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MULTIPARAMETER QUANTUM SCHUR DUALITY OF TYPE B

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Abstract. We establish a Schur-type duality between a coideal subalgebra of the quantum group of type A and the Hecke algebra of type B with two parameters. We identify the $\imath$-canonical basis on the tensor product of the natural representation with Lusztig's canonical basis of the type B Hecke algebra with unequal parameters associated to a weight function.

1. Introduction

1.1. Jimbo ([J86]) established a quantum Schur duality between the quantum group $\mathbf{U}$ of type A and the Hecke algebra $\mathcal{H}_{A_{m-1}}$, which asserts that their actions on the tensor product $\mathbb{V}^{\otimes m}$ of the natural representation $\mathbb{V}$ of $\mathbf{U}$ commute and form double centralizers. To facilitate further discussions, we take the base field to be $\mathbb{Q}(p,q)$ for two parameters p,q instead of $\mathbb{Q}(q)$.

Let $\mathcal{H}_m$ be the Hecke algebra of type B_m with two parameters p,q, which contains $\mathcal{H}_{A_{m-1}}$ as a subalgebra and admits one extra generator H_0; see (2.6) for the definition. The Hecke algebra $\mathcal{H}_m$ acts naturally on $\mathbb{V}^{\otimes m}$ as well, where H_0 acts on the first tensor factor only. On the other hand, there is a notion of quantum symmetric pair, $(\mathbf{U},\mathbf{U}^\imath)$, where $\mathbf{U}^\imath$ is a coideal subalgebra of $\mathbf{U}$. The algebra $\mathbf{U}^\imath$ allows for some freedom of choices of parameters; see Letzter [Le99] (also see [BK15]). We make a particular choice of the parameters for $\mathbf{U}^\imath$ in this paper depending on p and q. The coideal subalgebra $\mathbf{U}^\imath$ acts on $\mathbb{V}^{\otimes m}$ naturally.

Our first main result ($\imath$-Schur duality) asserts that the actions of $\mathbf{U}^\imath$ and $\mathcal{H}_m$ on $\mathbb{V}^{\otimes m}$ commute and form double centralizers. This double centralizer theorem was established by the first two authors in [BW13] in the specialization when $p=q$, and then by the first author in [B16] in the specialization when $p=1$ and q is generic. The multiparameter $(\mathbf{U}^\imath,\mathcal{H}_m)$-duality in this paper is a natural generalization and synthesis of these earlier cases.

1.2. Let us return to the setting of the type A Schur-Jimbo duality for a moment. The space $\mathbb{V}^{\otimes m}$ admits a (parabolic) Kazhdan-Lusztig basis via its identification as a direct sum of permutation modules for the Hecke algebra $\mathcal{H}_{A_{m-1}}$. By the work of Lusztig [L94], there exists a canonical basis on the tensor product $\mathbf{U}$-module $\mathbb{V}^{\otimes m}$. It is well known that the canonical basis and the Kazhdan-Lusztig basis on $\mathbb{V}^{\otimes m}$ coincide (cf. Frenkel-Khovanov-Kirillov [FKK98]).

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Let $\mathtt{L} : W_{B_m} \to \mathbb{Z}$ be a weight function on the Weyl group of type B; cf. [L03]. Via a homomorphism $v^{\mathtt{L}} : \mathbb{Q}(p,q) \to \mathbb{Q}(v)$ with an indeterminate v, we consider the specializations $\mathcal{H}_m^{\mathtt{L}}$, $\mathbf{U}_{\mathtt{L}}$, $\mathbf{U}_{\mathtt{L}}^{\imath}$ and $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ over $\mathbb{Q}(v)$. The multiparameter $(\mathbf{U}^{\imath}, \mathcal{H}_m)$-duality leads to the $(\mathbf{U}_{\mathtt{L}}^{\imath}, \mathcal{H}_m^{\mathtt{L}})$-duality on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ under such a specialization.

