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EVEN MORE ON PARTITIONING TRIPLES OF COUNTABLE ORDINALS

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(Communicated by Mirna Džamonja)

Dedicated to the memory of James E. Baumgartner, my mentor and friend. I miss you, Jim.

Abstract. We prove that $\omega_1 \to (\omega+\omega+1, n)^3$ for all $n < \omega$.

1. Some background

The conjecture below is generally attributed to P. Erdős, though it seems to have first appeared in print in [9] by E. C. Milner and K. Prikry.

Conjecture (ZFC). $\omega_1 \to (\alpha, n)^3$ *for all* $\alpha < \omega_1$ *and all* $n < \omega$.

In [5] A. Hajnal proved that

$$\omega_1 \not\to (\omega_1, 4)^3.$$

In [6, §4, Theorem 2] we[(a)] proved that

$$\omega_1 \not\to (\omega+2, \omega)^3.$$

Thus, if the conjecture is true, then it is sharp.

Some progress has been made on the conjecture. In [9, §1, Equation 1.8] E. C. Milner and K. Prikry proved that

$$\omega_1 \to (\omega+m, 4)^3$$

for all $m < \omega$. In [7, §3, Theorem 1] we proved that

$$\omega_1 \to (\omega+m, n)^3$$

for all $m < \omega$ and all $n < \omega$. In [10, §1, Equation 1.6] E. C. Milner and K. Prikry proved that

$$\omega_1 \to (\omega+\omega+1, 4)^3.$$

Here we prove that

$$\omega_1 \to (\omega+\omega+1, n)^3$$

for all $n < \omega$. To our knowledge, this result represents the best progress thus far on the conjecture above.

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(a) We think that this result was known before (as was claimed in [9]). We believe that this result is due to Erdős and Rado, but the nearest we could find in their published works was

$$\omega_1 \not\to (\omega+2, \omega+1)^3$$

which appeared in [3, §7, Theorem 41].

2. Our notation and terminology

In this section, we define many terms that we use in later sections, including the notion of *wondrous graph*, the family of all such graphs $\mathcal{W}$, basic notation like V_G, E_G, T_G, and $G[X]$ associated with a graph G.

If X is a set of ordinals and α is an ordinal, then $[X]^\alpha$ is the collection of subsets of X whose ordertype is α. In particular, $[X]^2$ and $[X]^3$ are the sets of pairs and triples of elements of X, respectively.

If X and Y are sets of ordinals and α and β are ordinals, then $[X,Y]^{\alpha,\beta}$ is the collection of subsets of $X \cup Y$ whose intersection with X has ordertype α and whose intersection with Y has ordertype β. In particular, $[X,Y]^{1,1}$ is the set of pairs of elements of $X \cup Y$ with one element from X and the other element from Y, and $[X,Y]^{2,1}$ is the set of triples of elements of $X \cup Y$ with two elements from X and one element from Y.

If G is a graph, then V_G is the set of vertices of G, E_G is the set of edges of G, and $T_G = \{X \in [V_G]^3 \mid [X]^2 \subseteq E_G\}$ is the set of triangles of G.

We say that G is a *subgraph* of H if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. We write $G \leq H$ to indicate this relation. (Note that if $G \leq H$, then $T_G \subseteq T_H$, too.)

If $X \subseteq V_G$, then $G[X]$ is the graph *induced* on X by G, the graph with vertices $V_{G[X]} = X$ and edges $E_{G[X]} = E_G \cap [X]^2$.

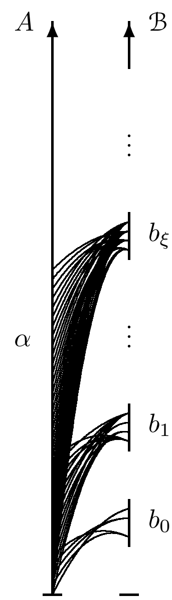
For us, a graph G is *large* if its vertices form an uncountable subset of ω_1, i.e., if $V_G \in [\omega_1]^{\omega_1}$. A large graph G is *boring* if it has an uncountable *independent set*, i.e., there is $X \in [V_G]^{\omega_1}$ with $[X]^2 \cap E_G = \emptyset$.

A large graph G is *wondrous* if for any $X \in [V_G]^{\omega_1}$ there are

$$A \in [X]^{\omega_1} \text{ and } \mathcal{B} = \{b_\xi \mid \xi < \omega_1\} \subseteq [X]^{<\omega}$$

such that

(1) $b_\xi \neq \emptyset$ for all $\xi < \omega_1$,
(2) $b_\xi < b_\eta$ for all $\xi < \eta < \omega_1$, and
(3) for each $\alpha \in A$ and $\xi < \omega_1$ with $\alpha < b_\xi$ there is $\beta \in b_\xi$ with $\{\alpha, \beta\} \in E_G$.



Call such A and $\mathcal{B}$ *wondrous witnesses* for X in G. Note that if A and $\mathcal{B}$ are wondrous witnesses for X in G, then they are wondrous witnesses for any superset of X in G, as well. Also, if A and $\mathcal{B}$ are wondrous witnesses for X in G, then the same is true for any uncountable subsets of A and $\mathcal{B}$.

Let $\mathcal{W}$ be the collection of all wondrous graphs.

Martin's axiom for ω_1 many dense sets (MA_{ω_1}) asserts that for any c.c.c. notion of forcing $\mathbb{P}$ and any collection $\{D_\alpha \mid \alpha < \omega_1\}$ of dense subsets of $\mathbb{P}$ there is a filter $\mathbb{G}$ on $\mathbb{P}$ such that $\mathbb{G} \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. We kindly refer the reader to K. Kunen [8] or M. Fremlin [4] for more information on Martin's Axiom. We do not use MA_{ω_1} directly, but we do use several well-known consequences of it. When we do, we provide specific references.

