kappa_\infty}(h + |t - s|)^{-\gamma_\infty}$$

and

$$\|U(t, \tau)U(s, \tau)^\star\|_{L^2(X) \to L^2(X)} \lesssim h^{-\kappa_2}(h + |t - s|)^{-\gamma_2}.$$

All other estimates follow by interpolation with these and by resolving the $t - s$ integral with either Young's inequality or Hardy-Littlewood-Sobolev. It is here we see the connection with Keel-Tao [6] abstract Strichartz estimates which can be proved in the same fashion.

For submanifold estimates there is an additional question of whether or not this special direction, $x_i = t$, lies along the submanifold. It turns out that we may assume it does, as this case gives all sharp estimates. That is, if $\Sigma = \{(y,z) \in M \mid z = 0\}$, we may assume that ξ_i is dual to y_1.

The interpolation argument of Step 3 gives an estimate of the form

$$\|U(t)U(s)^\star\|_{L^{p'} \to L^p} \lesssim h^{-\kappa_p}(h + |t-s|)^{-\gamma_p}.$$

We can think of this as a decay estimate for propagation time $|t-s|$. To generate sharp examples we then need to find what scale of $|t-s|$ makes the largest contribution to the estimate. The sharp example will then be the T_α whose long direction is equal to this critical scale $|t-s|_c$. That is, $|t-s|_c = h^{1-2\alpha}$.

Therefore the regime changes in the L^p estimates depend only on the power γ_p, the numerology of which depends only on the $L^1(X) \to L^\infty(X)$ estimates and the $L^2(X) \to L^2(X)$ estimates. To resolve the $t-s$ integral we estimate

$$\int (h + |\tau|)^{-\frac{\gamma_p p}{2}} d\tau.$$

- If $\frac{\gamma_p p}{2} > 1$ the major contribution comes from the smallest possible $\tau = \tau_{min}$.
- If $\frac{\gamma_p p}{2} < 1$ the major contribution comes from the largest possible τ_{max}.

In both cases we expect the sharp examples to be given by $T_{\alpha_{min}}$ and $T_{\alpha_{max}}$, where

$$\tau_{min} = h^{1-\alpha_{min}}, \quad \tau_{max} = h^{1-\alpha_{max}}.$$

Independent of X we can obtain an $L^1(X) \to L^\infty(X)$ estimate of

$$\|U(t,\tau)U(s,\tau)^\star\|_{L^1(X) \to L^\infty(X)} \lesssim h^{-\frac{n-1}{2}}(h + |t-s|)^{-\frac{n-1}{2}}, \tag{8}$$

so the key point is to obtain the $L^2(X) \to L^2(X)$ estimates. In [10] we see that these are given by the $L^2(X) \to L^2(X)$ mapping norms of an operator

$$W(t-s)u = \int W(x,y,t-s)u(y)dy,$$

$$W(x,y,t-s) = h^{-\frac{n-1}{2}}(h + |t-s|)^{-\frac{n-1}{2}} e^{\frac{i}{h}\phi(x,y,t-s)} b(t,s,x,y),$$

where the factor

$$e^{\frac{i}{h}\phi(x,y,t-s)}$$

oscillates with frequency $h^{-1}|t-s|^{-1}$. From considerations of almost orthogonality we expect that the $L^2(X) \to L^2(X)$ mapping norm of such an operator should be determined by the mapping norm on $h^{1/2}|t-s|^{1/2}$ boxes. This suggests a general heuristic for finding those p at which the behaviour of the $L^2(M) \to L^p(X)$ estimates change.

1. Calculate the $L^2(X) \to L^2(X)$ mapping norm of $U(t,\tau)U(s,\tau)^\star$ on the intersection of an $h^{1/2}|t-s|^{1/2}$ box with X.
2. Interpolate that result with the $L^1(X) \to L^\infty(X)$ estimate given by (8). This will give γ_p for all p.
3. Find the values of p for which $\frac{\gamma_p p}{2} = 1$. We expect regime changes at these p.
4. Determine τ_{min} and τ_{max} for each critical p. The functions $T_{\alpha_{min}}$ and $T_{\alpha_{max}}$ are expected to give sharp examples.

