PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 8, August 2018, Pages 3473–3483
http://dx.doi.org/10.1090/proc/13410
Article electronically published on April 17, 2018

SCALAR CURVATURE BOUND AND COMPACTNESS RESULTS FOR RICCI HARMONIC SOLITONS

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(Communicated by Guofang Wei)

Abstract. In this paper, we study the gradient Ricci harmonic soliton. For noncompact gradient shrinking Ricci harmonic solitons, we prove that the scalar curvature has at most quadratic decay. Given some curvature conditions, we prove that these shrinking solitons must be compact. In two dimensions, we can get similar results with weaker assumptions.

1. Introduction

Ricci flow was introduced by Hamilton in [9]. It has played an important role in the proof of the Poincaré Conjecture [14], [15], [16] and the sphere theorem [1] and has been widely used in the study of geometric and topological problems.

Ricci harmonic flow is defined as follows:

$$\begin{cases} \frac{\partial}{\partial t} g = -2Ric + 2d\phi \otimes d\phi, \\ \frac{\partial}{\partial t}\phi = \Delta\phi, \end{cases} \tag{1.1}$$

where $\phi : (M, g) \to \mathbb{R}$ is a smooth function. Ricci harmonic flow was introduced in [10], where the author proved short time existence and long time existence if ϕ is a smooth function from M to $\mathbb{R}$. Later, in [13], the author considered ϕ as a smooth map from (M, g) to (N, h) and proved some fundamental results for flow equations (1.1). The flow equations (1.1) come from static Einstein vacuum equations arising in the general relativity, and also arise as dimensional reductions of Ricci flow in higher dimensions.

The gradient shrinking Ricci soliton is the blow-up limit of a Type I solution of Ricci flow, and it plays an important role in the singularity analysis; see [2], [5]. Similar results hold for Ricci harmonic flow; see [7]. First we give the precise definition of a gradient Ricci harmonic soliton.

Definition 1.1. Assume that (M^n, g) is a smooth Riemannian manifold and let $\phi : M \to \mathbb{R}$ be a smooth function. If there is a smooth function $f : M \to \mathbb{R}$ and a constant λ such that

$$\begin{cases} Ric - d\phi \otimes d\phi + \nabla^2 f + \lambda g = 0, \\ \Delta\phi = \langle \nabla f, \nabla\phi \rangle, \end{cases} \tag{1.2}$$

then (M, g, ϕ, f) is called a gradient Ricci harmonic soliton and f the potential function. The soliton is called shrinking, steady or expanding when $\lambda < 0, = 0, > 0$, respectively.

Received by the editors January 19, 2016 and, in revised form, August 2, 2016.
2010 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C24.
Key words and phrases. Ricci harmonic soliton, scalar curvature bound, weighted equation.

For simplicity, we introduce a "Riemannian curvature" for Ricci harmonic solitons, which is denoted as Sm. In local coordinates,

$$S_{ijkl} := R_{ijkl} - \frac{1}{2}(\phi_i \phi_j g_{kl} + \phi_j \phi_k g_{il});$$

then $S_{ik} = g^{jl} R_{ijkl} = R_{ik} - \phi_i \phi_k$, i.e. $Sic = Ric - d\phi \otimes d\phi$. We use S to denote the trace of Sic, i.e. $S = R - |\nabla \phi|^2$. During the rest of the paper, for gradient shrinking Ricci harmonic solitons, we can always assume $\lambda = -\frac{1}{2}$ after rescaling g by a constant.

Definition 1.2. We say $Sec(Sm) \geq 0$ if $Sm(u, v, u, v) \geq 0$ for any orthonormal vectors u and v.

First we get a quadratic lower bound for S following the idea in [3].

Proposition 1.3. *Suppose (M, g, ϕ, f) is a complete noncompact nonflat gradient shrinking Ricci harmonic soliton. Then there exist $r > 0$ and $a > 0$ such that*

$$S \geq \frac{a}{f}$$

on $M \backslash B(p, r)$, where p is a critical point of f and a depends only on f.

Remark. The examples of noncompact Kähler Ricci shrinkers by Feldman, Ilmanen and Knopf [6] show that this result is sharp.