3. Exploring wondrous graphs

In this section we establish some basic facts about wondrous graphs. In particular, we define Axiom W, introduce simply wondrous graphs, and prove that if ZFC is consistent, then so is ZFC + MA_{ω_1} + Axiom W. Assuming ZFC + MA_{ω_1} + Axiom W, we prove that $G \to (\alpha)^2_n$ for each $\alpha < \omega_1$ and each $n < \omega$ for any wondrous graph G.

Lemma 3.1 (ZFC)**.** *A large graph G is wondrous if and only if all of its large induced subgraphs are wondrous. Moreover, if a graph G is large but not wondrous, then there is $X \in [V_G]^{\omega_1}$ such that no subgraph (induced or otherwise) of $G[X]$ is wondrous.*

Proof. These follow immediately from the definition of wondrous graph. □

Adding edges to a wondrous graph preserves wondrousness. Removing edges preserves non-wondrousness. On the other hand, adding *or* removing any countably many edges or vertices preserves both wondrousness and non-wondrousness.

Let Axiom W be the assertion that $\mathcal{W} \to (\mathcal{W})^2_n$ for all $n < \omega$. In other words, Axiom W states that if the edges of a wondrous graph are partitioned into finitely many classes, then it must include a homogeneous wondrous subgraph, i.e., a wondrous subgraph all of whose edges belong to the same class. Note that the homogeneous wondrous subgraph need not be an induced subgraph of the original graph. Axiom W is a kind of generalization of Ramsey's Theorem to the uncountable that hopes to avoid the obstructions discovered by Sierpiński.

If G is a large graph and $\prec$ is a linear ordering of its vertices, then $G(\prec)$ is the graph on the same vertices whose edges are *gated* by $\prec$.

$$V_{G(\prec)} = V_G \quad \text{and} \quad E_{G(\prec)} = \{\{\alpha, \beta\} \in E_G \mid \alpha < \beta \wedge \alpha \prec \beta\}.$$

Note that $E_{G(\succ)} = E_G \setminus E_{G(\prec)}$.

For any large graph G and linear orderings $\prec_1, \prec_2, \ldots, \prec_n$ of its vertices, the *Sierpiński graph*

$$G(\prec_1, \prec_2, \ldots, \prec_n)$$

is constructed by iterating this gating in the obvious way. Note that the order in which linear orderings are applied is immaterial.

A large graph is *simply wondrous* if for any $X \in [V_G]^{\omega_1}$ there are $A, B \in [X]^{\omega_1}$ such that $\{\alpha, \beta\} \in E_G$ whenever $\alpha \in A$, $\beta \in B$, and $\alpha < \beta$. The sets A and B are *simply wondrous witnesses* for X in G. Clearly, any graph that is simply wondrous is also wondrous. Note that all large complete graphs are simply wondrous.

If A and B are simply wondrous witnesses for X in G, then so are any uncountable subsets of A and B.

Lemma 3.2 (ZFC)**.** *A large graph G is simply wondrous if and only if all of its uncountable induced subgraphs are simply wondrous. Moreover, if a graph G is large but not simply wondrous, then there is $X \in [V_G]^{\omega_1}$ such that no subgraph (induced or otherwise) of $G[X]$ is simply wondrous.*

Proof. These follow immediately from the definition of simply wondrous. □

Lemma 3.3 (ZFC)**.** *If G is a simply wondrous graph and $\prec$ is a linear ordering of the vertices of G, then at least one of $G(\prec)$ or $G(\succ)$ includes an induced simply wondrous subgraph.*

Proof. For $\gamma \in V_G$ and $X \subseteq V_G$, let

$$X_{[\gamma]} = \{\xi \in X \mid \xi \prec \gamma\} \quad \text{and} \quad X^{[\gamma]} = \{\xi \in X \mid \xi \succ \gamma\}.$$

In particular,

$$X_{[\gamma]} \prec \gamma \prec X^{[\gamma]}$$

for all $\gamma \in V_G$ and all $X \subseteq V_G$.

There are three cases to consider.

Case 3.3.1. There is $X \in [V_G]^{\omega_1}$ such that $\alpha < \beta$ implies $\alpha \prec \beta$ for all $\alpha, \beta \in X$.

In this case, the subgraph of $G(\prec)$ induced on X is the same as the subgraph of G induced on X, which is simply wondrous by Lemma 3.2.

Case 3.3.2. There is $X \in [V_G]^{\omega_1}$ such that $\alpha < \beta$ implies $\alpha \succ \beta$ for all $\alpha, \beta \in X$.

Once again, the subgraph of $G(\succ)$ induced on X is the same as the subgraph of G induced on X, which is simply wondrous by Lemma 3.2.

Case 3.3.3. For all $X \in [V_G]^{\omega_1}$ there are $\alpha, \beta \in X$ with $\alpha < \beta$ and $\alpha \prec \beta$ and there are $\gamma, \delta \in X$ with $\gamma < \delta$ and $\gamma \succ \delta$.

In this case, we suppose that $G(\succ)$ has no induced simply wondrous subgraphs and argue that $G(\prec)$ must have an induced simply wondrous subgraph.

Because $G(\succ)$ is not simply wondrous, there must exist $X \in [V_G]^{\omega_1}$ such that for all $A, B \in [X]^{\omega_1}$ there are $\alpha \in A$ and $\beta \in B$ with $\alpha < \beta$ and $\{\alpha, \beta\} \notin E_{G(\prec)}$ (that is, either $\{\alpha, \beta\} \notin E_G$ or $\alpha \prec \beta$). In other words, there must be $X \in [V_G]^{\omega_1}$ which has no simply wondrous witnesses in $G(\succ)$. We prove that the subgraph of $G(\prec)$ induced on X is simply wondrous.