4.1. **Whole and submanifold estimates.** We apply the heuristic and consider the $L^2(X) \to L^2(X)$ norm on the intersection between X and an $h^{1/2}|t-s|^{1/2}$ box. We obtain, for X a k-dimensional submanifold,

$$\begin{aligned}\|W(t-s)\|_{L^2(X)\to L^2(X)} &\lesssim h^{-\frac{n-1}{2}}(h+|t-s|)^{-\frac{n-1}{2}}(h^{\frac{1}{2}}|t-s|^{1/2})^{k-1}\\ &= h^{-\frac{n-k}{2}}(h+|t-s|)^{-\frac{n-k}{2}}.\end{aligned}$$

From the interpolation numerology we obtain that for whole manifolds and hypersurfaces there is only one p so that $\frac{\gamma_p p}{2} = 1$ ($p = \frac{2(n+1)}{n-1}$ and $p = \frac{2n}{n-1}$, respectively). For a lower-dimensional submanifold $\frac{\gamma_p p}{2} \geq 1$ for all $p \geq 2$. Since we truncate at $|t-s| \leq h$ the smallest effective scale is $\tau_{min} = h$, and since we are dealing with compact sets, $\tau_{max} = 1$. Therefore our sharp examples will come from T_0 and $T_{1/2}$ for the whole manifold and hypersurface case, and from T_0 alone for the lower submanifolds.

4.2. Σ_β **estimates.** By considering Σ_β we introduce a new scale (namely h^β) into the problem. If $h^\beta \geq h^{1/2}|t-s|^{1/2}$ an $h^{1/2}|t-s|^{1/2}$ box can lie fully in Σ_β, and therefore we get the $L^2 \to L^2$ estimate

$$\|W(t-s)\|_{L^2(\Sigma_\beta)\to L^2(\Sigma_\beta)} \lesssim 1, \qquad |t-s| \leq h^{2\beta-1},$$

which is the same as over the whole manifold. If on the other hand $h^\beta \leq h^{1/2}|t-s|^{1/2}$, the $h^{1/2}|t-s|^{1/2}$ box does not lie fully in Σ_β, so we obtain

$$\|W(t-s)\|_{L^2(\Sigma_\beta)\to L^2(\Sigma_\beta)} \lesssim h^{-\frac{n-k}{2}+\frac{\beta(n-k)}{2}}(h+|t-s|)^{-\frac{n-k}{2}}, \quad |t-s| \geq h^{2\beta-1}.$$

Therefore we potentially have two points at which $\frac{\gamma_p p}{2} = 1$. Where Σ is a hypersurface there are two critical points. The first arises from the $|t-s| \leq h^{2\beta-1}$ estimates and is at $p = \frac{2(n+1)}{n-1}$. Therefore for this critical point $\tau_{min} = h$ and $\tau_{max} = h^{2\beta-1}$. The second point arises from the $|t-s| \geq h^{2\beta-1}$ estimate and is at $p = \frac{2n}{n-1}$. For this critical point $\tau_{min} = h^{2\beta-1}$ and $\tau_{max} = 1$. Therefore sharp behaviour should be determined by T_0, $T_{1-\beta}$, and $T_{1/2}$. For lower-dimensional submanifolds we obtain a critical p coming from the $|t-s| \leq h^{2\beta-1}$ estimate again with $\tau_{min} = h$, $\tau_{max} = h^{2\beta-1}$. However, from the long-time estimates we always have $\frac{\gamma_p p}{2} \geq 1$; therefore we only need to examine τ_{min}, in this case $h^{2\beta-1}$. Therefore sharp examples should come from T_0 and $T_{1-\beta}$ alone.

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