Next we consider the compactness results for gradient shrinking Ricci harmonic solitons. We borrow the idea from [12], where the authors consider similar problems for gradient shrinking Ricci solitons.

Theorem 1.4. *Suppose (M^n, g, ϕ, f) is a complete gradient shrinking Ricci harmonic soliton* (1.2)*. If we assume $Sec(Sm) \geq 0$, $Sic > 0$ and ϕ is convex, then (M^n, g) must be compact.*

In two dimensions, because sectional curvature can be expressed in terms of scalar curvature, we can get similar results with weaker assumptions.

Corollary 1.5. *Suppose we have a two dimensional complete gradient shrinking Ricci harmonic soliton. If $R \geq 4|\nabla \phi|^2$, then (M^2, g) is compact.*

2. Basic facts about Ricci harmonic solitons

Lemma 2.1. *Suppose that (M, g, ϕ, f) is a complete gradient shrinking Ricci harmonic soliton. Then*

$$dS = 2Sic(\nabla f). \tag{2.3}$$

Proof. Using Ricci identity and soliton equation (1.2), we get

$$\begin{aligned}
\nabla_i S &= -\nabla_i \Delta f \\
&= -\nabla_i \nabla_k \nabla_k f \\
&= -(\nabla_k \nabla_i \nabla_k f + R_{ikkl} \nabla_l f) \\
&= \nabla_k(\lambda g_{ik} + S_{ik}) + R_{il}\nabla_l f \\
&= \nabla_k S_{ik} + R_{il}\nabla_l f \\
&= \nabla_k(R_{ik} - \nabla_i\phi\nabla_k\phi) + R_{il}\nabla_l f \\
&= \frac{1}{2}\nabla_i R - \nabla_k\nabla_i\phi\nabla_k\phi - \nabla_i\phi\Delta\phi + R_{il}\nabla_l f \\
&= \frac{1}{2}\nabla_i R - \frac{1}{2}\nabla_i|\nabla\phi|^2 + R_{il}\nabla_l f - \nabla_i\phi\langle\nabla f, \nabla\phi\rangle \\
&= \frac{1}{2}\nabla_i S + S_{il}\nabla_l f.
\end{aligned}$$

So

$$\nabla_i S = 2S_{il}\nabla_l f,$$

i.e. $dS = 2Sic(\nabla f)$. □

Lemma 2.2. *Under the same assumption as Lemma* 2.1*, we have*

$$S + |\nabla f|^2 + 2\lambda f = C.$$

Proof. Using Lemma 2.1, we get

$$\begin{aligned}
& \nabla_i(S + |\nabla f|^2 + 2\lambda f) \\
= \ & \nabla_i S + 2\nabla_i\nabla_k f\nabla_k f + 2\lambda\nabla_i f \\
= \ & 2S_{il}\nabla_l f + 2\nabla_i\nabla_k f\nabla_k f + 2\lambda\nabla_i f \\
= \ & 2S_{il}\nabla_l f + (-2\lambda g_{ik} - 2S_{ik})\nabla_k f + 2\lambda\nabla_i f = 0.
\end{aligned}$$

So $S + |\nabla f|^2 + 2\lambda f = C$. □

Lemma 2.3. *Suppose that* (M, g, ϕ, f) *is a compact gradient Ricci harmonic soliton to equation* (1.2) *and* ϕ *is a smooth function from* M *to* $\mathbb{R}$*. Then* ϕ *is a constant; i.e.* (1.2) *reduces to a gradient Ricci soliton.*

Proof. Because (M, g) is compact, we can apply the strong maximum principle to equation $\Delta\phi = \langle\nabla f, \nabla\phi\rangle$, so ϕ is constant. □

Corollary 2.4. *Suppose that* (M, g, ϕ, f) *is a compact gradient steady or expanding Ricci harmonic soliton to equation* (1.2)*. Then* f *is constant; i.e.* (1.2) *reduces to*

$$Ric + \lambda g = 0.$$

Proof. First we prove the expanding case.