Let Y be an arbitrary uncountable subset of X.

Claim 3.3.1. For any $B \in [Y]^{\omega_1}$ and any $\gamma \in Y$, either $\left|B_{[\gamma]}\right| = \omega_1$ or $\left|B^{[\gamma]}\right| = \omega_1$.

Proof. The union of finitely many countable sets is countable. □

Claim 3.3.2. For any $A \in [Y]^{\omega_1}$ there is $\gamma \in Y$ with $\left|A_{[\gamma]}\right| = \left|A^{[\gamma]}\right| = \omega_1$.

Proof. This (and more) was proved by P. Erdős and R. Rado in [3, §7, Lemma 1]. □

Because G is simply wondrous, there must be $A, B \in [Y]^{\omega_1}$ which are simply wondrous witnesses for Y in G. In particular, we know that for every $\alpha \in A$ and $\beta \in B$, if $\alpha < \beta$, then $\{\alpha, \beta\} \in E_G$.

If only it were true that $A \prec B$, then A and B would be simply wondrous witnesses for Y in the subgraph of $G(\prec)$ induced on X. This is not necessarily true, but we *can* use our assumptions and the claims above to find subsets of A and B which do the trick.

By Claim 3.3.2, there must be $\gamma \in Y$ with $\left|A_{[\gamma]}\right| = \left|A^{[\gamma]}\right| = \omega_1$. By Claim 3.3.1, either $\left|B_{[\gamma]}\right| = \omega_1$ or $\left|B^{[\gamma]}\right| = \omega_1$.

Claim 3.3.3. In fact, $\left|B_{[\gamma]}\right| < \omega_1$ and $\left|B^{[\gamma]}\right| = \omega_1$.

Proof. Otherwise, $A^{[\gamma]}$ and $B_{[\gamma]}$ would be simply wondrous witnesses for X in $G(\succ)$, contradicting our assumption that X had no such witnesses. □

It now follows that $A_{[\gamma]}$ and $B^{[\gamma]}$ are simply wondrous witnesses for Y in the subgraph of $G(\prec)$ induced on X.

As our choice of $Y \in [X]^{\omega_1}$ was arbitrary, we conclude that the subgraph of $G(\prec)$ induced on X is simply wondrous. The lemma follows. □

Lemma 3.4 (ZFC)**.** *There is a proper notion of forcing which forces ZFC+MA_{ω_1}+ Axiom W.*

Proof. Let Axiom T be the assertion that for any partition $[\omega_1]^2 = K_0 \cup K_1$ either there is $X \in [\omega_1]^{\omega_1}$ with $[X]^2 \subseteq K_0$, or there are $A \in [\omega_1]^{\omega_1}$ and $\mathcal{B} = \{b_\xi \mid \xi < \omega_1\} \subseteq [\omega_1]^{<\omega}$ such that

(1) $b_\xi \neq \emptyset$ for all $\xi < \omega_1$,
(2) $b_\xi < b_\eta$ for all $\xi < \eta < \omega_1$, and
(3) for each $\alpha \in A$ and $\xi < \omega_1$ with $\alpha < b_\xi$ there is $\beta \in b_\xi$ with $\{\alpha, \beta\} \in K_1$.

It is easily checked that Axiom T is equivalent to the assertion that every large graph is either boring or wondrous, that $\omega_1 \to (\omega_1, \mathcal{W})^2$. S. Todorčević demonstrated in [12] that there is a proper notion of forcing which forces $\mathrm{ZFC} + \mathrm{MA}_{\omega_1} +$ Axiom T.

It suffices then to prove that Axiom T implies Axiom W. Suppose that G is a wondrous graph and that $E_G = K_0 \cup K_1$. Consider the graph $H = \langle V_G, K_1 \rangle$. By Axiom T, we know that H is either wondrous or boring. If H were wondrous, then H would be a monochromatic wondrous subgraph of G. On the other hand, if H were boring, then there would be $X \in [V_G]^{\omega_1}$ with $[X]^2 \cap K_1 = \emptyset$. The graph $G[X]$ would be wondrous by Lemma 3.1 and monochromatic because $E_{G[X]} = E_G \cap [X]^2 \subseteq K_0$. □

Lemma 3.5 (ZFC+MA_{ω_1})**.** $(\alpha : \omega_1) \to (\alpha : \omega_1)^{1,1}_n$ *for all additively indecomposable $\alpha < \omega_1$ and all $n < \omega$. In other words, if $\alpha < \omega_1$ is additively indecomposable and $[\alpha, \omega_1]^{1,1}$ is partitioned into finitely many classes, then there must be $A \in [\alpha]^\alpha$ and $B \in [\omega_1]^{\omega_1}$ with $[A, B]^{1,1}$ contained in a single class.*

Proof. This was proven by J. Baumgartner and A. Hajnal in [1, §4, Corollary 2]. □

Proposition 3.6 ($\mathrm{ZFC} + \mathrm{MA}_{\omega_1}$)**.** *For any wondrous graph G and any ordinal $\alpha < \omega_1$ there is $A \in [V_G]^\alpha$ with $[A]^2 \subseteq E_G$.*

Proof. By induction on $\alpha < \omega_1$. Suppose that the theorem is true for all $\beta < \alpha$. Let G be an arbitrary wondrous graph.