Lusztig [L03] constructed a distinguished bar invariant basis of $\mathcal{H}_m^{\mathtt{L}}$ (called an $\mathtt{L}$-basis in this paper), which specializes to the Kazhdan-Lusztig (KL) basis ([KL79]) when $\mathtt{L}$ is the length function ℓ of a Weyl group. It is straightforward to adapt Lusztig's construction to the parabolic setting; cf. [Deo87]. Thus $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ admits an $\mathtt{L}$-basis through its identification as a direct sum of permutation modules over $\mathcal{H}_m^{\mathtt{L}}$.

An $\imath$-canonical basis on a tensor product $\mathbf{U}$-module (for example, $\mathbb{V}^{\otimes m}$) when $p = q$ was constructed in [BW13], which is invariant with respect to a new bar involution introduced therein. Moreover, the $\imath$-canonical basis on $\mathbb{V}^{\otimes m}$ when $p = q$ (which corresponds to the case when $\mathtt{L} = \ell$) is identified with the type B KL-basis. An easy modification of the construction in [BW13] leads to an $\imath$-canonical basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$, for any weight function $\mathtt{L}$. Note that the $\imath$-canonical basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ depends on $\mathtt{L}$, since the algebra $\mathbf{U}_{\mathtt{L}}^{\imath}$ depends on $\mathtt{L}$. We refer to [BW16] for a general theory of $\imath$-canonical bases for quantum symmetric pairs with parameters.

The second main result of this paper is that the $\imath$-canonical basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ coincides with the $\mathtt{L}$-basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$, for any weight function $\mathtt{L}$. For another distinguished choice of $\mathtt{L}$ (which corresponds to taking $p = 1$), the $\mathtt{L}$-basis (which is also the $\imath$-canonical basis) is the KL-basis of type D; see [B16].

1.3. The constructions and proofs in this paper are mostly adapted from [BW13], some of which have been known to us for some time. Nevertheless, the new setting does require various new nontrivial 2-parameter formulas, and thus we present explicitly the precise details which are new in our setting. The detailed constructions in this paper (where $\mathbf{U}$ is the quantum group for $\mathfrak{sl}_k$) depend much on the parity of k, so we treat the two cases separately. In sections 2 and 3, we treat the case when $k = 2r + 2$ is even. We establish in section 2 the $(\mathbf{U}^{\imath}, \mathcal{H}_m)$-duality on $\mathbb{V}^{\otimes m}$. In section 3, we study the $\imath$-canonical basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ associated to a weight function $\mathtt{L}$ and show that it coincides with the $\mathtt{L}$-basis. In section 4 we present the analogous constructions when $k = 2r + 1$ is odd.

2. The $\imath$-Schur duality with two parameters

In this section, we establish a Schur-type duality between a coideal subalgebra $\mathbf{U}^{\imath}$ of the quantum group for $\mathfrak{sl}_{2r+2}$ and the Hecke algebra of type B in two parameters.

2.1. **The quantum symmetric pair $(\mathbf{U}, \mathbf{U}^{\imath})$.** Let $r \geq 0$ be an integer. We set

$$\mathbb{I} = \{-r, -r+1, \dots, r\}, \qquad \mathbb{I}^{\imath} = \{1, \dots, r\}.$$

Let $\Pi = \{\alpha_i = \varepsilon_{i-\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}} \mid i \in \mathbb{I}\}$ be the simple system of type A_{2r+1}, and Φ the associated root system. Denote the weight lattice by

$$\Lambda = \sum_{i \in \mathbb{I}} \big(\mathbb{Z}\varepsilon_{i-\frac{1}{2}} + \mathbb{Z}\varepsilon_{i+\frac{1}{2}}\big).$$

Let p and q be indeterminates. Let $\mathbf{U}_q(\mathfrak{sl}_{2r+2})$ denote the quantum group of type A_{2r+1} over $\mathbb{Q}(q)$ with the standard generators E_i, F_i and $K_i^{\pm 1}$ for $i \in \mathbb{I}$ (see, e.g., [BW13, §1.2] for a precise definition). Let

$$\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_{2r+2}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(p,q).$$

We denote by ψ the bar involution on $\mathbf{U}$ (more conventionally denoted by $^{-}$), that is, a $\mathbb{Q}$-algebra involution of $\mathbf{U}$ such that