Taking trace over equation (1.2) and using Lemma 2.2, we get

$$\begin{cases} S + |\nabla f|^2 + f = C, \\ S + \Delta f + \frac{n}{2} = 0. \end{cases}$$

So

$$\Delta f - f = |\nabla f|^2 - C - \frac{n}{2}.$$

Assume $f(p) = \max_{x\in M} f(x)$, $f(q) = \min_{x\in M} f(x)$. Then

$$\Delta f(p) - f(p) = \Delta f(q) - f(q),$$

i.e.

$$0 \geq \Delta f(p) - \Delta f(q) = f(p) - f(q) \geq 0,$$

so $f(p) = f(q)$, i.e. $f = const$. So (1.2) reduces to

$$\begin{cases} Ric - d\phi \otimes d\phi + \frac{1}{2}g = 0, \\ \Delta\phi = 0. \end{cases}$$

Because M is compact, we know ϕ must be constant.

In the steady case, similarly taking trace over equation (1.2) and using Lemma 2.2,

$$\begin{cases} S + |\nabla f|^2 = C, \\ S + \Delta f = 0. \end{cases}$$

So

$$|\nabla f|^2 - \Delta f = C.$$

Multiplying e^{-f} on both sides,

$$\Delta e^{-f} = Ce^{-f},$$

it is easy to see that $C = 0$, so $\Delta e^{-f} = 0$, i.e. $f =$ constant. Due to the compactness of M we know ϕ is constant. □

3. Elliptic equation of Sic

Ricci harmonic solitons generate the solution of Ricci harmonic flow automatically. In the following, we will focus on the gradient shrinking Ricci harmonic soliton. Suppose we have the following equation:

$$\begin{cases} Sic + \nabla^2 f - \frac{1}{2}g = 0, \\ \Delta\phi = \langle\nabla f, \nabla\phi\rangle. \end{cases} \tag{3.4}$$

Consider the gradient flow generated by ∇f:

$$\begin{cases} \frac{\partial F}{\partial t} = -\nabla f(F), \\ F(0) = Id. \end{cases} \tag{3.5}$$

Proposition 3.1. *When $t \in (-\infty, 0)$,*

$$\begin{cases} g(t) = (-t)F^*_{log(-t)}g, \\ \phi(t) = F^*_{log(-t)}\phi \end{cases}$$

satisfies

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + 2d\phi(t) \otimes d\phi(t), \\ \frac{\partial}{\partial t}\phi(t) = \Delta_{g(t)}\phi(t). \end{cases}$$

Proof. By soliton equation (3.4) and flow equation (3.5), we get

$$\begin{aligned}
\frac{\partial}{\partial t} g(t) &= -F^*_{log(-t)} g + t \cdot \frac{-1}{-t} F^*_{log(-t)} (L_{\nabla f} g) \\
&= -F^*_{log(-t)} g + F^*_{log(-t)} (g - 2Sic(g)) \\
&= -2F^*_{log(-t)} (Ric(g) - d\phi \otimes d\phi) \\
&= -2(Ric(F^*_{log(-t)} g) - dF^*_{log(-t)} \phi \otimes dF^*_{log(-t)} \phi) \\
&= -2(Ric(g(t)) - d\phi(t) \otimes \phi(t))
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} \phi(t) &= \frac{\partial}{\partial t} F^*_{log(-t)} \phi \\
&= \frac{-1}{-t} \cdot (-1) F^*_{log(-t)} L_{\nabla f} \phi \\
&= \frac{-1}{t} F^*_{log(-t)} \langle \nabla f, \nabla \phi \rangle \\
&= \frac{1}{-t} F^*_{log(-t)} (\Delta_g \phi) \\
&= \frac{1}{-t} \Delta_{F^*_{log(-t)} g} F^*_{log(-t)} \phi \\
&= \Delta_{(-t) F^*_{log(-t)} g} F^*_{log(-t)} \phi = \Delta_{g(t)} \phi(t).
\end{aligned}$$

□

In this paper, we need the evolution equation of Sic along the Ricci harmonic flow (1.1).