For each $X \in [V_G]^{\omega_1}$ and each additively indecomposable ordinal $\beta < \alpha$, define $A(X, \beta) \in [X]^\beta$ and $B(X, \beta) \in [X]^{\omega_1}$ as follows. First, let $U \in [X]^{\omega_1}$ and $\mathcal{V} \subseteq [X]^{<\omega}$ be wondrous witnesses for X in G. $G[U]$ is wondrous by Lemma 3.1. Hence, there must be $Y \in [U]^\beta$ with $[Y]^2 \subseteq E_{G[U]} \subseteq E_G$. Choose $n < \omega$ and $\mathcal{Z} = \{z_\eta \mid \eta < \omega_1\} \subseteq \mathcal{V}$ such that $|z_\eta| = n$ for all $\eta < \omega_1$ and $z_\xi < z_\eta$ for all $\xi < \eta < \omega_1$. For each $\eta < \omega_1$ and $i < n$, let $z_{\eta,i}$ be the ith element of z_η in increasing order. Since U and $\mathcal{V}$ are wondrous witnesses, for each $\upsilon \in Y$ and $\eta < \omega_1$, there must be $i < n$ with $\{\upsilon, z_{\eta,i}\} \in E_G$. Let $i(\upsilon, \eta) = i$. By Lemma 3.5, there must $i < n$, $A \in [Y]^\beta$, and $Z \in [\omega_1]^{\omega_1}$ with $i(\upsilon, \eta) = i$ for each $\upsilon \in A$ and each $\eta \in Z$. Let $B = \{z_{\eta,i} \mid \eta \in Z\}$. Note that $[A]^2 \cup [A, B]^{1,1} \subseteq E_G$. Finally, let $A(X, \beta) = A$ and $B(X, \beta) = B$.

Choose $\mu \leq \omega$ and additively indecomposable ordinals $\{\alpha_k \mid k < \mu\} \subseteq \alpha$ such that $\alpha = \sum\{\alpha_k \mid k < \mu\}$. Let $B_0 = V_G$. For each $k < \mu$, assuming that $B_k \in [V_G]^{\omega_1}$ is defined, let $A_k = A(B_k, \alpha_k)$ and $B_{k+1} = B(B_k, \alpha_k)$. Let $A = \bigcup\{A_k \mid k < \mu\}$. It is easily checked that $A \in [V_G]^\alpha$ and $[A]^2 \subseteq E_G$. □

Corollary 3.7 ($\mathrm{ZFC} + \mathrm{MA}_{\omega_1} +$ Axiom W)**.** $\mathcal{W} \to (\alpha)^2_n$ *for all $n < \omega$.*

Though we do not need the result, we remark that the assumption of Axiom W in Corollary 3.7 is unnecessary.

4. Partitioning triangles in wondrous graphs

In this section, we prove that $\mathrm{ZFC} + \mathrm{MA}_{\omega_1} + \text{Axiom } W$ implies that

$$\mathcal{W} \to (\omega + \omega + 1, n)^3$$

for each $n < \omega$. Since the complete graph on ω_1 is wondrous, it follows that $\mathrm{ZFC} + \mathrm{MA}_{\omega_1} + \text{Axiom } W$ implies that $\omega_1 \to (\omega + \omega + 1, n)^3$ for each $n < \omega$.

Lemma 4.1 ($\mathrm{ZFC} + \mathrm{MA}_{\omega_1}$)**.** *$(\omega : \omega_1) \to (\omega : \omega_1)^{m,1}_n$ for all $m, n < \omega$. In other words, whenever $[\omega, \omega_1]^{m,1}$ is partitioned into finitely many classes, there must be an infinite $A \in [\omega]^\omega$ and an uncountable $B \in [\omega_1]^{\omega_1}$ with $[A, B]^{m,1}$ contained in a single class.*

Proof. This was proven by J. Baumgartner and A. Hajnal in [1, §4, Corollary 4]. □

Lemma 4.2 ($\mathrm{ZFC}+\mathrm{MA}_{\omega_1}$)**.** *$\mathcal{W} \to ((\omega : \omega_1)^{2,1}, n)^3$ for each $n < \omega$. In other words, for each wondrous graph G and each partition $T_G = K_0 \cup K_1$ of the triangles of G, either*

(a) there are $A \in [V_G]^\omega$ and $B \in [V_G]^{\omega_1}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$

or

(b) for each $n < \omega$ there is $C \in [V_G]^n$ with $[C]^3 \subseteq K_1$.

Note that in (a) *having $[A, B]^{2,1} \subseteq K_0$ also implies that $[A]^2 \cup [A, B]^{1,1} \subseteq E_G$ and that in* (b) *having $[C]^3 \subseteq K_1$ also implies that $[C]^2 \subseteq E_G$.*

Proof. For each $n < \omega$ and each $A \in [V_G]^{\omega^n}$ there is a unique order isomorphism between $\langle [\omega]^n, <_{\mathrm{lex}} \rangle$ and $\langle A, < \rangle$. For each $x \in [\omega]^n$, let $A(x)$ be the element of A identified with x via this isomorphism.