Proposition 3.2 ([13])**.** *Let $(M, g(t), \phi(t))$ be a solution to the Ricci harmonic flow* (1.1)*. Then we have the following evolution equations:*

$$\frac{\partial}{\partial t} S_{ij} = \Delta S_{ij} + 2R_{ikjl} S_{kl} - S_{ik} R_{kj} - R_{ik} S_{kj} + 2\Delta\phi \nabla_i \nabla_j \phi, \tag{3.6}$$

$$\frac{\partial}{\partial t} S = \Delta S + 2|Sic|^2 + 2|\Delta\phi|^2. \tag{3.7}$$

Proposition 3.3. *Suppose (M, g, ϕ, f) satisfies gradient shrinking Ricci harmonic soliton equation* (3.4)*. Then*

$$\Delta_f S_{ij} = S_{ij} - 2R_{ikjl} S_{kl} - 2\Delta\phi \nabla_i \nabla_j \phi + \phi_i \phi_k S_{kj} + \phi_k \phi_j S_{ik}, \tag{3.8}$$

where $\Delta_f Sic = \Delta Sic - \nabla_{\nabla f} Sic$.

Proof. By the diffeomorphism invariance,

$$\begin{aligned}
&\frac{\partial}{\partial t} Sic(g(t))|_{t=-1} \\
&= \frac{\partial}{\partial t} F^*_{log(-t)} Sic(g)|_{t=-1} \\
&= \frac{-1}{-t} (-1) L_{\nabla f} Sic(g)|_{t=-1} \\
&= L_{\nabla f} Sic(g).
\end{aligned}$$

Choose normal coordinates $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$ at point $q \in M$. We have

$$\begin{aligned}
L_{\nabla f} S_{ij} &= (L_{\nabla f} Sic)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= (\nabla_{\nabla f} Sic)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + Sic\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla f, \frac{\partial}{\partial x^j}\right) + Sic\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^j}} \nabla f\right) \\
&= (\nabla_{\nabla_f} Sic)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + Sic\left(\frac{1}{2}\frac{\partial}{\partial x^i} - Sic(\frac{\partial}{\partial x^i}), \frac{\partial}{\partial x^j}\right) \\
&\quad + Sic\left(\frac{\partial}{\partial x^i}, \frac{1}{2}\frac{\partial}{\partial x^j} - Sic(\frac{\partial}{\partial x^j})\right) \\
&= \nabla_{\nabla f} S_{ij} + S_{ij} - 2S_{ik}S_{kj}.
\end{aligned}$$

On the other hand, by the evolution equation of Ricci tensor along the Ricci harmonic flow (3.6), we get

$$\frac{\partial}{\partial t} S_{ij} = \Delta S_{ij} + 2R_{ikjl}S_{kl} - S_{ik}R_{kj} - R_{ik}S_{kj} + 2\Delta\phi\nabla_i\nabla_j\phi,$$

so

$$\begin{aligned}
\Delta_f S_{ij} &= \Delta S_{ij} - (\nabla_{\nabla f} Sic)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\
&= -2R_{ikjl}S_{kl} + S_{ij} + S_{ik}R_{kj} + R_{ik}S_{kj} - 2S_{ik}S_{kj} - 2\Delta\phi\nabla_i\nabla_j\phi \\
&= S_{ij} - 2R_{ikjl}S_{kl} - 2\Delta\phi\nabla_i\nabla_j\phi + \phi_i\phi_k S_{kj} + \phi_k\phi_j S_{ik}.
\end{aligned}$$

□

4. Main results

4.1. Scalar curvature bound.

Proposition 4.1 (Yang-Shen, [18]). *Suppose that (M, g, ϕ, f) is a complete noncompact shrinking gradient Ricci harmonic soliton. Then the potential function f satisfies*

$$\frac{1}{4}(d(x,p) - C_1)^2 \le f(x) \le \frac{1}{4}(d(x,p) + C_2)^2,$$

where p is a critical point of f and C_1, C_2 are positive constants depending only on the dimension of the manifold and the geometry of the unit ball $B_1(p)$.

Proposition 4.2 (Yang-Shen, [18]). *Suppose that (M, g, ϕ, f) is a complete noncompact shrinking gradient Ricci harmonic soliton. Then $S = R - |\nabla\phi|^2 \ge 0$.*

Proposition 4.3. *Suppose (M, g, ϕ, f) is a complete noncompact nonflat gradient shrinking Ricci harmonic soliton. Then there exists $r > 0$ and $a > 0$ such that*

$$S \ge \frac{a}{f}$$

on $M \backslash B(p,r)$, where p is a critical point of f and a depends on f.

Proof. Due to the asymptotic behavior of f, we know on $M \setminus B(p,r)$ where r is large enough:

$$\begin{aligned}\Delta_f f^{-1} &= -\Delta_f f \cdot f^{-2} + 2|\nabla f|^2 f^{-3} \\ &= (f - \frac{n}{2})f^{-2} + 2|\nabla f|^2 \cdot f^{-3} \\ &\geq f^{-1} - \frac{n}{2} f^{-2},\end{aligned}$$

$$\Delta_f f^{-2} = 2(f - \frac{n}{2})f^{-3} + 6|\nabla f|^2 f^{-4} \geq \frac{3}{2} f^{-2}.$$

Define $u = S - \frac{a}{f} - \frac{na}{f^2}$, where $a = \min_{\partial B(p,r)} S$. Applying the asymptotic behavior of f again, we know u is positive on $\partial B(p,r)$ if r is large enough.

On $M \backslash B(p,r)$,

$$\begin{aligned}\Delta_f u &= \Delta_f S - \Delta_f \left(\frac{a}{f}\right) - \Delta_f \left(\frac{na}{f^2}\right) \\ &\leq S - a(f^{-1} - \frac{n}{2} f^{-2}) - \frac{3}{2} na f^{-2} \\ &= S - \frac{a}{f} - \frac{na}{f^2} = u.\end{aligned}$$

Next we claim $u \geq 0$ on $M \backslash B(p,r)$.

If there exists x_0 such that $u(x_0) < 0$, then due to $\min_{\partial B(p,r)} u > 0$ and $\varliminf_{x \to \infty} u(x) \geq 0$, there is y_0 such that $u(y_0) = \min_{M \backslash B(p,r)} u < 0$. So

$$0 \leq \Delta_f u(y_0) \leq u(y_0) < 0,$$

a contradiction.

Hence we get $u \geq 0$ on $M \backslash B(p,r)$, i.e. $S \geq \frac{a}{f}$.

4.2. **Compactness of gradient shrinking Ricci harmonic solitons.**

Proposition 4.4. *Suppose* (M, g, ϕ, f) *is a complete gradient shrinking Ricci harmonic soliton* (1.2)*. If we assume* $Sec(Sm) \geq 0$*,* $Sic > 0$ *and* ϕ *is convex, then it must be compact.*

Proof. Assume that (M, g, ϕ, f) is noncompact. From Lemma 2.2, we know that $S + |\nabla f|^2 - f = C$, where S is the trace of Sic. By adding a constant to f if necessary, we can assume that $S + |\nabla f|^2 = f$. Concerning the potential f, Yang and Shen [18] proved that

$$\frac{1}{4}(d(x,p) - C_1)^2 \leq f(x) \leq \frac{1}{4}(d(x,p) + C_2)^2, \tag{4.9}$$

where C_1 and C_2 are positive constants depending only on the dimension of the manifold and the geometry of the unit ball $B_p(1)$.

From (3.8), we know that

$$\Delta_f S_{ij} = S_{ij} - 2R_{ikjl}S_{kl} - 2\Delta\phi \nabla_i \nabla_j \phi + \phi_i \phi_k S_{kj} + \phi_k \phi_j S_{ik}. \tag{4.10}$$

Recall that we have defined the "Riemannian Curvature" Sm for the Ricci harmonic soliton, which is rewritten as

$$S_{ijkl} := R_{ijkl} - \frac{1}{2}(\phi_i \phi_j g_{kl} + \phi_j \phi_k g_{il}).$$

Then $S_{ik} = g^{jl}R_{ijkl} = R_{ik} - \phi_i\phi_k$, i.e. $Sic = Ric - d\phi \otimes d\phi$. Using the above notation, (4.10) can be rewritten as

$$\Delta_f S_{ij} = S_{ij} - 2S_{ikjl}S_{kl} - 2\Delta\phi\nabla_i\nabla_j\phi. \tag{4.11}$$