For each $B \in [V_G]^{\omega_1}$, each $n < \omega$, each $A \in [V_G]^{\omega^n}$ with $A < B$ and $[A]^2 \cup [A, B]^{1,1} \subseteq E_G$, and each $m < n$, define the partition

$$[\omega, \omega_1]^{2n-m,1} = K_0^{B,A,m} \cup K_1^{B,A,m}$$

as follows. Decompose each $a \in [\omega]^{2n-m}$ as $a = u \cup v_0 \cup v_1$ with $u < v_0 < v_1$, $|u| = m$, and $|v_0| = |v_1| = n - m$. For each $\beta < \omega_1$, let $B(\beta)$ be the βth element of B in increasing order. For each $i \in \{0, 1\}$, let

$$K_i^{B,A,m} = \{\langle a, \beta \rangle \in [\omega, \omega_1]^{2n-m,1} \mid \{A(u \cup v_0), A(u \cup v_1), B(\beta)\} \in K_i\}.$$

By Lemma 4.1, there must be $X = X^{B,A,m} \in [\omega]^\omega$, $Y = Y^{B,A,m} \in [B]^{\omega_1}$, and $i = i^{B,A,m} \in \{0, 1\}$ with

$$[X, Y]^{2n-m,1} \subseteq K_i^{B,A,m}.$$

For each $n < \omega$, let $\mathcal{B}_n$ be the collection of all $B \in [V_G]^{\omega_1}$ for which there are $A \in [V_G]^{\omega^n}$ and $m < n$ with $A < B$, with $[A]^2 \cup [A, B]^{1,1} \subseteq E_G$, and with $i^{B,A,m} = 0$.

Note that $\mathcal{B}_0$ is always empty.

Claim 4.2.1. If (a) fails, then $\mathcal{B}_n$ is empty for all $n < \omega$.

Proof. We prove the contrapositive. Suppose $\mathcal{B}_n$ is non-empty for some $n < \omega$. By the definition of $\mathcal{B}_n$, there are $B \in [V_G]^{\omega_1}$, $A \in [V_G]^{\omega^n}$, and $m < n$ with $A < B$, $[A]^2 \cup [A, B]^{1,1} \subseteq E_G$, and $i^{B,A,m} = 0$. Let $X = X^{B,A,m}$ and $Y = Y^{B,A,m}$. Choose $u \in [X]^m$ and $\{v_i \mid i < \omega\} \subseteq [X]^{n-m}$ with

$$u < v_0 < v_1 < v_2 < \dots.$$

Let $\bar{A} = \{A(u \cup v_i) \mid i < \omega\}$ and $\bar{B} = \{B(\beta) \mid \beta \in Y\}$. Note that $\bar{A} \in [V_G]^{\omega}$, $\bar{B} \in [V_G]^{\omega_1}$, $\bar{A} < \bar{B}$, and $[\bar{A}, \bar{B}]^{2,1} \subseteq K_0$. Thus (a) holds. □

For each $n < \omega$ and each $x, y \in [\omega]^n$, put $x \ll y$ if and only if there are $m < n$, $u \in [\omega]^m$, and $v_0, v_1 \in [\omega]^{n-m}$ with $u < v_0 < v_1$ and such that $x = u \cup v_0$, and $y = u \cup v_1$.

Let $\mathcal{A}_n$ be the collection of all $A \in [V_G]^{\omega^n}$ with $[A]^2 \subseteq E_G$ and $\{A(a), A(b), A(c)\} \in K_1$ for all $a, b, c \in [\omega]^n$ with $a \ll b \ll c$ and $|a \cap b| > |b \cap c|$.

Claim 4.2.2. If $\mathcal{B}_n$ is empty for all $n < \omega$, then $\mathcal{A}_n$ is non-empty for all $n < \omega$.

Proof. For each $X \in [V_G]^{\omega_1}$ and each $n < \omega$, let $\mathcal{A}_n(X) = \mathcal{A}_n \cap [X]^{\omega^n}$. In particular, $\mathcal{A}_n = \mathcal{A}_n(V_G)$.

Suppose that $\mathcal{B}_n$ is empty for all $n < \omega$. We prove by induction on $n < \omega$ that $\mathcal{A}_n(X)$ is non-empty for every $X \in [\omega_1]^{\omega_1}$.

We first note that $\mathcal{A}_0(X) = [X]^1$ (and is therefore non-empty) for every $X \in [V_G]^{\omega_1}$. We then fix $n < \omega$ and suppose that $\mathcal{A}_n(X)$ is non-empty for every $X \in [V_G]^{\omega_1}$. We must prove that $\mathcal{A}_{n+1}(X)$ is non-empty for every $X \in [V_G]^{\omega_1}$.

Subclaim 4.2.2.1. For each $X \in [V_G]^{\omega_1}$ there are $A \in \mathcal{A}_n(X)$ and $B \in [X]^{\omega_1}$ with $A < B$ and $\{A(x), A(y), \beta\} \in K_1$ for all $x, y \in [\omega]^n$ with $x \ll y$ and all $\beta \in B$.

Proof. Let U and $\mathcal{V} = \{v_\xi \mid \xi < \omega_1\}$ be wondrous witnesses for X in G. Choose $W \in \mathcal{A}_n(U)$. There must be a non-zero $r < \omega$ such that

$$R = \{\xi < \omega_1 \mid W < v_\xi \wedge |v_\xi| = r\}$$

is uncountable. For each $\xi < \omega_1$, let $R(\xi)$ be the ξth element of R in increasing order. For each $\xi < \omega_1$ and each $i < |v_\xi|$, let $v_\xi(i)$ be the ith element of v_ξ in increasing order. Choose a coloring $c : [\omega, \omega_1]^{n,1} \to r$ for which

$$\{W(x), v_{R(\xi)}(c(x, \xi))\} \in E_G$$

for all $x \in [\omega]^n$ and all $\xi < \omega_1$.