Denote $\lambda(x)$ as the minimal eigenvalue of Sic at x, and suppose v is the eigenvector corresponding to $\lambda(x)$. Then

$$S_{ikjl}S_{kl}v_iv_j = Sm(v, e_k, v, e_l)S_{kl}.$$

Diagonalizing Sic so that $S_{kl} = \lambda_k\delta_{kl}$, it follows from the assumption that

$$S_{ikjl}S_{kl}v_iv_j = Sec(Sm)(v, e_k)\lambda_k \geq 0. \tag{4.12}$$

From $\Delta\phi\nabla^2\phi \geq 0$, (4.11) and (4.12), it follows that λ satisfies the following differential inequality in the sense of barrier:

$$\Delta_f\lambda \leq \lambda.$$

Actually, this means that at any point x, we can find a smooth function $\widetilde{u}$ such that $\widetilde{u} \geq \lambda$ on $B(x,\delta)$ and $\Delta_f\widetilde{u}(x) \leq \widetilde{u}(x)$, where δ is a small positive constant.

Choose a geodesic ball $B(p, r)$ of radius r large enough, and let

$$a = \min_{\partial B(p,r)} \lambda > 0. \tag{4.13}$$

Similar to the proof of Proposition 4.3, we define

$$u = \lambda - \frac{a}{f} - \frac{na}{f^2}.$$

From (4.13) and (4.9), it follows that if r is large enough, then $u > 0$ on $\partial B(p,r)$. From (4.9) and Proposition 4.2, we get $\underline{\lim}_{x\to\infty}u(x) \geq 0$. As in the proof of Proposition 4.3, we get

$$\begin{aligned}\Delta_f u &= \Delta_f\lambda - \Delta_f(\frac{a}{f}) - \Delta_f(\frac{na}{f^2})\\ &\leq \lambda - a(f^{-1} - \frac{n}{2}f^{-2}) - \frac{3}{2}naf^{-2}\\ &= \lambda - \frac{a}{f} - \frac{na}{f^2} = u.\end{aligned}$$

We have now proved

$$\Delta_f u \leq u$$

on $M\backslash B(p,r)$ if r is large enough.

Next we prove $u \geq 0$ on $M\backslash B(p,r)$. If there is a point $y_0 \in M\backslash B(p,r)$ such that $u(y_0) < 0$, then there must exist a point $x_0 \in M\backslash B(p,r)$ such that $u(x_0) = \min_{y\in\{M\backslash B(p,r)\}} u(y)$.

At x_0, suppose v is the eigenvector corresponding to $\lambda(x_0)$. Take parallel translation along all the unit geodesics starting from x_0. Then in a small ball $B(x_0,\delta)$ we get a smooth vector field V with $V(x_0) = v$. Define $\widetilde{u} = Sic(V(y), V(y)) - \frac{a}{f} - \frac{na}{f^2}$. Then for any $y \in B(x_0,\delta)$, $\widetilde{u}(y) \geq u(y)$ and $\widetilde{u}(x_0) = u(x_0)$, so

$$0 \leq \Delta_f\widetilde{u}(x_0) \leq \widetilde{u}(x_0) < 0.$$

This is a contradiction, so $u \geq 0$ on $M\backslash B(p,r)$, i.e. $Sic \geq \frac{a}{f}$. As in the argument in [12], we get $S \geq b \cdot \log f$ for some $b > 0$.

Next we give another proof of the above asymptotic bound of S assuming that f has no critical point on $M\backslash B(p,r)$. Choose level set $\{f=r_0\}$ which is contained in $M\backslash B(p,r)$, and consider the flow line starting from $\partial\{f=r_0\}$,

$$\begin{cases} \frac{\partial F}{\partial s}=\frac{\nabla f}{|\nabla f|^2}, \\ F(q,r_0)=Id, q\in\partial\{f=r_0\}. \end{cases} \tag{4.14}$$