By Lemma 4.1 there must be $i < r$, $P \in [\omega]^{\omega}$, and $Q \in [R]^{\omega_1}$ so that $c(x, \xi) = i$ for all $x \in [P]^n$ and $\xi \in Q$. Let

$$C = \{W(x) \mid x \in [P]^n\} \text{ and } D = \{v_{R(\xi)}(i) \mid \xi \in Q\}.$$

Define $C_k \in [C]^{\omega^n}$ and $D_k \in [D]^{\omega_1}$ recursively for each $m \leq n$ as follows. Let $C_0 = C$ and $D_0 = D$. Given C_m and D_m, let

$$C_{m+1} = \{C_m(x) \mid x \in \left[X^{D_m, C_m, m}\right]^n\} \text{ and } D_{m+1} = Y^{D_m, C_m, m}.$$

Remember that $i^{D_m, C_m, m} = 1$ for each $m < n$ because $\mathcal{B}_n$ is empty. Let $A = C_n$ and $B = D_n$. It is now straightforward to verify that A and B satisfy the requirements of the subclaim. □

Define by recursion sets A_j and B_j for $j < \omega$ by first applying the subclaim to X to get A_0 and B_0 and then applying the subclaim to each B_j to get A_{j+1} and B_{j+1}. Note that $A_j < A_k$ for all $j < k < \omega$.

Set $A = \bigcup_{j<\omega} A_j$. Note that $A \in [X]^{\omega^{n+1}}$ and $[A]^2 \subseteq E_G$. To see that $A \in \mathcal{A}_{n+1}(X)$, consider $a \ll b \ll c \in [\omega]^{n+1}$ with $|a \cap b| > |b \cap c|$. Set $j = \min(a \cap b) \leq k = \min c$. Then $A(a), A(b) \in A_j$ and $A(c) \in A_k$. If $j = k$, then $\{A(a), A(b), A(c)\} \in K_1$ because $A_j \in \mathcal{A}_n(X)$. If $j < k$, then $\{A(a), A(b), A(c)\} \in K_1$ by construction. □

Claim 4.2.3. If $\mathcal{A}_n$ is non-empty for all $n < \omega$, then (b) holds. (More specifically, for each $n < \omega$, if $\mathcal{A}_n$ is non-empty, then there is $Y \in [\omega_1]^{n+1}$ with $[Y]^3 \subseteq K_1$.)

Proof. Fix $n < \omega$. Suppose $A \in \mathcal{A}_n$. For each $k \leq n$ let

$$x_{n,k} = \{c \,|\, c < n - k\} \cup \{kn + c \,|\, c < k\}.$$

For all $i < j < k \leq n$ it is easily verified that $x_{n,i} \ll x_{n,j} \ll x_{n,k}$ and $|x_{n,i} \cap x_{n,k}| = n - k < n - j = |x_{n,i} \cap x_{n,j}|$. Let $Y = \{A(x_{n,k}) \,|\, k \leq n\}$. Then $Y \in [\omega_1]^{n+1}$ and $[Y]^3 \subseteq K_1$. □

The lemma now follows directly from the three claims above. By Claim 4.2.1, if (a) fails, then $\mathcal{B}_n$ is empty for all $n < \omega$. By Claim 4.2.2, if $\mathcal{B}_n$ is empty for all $n < \omega$, then $\mathcal{A}_n$ is non-empty for all $n < \omega$. By Claim 4.2.3, if $\mathcal{A}_n$ is non-empty for all $n < \omega$, then (b) holds. Thus, either (a) holds or (b) holds, and the lemma is proven. □

Proposition 4.3 (ZFC + MA_{ω_1} + Axiom W)**.** *For all $n < \omega$, $\mathcal{W} \to (\omega + \omega + 1, n)^3$. In other words, for each wondrous graph G and each partition $T_G = K_0 \cup K_1$ of the triangles of G, either*

(a) there is $A \in [V_G]^{\omega+\omega+1}$ with $[A]^3 \subseteq K_0$, or
(b) for each $n < \omega$ there is $B \in [V_G]^n$ with $[B]^3 \subseteq K_1$.

Note that having $[A]^3 \subseteq K_0$ in (a) also implies that $[A]^2 \subseteq E_G$ and that having $[B]^3 \subseteq K_1$ in (b) also implies that $[B]^2 \subseteq E_G$.

Proof. By induction on $n < \omega$. Suppose that

$$\mathcal{W} \to (\omega + \omega + 1, n)^3.$$

Let G be an arbitrary wondrous graph and $T_G = K_0 \cup K_1$ be an arbitrary partition of the triangles of G into two classes.

Claim 4.3.1. If there are $\alpha \in V_G$ and a wondrous subgraph H of G with $\{\alpha\} \cup e \in K_1$ for all $e \in E_H$, then either there is $X \in [V_H]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$ or there is $Y \in [V_G]^{n+1}$ with $[Y]^3 \subseteq K_1$.

Proof. Since $\mathcal{W} \to (\omega + \omega + 1, n)^3$, if there is no $X \in [V_H]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$, then there is $\bar{Y} \in [V_H]^n$ with $[\bar{Y}]^3 \subseteq K_1$. Let $Y = \{\alpha\} \cup \bar{Y}$. Clearly, $Y \in [V_G]^{n+1}$ and $[Y]^3 \subseteq K_1$. □

Claim 4.3.2. If there are $A \in [V_G]^\omega$, a non-principal ultrafilter $\mathcal{U}$ on A, and a wondrous subgraph H of G such that $A_e = \{\alpha \in A \,|\, \{\alpha\} \cup e \in K_1\} \in \mathcal{U}$ for all $e \in E_H$, then either there is $X \in [V_H]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$ or there is $Y \in [V_G]^{n+1}$ with $[Y]^3 \subseteq K_1$.