The advantage of this flow is that it maps one level set of f to another one. Along this flow,

$$\begin{aligned} \frac{d}{ds}S\circ F(q,s) &= \langle\nabla S,\frac{\partial F}{\partial s}\rangle \\ &= \langle\nabla S,\frac{\nabla f}{|\nabla f|^2}\rangle \\ &= 2Sic(\frac{\nabla f}{|\nabla f|},\frac{\nabla f}{|\nabla f|}) \\ &\geq \frac{2a}{f}=\frac{2a}{s}, \end{aligned}$$

so

$$\begin{aligned} S\circ F(q,s) &\geq S\circ F(q,r_0)+\int_{r_0}^{s}\frac{2a}{\tau}d\tau \\ &= S\circ F(q,r_0)+2a\log\frac{s}{r_0}. \end{aligned}$$

Hence $S\geq b\cdot\log f$ on $M\backslash B(p,r)$ for some $b>0$ if r is large enough.

From the soliton equation (1.2), we get $\Delta f+S=\frac{n}{2}$. Consider the sublevel set $\{f\leq c\}$ of f, integrating over it, we get

$$\begin{aligned} \int_{\{f\leq c\}}\frac{n}{2} &= \int_{\{f\leq c\}}(\Delta f+S) \\ &= \int_{\{f=c\}}|\nabla f|+\int_{\{f\leq c\}}S. \end{aligned}$$

So the average of S over $\{f\leq c\}$ is less than $\frac{n}{2}$.

On the other hand, choose c_0 such that $b\log c_0>n$, when c is large enough:

$$\begin{aligned} & \frac{1}{Vol\{f\leq c\}}\int_{\{f\leq c\}}S \\ = \;& \frac{Vol\{c_0\leq f\leq c\}}{Vol\{f\leq c\}}\frac{\int_{\{f\leq c\}}S}{Vol\{c_0\leq f\leq c\}} \\ \geq \;& \frac{Vol\{c_0\leq f\leq c\}}{Vol\{f\leq c\}}\frac{\int_{\{c_0\leq f\leq c\}}S}{Vol\{c_0\leq f\leq c\}} \\ \geq \;& \frac{Vol\{c_0\leq f\leq c\}}{Vol\{f\leq c\}}b\log c_0 \\ > \;& \frac{n}{2}. \end{aligned}$$

This contradicts the fact that the average of S over $\{f\leq c\}$ is less than $\frac{n}{2}$. In the last inequality we use the fact that $Vol(M)=\infty$, so $\frac{Vol\{c_0\leq f\leq c\}}{Vol\{f\leq c\}}$ is close to 1 when c is large enough. Actually in [17] the authors proved that any noncompact gradient shrinking Ricci harmonic soliton has at least linear volume growth. □

Remark. Actually, in the last step of the above proof, we can use the volume comparison as Munteanu-Wang [12] did to get a similar contradiction, because we have $\mathrm{Ric} \geq Sic > 0$.

In two dimensions, we can get similar results with weaker assumptions.

Corollary 4.5. *Suppose we have a two dimensional complete gradient shrinking Ricci harmonic soliton. If $R \geq 4|\nabla\phi|^2$, then (M^2, g) must be compact.*

Proof. By Proposition 4.3, we know that $S \geq \frac{a}{f}$ on $M\backslash B(p,r)$ for some $r > 0$ and $a > 0$.

If $R \geq 4|\nabla\phi|^2$, then

$$\begin{aligned} Sic &= Ric - d\phi \otimes d\phi \\ &= \frac{1}{2}Rg - |\nabla\phi|^2 \\ &\geq \frac{1}{3}(R - |\nabla\phi|^2)g = \frac{1}{3}Sg \geq \frac{a}{3f}g. \end{aligned}$$

Then arguing as in the above proposition we get that (M^2, g) is compact. $\square$