Proof. Since $\mathcal{W} \to (\omega+\omega+1, n)^3$, if there is no $X \in [V_H]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$, then there is $\bar{Y} \in [V_H]^n$ with $[\bar{Y}]^3 \subseteq K_1$. Let $\bar{A} = \bigcap\{A_e \mid e \in [\bar{Y}]^2\}$. Choose $\alpha \in \bar{A}$. Let $Y = \{\alpha\} \cup \bar{Y}$. Clearly, $Y \in [V_G]^{n+1}$ and $[Y]^3 \subseteq K_1$. □

Suppose that there is no $B \in [V_G]^{n+1}$ with $[B]^3 \subseteq K_1$. By Lemma 4.2 there must then be $A \in [V_G]^{\omega}$ and $B \in [V_G]^{\omega_1}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$.

Let $\mathcal{U}$ be a non-principal ultrafilter over A. For each edge $e \in E_{G[B]}$, there are then $i_e \in \{0,1\}$ and $A_e \in \mathcal{U}$ with $[A_e, e]^{1,2} \subseteq K_{i_e}$.

By Lemma 3.1, $G[B]$ is wondrous. Hence, by Axiom W, there are $i \in \{0,1\}$ and a wondrous subgraph H of $G[B]$ such that $i_e = i$ for all $e \in E_H$. By Claim 4.3.2, we may assume that $i = 0$ without loss of generality.

For each $\beta \in B$, call $\langle x, y\rangle$ a *good pair* for β if

(1) $x \in [A]^{<\omega}$, $y \in [V_H]^{<\omega}$, and $\max y < \beta$,
(2) $[x,y]^{1,2} \cup [x,y,\{\beta\}]^{1,1,1} \cup [y,\{\beta\}]^{2,1} \subseteq K_0$.

If $\langle x, y\rangle$ and $\langle x', y'\rangle$ are both good pairs for β, then put $\langle x,y\rangle \prec \langle x',y'\rangle$ if $x \subsetneq x'$ and $y \subsetneq y'$, but $\max x < \min(x' \setminus x)$ and $\max y < \min(y' \setminus y)$. Note that $\langle\emptyset,\emptyset\rangle$ is a good pair for each $\beta \in V_H$.

Claim 4.3.3. If for some $\beta \in V_H$ there is an infinite increasing sequence $\langle x_0, y_0\rangle \prec \langle x_1, y_1\rangle \prec \langle x_2, y_2\rangle \prec \cdots$ of good pairs for β, then either there is $A \in [V_G]^{\omega+\omega+1}$ with $[A]^3 \subseteq K_0$ or there is $B \in [V_G]^{n+1}$ with $[B]^3 \subseteq K_1$.

Proof. Let $X = \bigcup\{x_n \mid n < \omega\}$ and $Y = \bigcup\{y_n \mid n < \omega\}$. Note that $X, Y \in [V_G]^{\omega}$ because $\langle x_k, y_k\rangle \prec \langle x_{k+1}, y_{k+1}\rangle$ for each $k < \omega$. Note that $X < Y$ and $[X, Y \cup \{\beta\}]^{2,1} \subseteq K_0$ because $X \subseteq A$ and $Y \cup \{\beta\} \subseteq V_H \subseteq B$. Also, $[X,Y]^{1,2} \cup [X,Y,\{\beta\}]^{1,1,1} \cup [Y,\{\beta\}]^{2,1} \subseteq K_0$ because each $\langle x_k, y_k\rangle$ is a good pair for β. This is almost enough to ensure that $[X \cup Y \cup \{\beta\}]^3 \subseteq K_0$; all that is lacking is that $[X]^3 \subseteq K_0$ and $[Y]^3 \subseteq K_0$. Note that the preceding already guarantees that $[X \cup Y \cup \{\beta\}]^2 \subseteq E_G$.

But because $\omega \to (\omega, n+1)^3$, if there is no $B \in [X]^{n+1} \cup [Y]^{n+1}$ with $[B]^3 \subseteq K_1$, then there must be $\bar{X} \in [X]^{\omega}$ and $\bar{Y} \in [Y]^{\omega}$ with $[\bar{X}]^3 \subseteq K_0$ and $[\bar{Y}]^3 \subseteq K_0$. Thus, $\bar{X} \cup \bar{Y} \cup \{\beta\} \in [V_G]^{\omega+\omega+1}$ and $[\bar{X} \cup \bar{Y} \cup \{\beta\}]^3 \subseteq K_0$. □

Without loss of generality, we may therefore assume that for each $\beta \in V_H$ there is a $\prec$-maximal good pair $\langle x_\beta, y_\beta\rangle$. By pressing down we can find $C \in [V_H]^{\omega_1}$, $x \in [A]^{<\omega}$, and $y \in [V_H]^{<\omega}$ with $\langle x_\beta, y_\beta\rangle = \langle x, y\rangle$ for all $\beta \in C$.

Note that this implies that for each edge $\{\beta, \gamma\} \in E_{H[C]}$, there is $\alpha \in x \cup y$ with $\{\alpha, \beta, \gamma\} \in K_1$. (Otherwise, $\langle x \cup \{\alpha\}, y \cup \{\beta\}\rangle$ is a good pair for γ, where α is any element of $\bigcap\{A_e \mid e \in [y \cup \{\beta,\gamma\}]^2\}$.)

By Lemma 3.1, $H[C]$ is wondrous. It follows from Axiom W that $H[C] \to (\mathcal{W})^2_{|x|+|y|}$, so there must be $\alpha \in x \cup y$ and a wondrous subgraph J of $H[C]$ such that $\{\alpha, \beta, \gamma\} \in K_1$ for each pair $\{\beta, \gamma\} \in E_J$. By Claim 4.3.1, either there is $A \in [V_J]^{\omega+\omega+1}$ with $[A]^3 \subseteq K_0$ or there is $B \in [V_G]^{n+1}$ with $[B]^3 \subseteq K_1$. □

5. Partitioning triples of countable ordinals

Lemma 5.1 (ZFC)**.** *If there is a proper notion of forcing that forces*

$$ZFC + \ulcorner \omega_1 \to (\omega+\omega+1, n)^3 \text{ for all } n < \omega \urcorner,$$

then $\omega_1 \to (\omega+\omega+1, n)^3$ for all $n < \omega$. In other words, to prove that $\omega_1 \to (\omega+\omega+1, n)^3$ for all $n < \omega$, it suffices to prove that it is forced by a proper notion of forcing.

Proof. Proofs of this result (and much more) appear in both [1] and [11]. □

Proposition 5.2 (ZFC)**.** *For all $n < \omega$*

$$\omega_1 \to (\omega+\omega+1, n)^3.$$

Proof. By Lemma 3.4, there is a proper notion of forcing which forces ZFC+MA_{ω_1}+ Axiom W. By Proposition 4.3, this same notion also forces that

$$\mathcal{W} \to (\omega+\omega+1, n)^3$$

for all $n < \omega$. In particular, it forces that $\omega_1 \to (\omega+\omega+1, n)^3$ for all $n < \omega$. By Lemma 5.1, it follows that $\omega_1 \to (\omega+\omega+1, n)^3$ for all $n < \omega$. □

6. Final remarks

We note that each instance of MA_{ω_1} above and below could be replaced with MA_{ω_1}(σ-centered) or its equivalent (by a straightforward generalization of a result of M. G. Bell in [2]), the cardinal inequality $\mathfrak{p} > \omega_1$.

Question 1. Does ZFC prove that $\omega_1 \to (\alpha, n)^3$ for all $\alpha < \omega_1$ and $n < \omega$? By the results presented above, the simplest open problem here is whether or not the relation

$$\omega_1 \to (\omega+\omega+2, 4)^3$$

is decided by ZFC.

Question 2. Does ZFC + MA_{ω_1} prove that $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$? By an unpublished result of J. Hirschorn, the simplest open problem here is whether or not the relation

$$\omega_1 \to (\omega_1, \omega^2+2)^2$$

is decided by ZFC + MA_{ω_1}.

Question 3. Does ZFC prove that $\omega_1 \to ((\alpha : \omega_1)^{2,0\vee 1,1})^2_n$ for all $\alpha < \omega_1$? (Here, $[X,Y]^{2,0\vee 1,1} = [X,Y]^{2,0} \cup [X,Y]^{1,1} = [X]^2 \cup [X,Y]^{1,1}$ and the partition symbol has the corresponding meaning.)

Question 4. Does ZFC prove that if G is wondrous and $\prec$ is a linear ordering of its vertices, then either $G(\prec)$ or $G(\succ)$ includes a wondrous graph? An induced wondrous subgraph?

Question 5. Is Axiom W a consequence of ZFC? In other words, does ZFC prove that $\mathcal{W} \to (\mathcal{W})^2_n$ for all $n < \omega$? If not, then what if we assume ZFC + MA_{ω_1}? (If so, then all the results presented above would follow from ZFC + MA_{ω_1} alone.)

Question 6. It follows from ZFC + MA_{ω_1} + Axiom W that $\mathcal{W} \to (\mathcal{W}, \alpha)^2$ for all $\alpha < \omega_1$. Does ZFC alone prove this? If not, then what if we assume ZFC + MA_{ω_1} but not Axiom W? (Note that ZFC *does* prove that $\omega_1 \to (\mathcal{W}, \alpha)^2$ for all $\alpha < \omega_1$.)

Question 7. The assumption of Axiom W in Corollary 3.7 is unnecessary. It follows from ZFC + MA_{ω_1} alone that $\mathcal{W} \to (\alpha)^2_n$ for all $\alpha < \omega_1$ and all $n < \omega$. Can the assumption of MA_{ω_1} be removed, as well?

Question 8. Do c.c.c. forcings preserve wondrousness? In other words, if G is a wondrous graph, then does each c.c.c. notion of forcing force that G is wondrous? (If so, then the answer to Question 7 would be affirmative, and the assumption of $\mathrm{MA}_{\omega_1} + \text{Axiom } W$ in Corollary 3.7 would be unnecessary.) If not, then what about σ-centered or σ-finite-c.c. forcings?

Question 9. Does it follow from $\mathrm{ZFC} + \mathrm{MA}_{\omega_1} + \text{Axiom } W$ that $\mathcal{W} \to (\mathcal{W}, n)^3$ for all $n < \omega$? (If so, then ZFC proves that $\omega_1 \to (\alpha, n)^3$ for all $\alpha < \omega_1$ and all $n < \omega$.)

Question 10. Is there a graph G on ω_1 such that $G \to (\omega + 1, \omega)^2$ but $G \nrightarrow (\alpha)^2_n$ for some $\alpha < \omega_1$ and some $n < \omega$? Is there a graph G on ω_1 such that $G \to (\alpha)^2_n$ for each $\alpha < \omega_1$ and each $n < \omega$ but $G \nrightarrow (\beta, n)^3$ for some $\beta < \omega_1$ and some $n < \omega$?

Question 11. Let $\mathcal{B}$ denote the collection of graphs G on uncountable subsets of ω_1 for which $G \to (\alpha)^2_n$ for all $n < \omega$. It is easily seen that $\mathcal{B} \to (\mathcal{B})^2_n$ for all $n < \omega$. Is it true (or consistent) that $\mathcal{W}$ is a basis for $\mathcal{B}$, that every element of $\mathcal{B}$ includes a subgraph in $\mathcal{W}$?

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