Appendix

In this appendix we give another proof of the equation (3.8) without using the evolution equation of Ricci harmonic flow. Because a Ricci harmonic soliton equation is "elliptic", we can give a proof just using Ricci identity:

$$\begin{aligned} \Delta S_{ij} &= -\nabla_i\nabla_j\Delta f - \nabla_i R_{jl}\nabla_l f - \nabla_j R_{il}\nabla_l f - R_{il}\nabla_j\nabla_l f - R_{jl}\nabla_i\nabla_l f \\ &\quad +\nabla_{\nabla f}R_{ij} - 2R_{kijl}\nabla_k\nabla_l f \\ &= \nabla_i\nabla_j S + \nabla_{\nabla f}(S_{ij} + \nabla_i\phi\nabla_j\phi) + 2R_{kijl}(\frac{1}{2}g_{kl} - S_{kl}) \\ &\quad -\nabla_i(S_{jl} + \nabla_j\phi\nabla_l\phi)\nabla_l f - \nabla_j(S_{il} + \nabla_i\phi\nabla_l\phi)\nabla_l f \\ &\quad -(S_{il} + \nabla_i\phi\nabla_l\phi)\nabla_j\nabla_l f - (S_{jl} + \nabla_j\phi\nabla_l\phi)\nabla_i\nabla_l f \\ &= \nabla_i\nabla_j S + \nabla_{\nabla f}S_{ij} + \nabla_{\nabla f}(\nabla_i\phi\nabla_j\phi) + R_{ij} - 2R_{kijl}S_{kl} \\ &\quad -\nabla_i S_{jl}\nabla_l f - \nabla_i(\nabla_j\phi\nabla_l\phi)\nabla_l f - \nabla_j S_{il}\nabla_l f - \nabla_j(\nabla_i\phi\nabla_l\phi)\nabla_l f \\ &\quad -(\frac{1}{2}g_{il} - \nabla_i\nabla_l f + \nabla_i\phi\nabla_l\phi)\nabla_j\nabla_l f - (\frac{1}{2}g_{jl} - \nabla_j\nabla_l f + \nabla_j\phi\nabla_l\phi)\nabla_i\nabla_l f \\ &= \nabla_i\nabla_j S + 2\nabla_i\nabla_l f\nabla_j\nabla_l f - \nabla_i\nabla_j f + \nabla_i\nabla_j\nabla_l f\nabla_l f + \nabla_j\nabla_i\nabla_l f\nabla_l f \\ &\quad +\nabla_{\nabla f}S_{ij} + \nabla_{\nabla f}(\nabla_i\phi\nabla_j\phi) + R_{ij} - 2R_{ikjl}S_{kl} - \nabla_i(\nabla_j\phi\nabla_l\phi)\nabla_l f \\ &\quad -\nabla_j(\nabla_i\phi\nabla_l\phi)\nabla_l f - \nabla_i\phi\nabla_l\phi\nabla_j\nabla_l f - \nabla_j\phi\nabla_l\phi\nabla_i\nabla_l\phi \\ &= -2\nabla_i\nabla_j\phi\Delta\phi - \nabla_j\phi\nabla_i\nabla_l\phi\nabla_l f - \nabla_i\phi\nabla_j\nabla_l\phi\nabla_l f - \nabla_i\phi\nabla_l\phi(\frac{1}{2}g_{jl} - S_{jl}) \\ &\quad -\nabla_j\phi\nabla_l\phi(\frac{1}{2}g_{il} - S_{il}) + \nabla_{\nabla f}(\nabla_i\phi\nabla_j\phi) + \nabla_{\nabla f}S_{ij} + R_{ij} - 2R_{ikjl}S_{kl} \\ &= S_{ij} - 2\nabla_i\nabla_j\phi\Delta\phi + \nabla_{\nabla f}S_{ij} - 2R_{ikjl}S_{kl} + \nabla_i\phi\nabla_k\phi S_{jk} + \nabla_j\phi\nabla_k\phi S_{ik}. \end{aligned}$$

So

$$\begin{aligned} \Delta_f S_{ij} &= \Delta S_{ij} - \nabla_{\nabla_f}S_{ij} \\ &= S_{ij} - 2R_{ikjl}S_{kl} - 2\nabla_i\nabla_j\phi\Delta\phi + \nabla_i\phi\nabla_k\phi S_{jk} + \nabla_j\phi\nabla_k\phi S_{ik}. \end{aligned}$$

$\square$

Acknowledgments

The author would like to thank Doctor Peng Wu and Doctor Yi Li for many helpful discussions. The author is also grateful to Prof. Xianzhe Dai, Prof. Yu Zheng and Prof. Xi-nan Ma for their constant encouragement.

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