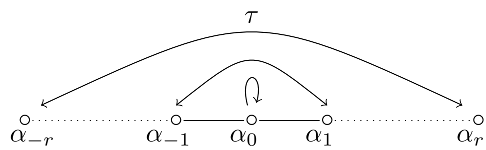
$$\psi(q) = q^{-1}, \quad \psi(p) = p^{-1}, \quad \psi(E_i) = E_i, \quad \psi(F_i) = F_i, \quad \psi(K_i) = K_i^{-1}. \tag{2.1}$$

We shall use the comultiplication $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ as follows:

(2.2)
$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1},\ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i,\ \Delta(K_i) = K_i \otimes K_i.$$

We review the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ over $\mathbb{Q}(p,q)$ [Le99] with the following Satake diagram:

A_{2r+1} :



Let $\mathbf{U}^\imath$ be the $\mathbb{Q}(p,q)$-subalgebra of $\mathbf{U}$ generated by (for $i \in \{1, \ldots, r\}$)

$$\begin{aligned} k_i &= K_i K_{-i}^{-1}, \quad t = E_0 + qF_0K_0^{-1} + \frac{p-p^{-1}}{q-q^{-1}} K_0^{-1}, \\ e_i &= E_i + F_{-i}K_i^{-1}, \quad f_i = E_{-i} + K_{-i}^{-1}F_i. \end{aligned} \tag{2.3}$$

Note that $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$; that is, we have $\Delta(\mathbf{U}^\imath) \subset \mathbf{U}^\imath \otimes \mathbf{U}$.

The algebra $\mathbf{U}^\imath$ has a presentation as an algebra over $\mathbb{Q}(p,q)$ generated by e_i, f_i, $k_i^{\pm 1}$ ($i \in \{1, \ldots, r\}$), and t, subject to the following relations for $i, j \in \{1, \ldots, r\}$ (see [BW13, BK15]):

$$\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\
k_i e_j k_i^{-1} &= q^{(\alpha_i - \alpha_{-i}, \alpha_j)} e_j, \\
k_i f_j k_i^{-1} &= q^{-(\alpha_i - \alpha_{-i}, \alpha_j)} f_j, \quad k_i t k_i^{-1} = t, \\
e_i f_j - f_j e_i &= \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i, && \text{if } |i-j| > 1, \\
e_i t &= t e_i, \quad f_i t = t f_i, && \text{if } i > 1, \\
e_i^2 e_j + e_j e_i^2 &= (q+q^{-1}) e_i e_j e_i, \quad f_i^2 f_j + f_j f_i^2 = (q+q^{-1}) f_i f_j f_i, && \text{if } |i-j| = 1, \\
e_1^2 t + t e_1^2 &= (q+q^{-1}) e_1 t e_1, \quad f_1^2 t + t f_1^2 = (q+q^{-1}) f_1 t f_1, \\
t^2 e_1 + e_1 t^2 &= (q+q^{-1}) t e_1 t + e_1, \\
t^2 f_1 + f_1 t^2 &= (q+q^{-1}) t f_1 t + f_1.
\end{aligned}$$

The next lemma follows by inspection using the above presentation of $\mathbf{U}^\imath$.

Lemma 2.1. *There exists a unique $\mathbb{Q}$-algebra bar involution $\psi_\imath$ on $\mathbf{U}^\imath$ such that $p \mapsto p^{-1}$, $q \mapsto q^{-1}$, $k_i \mapsto k_i^{-1}$, $e_i \mapsto e_i$, $f_i \mapsto f_i$, and $t \mapsto t$, for $i \in \mathbb{I}^\imath$.*

Remark 2.2. The presentation of the algebra $\mathbf{U}^\imath$ here is the same as the algebra in almost the same notation in [BW13, §2.1]. The parameter $\kappa \in \mathbb{Q}(p,q)$ in the embedding (see (2.3)), $t = E_0 + qF_0K_0^{-1} + \kappa K_0^{-1}$, is irrelevant to the presentation of the algebra $\mathbf{U}^\imath$. This phenomenon was first observed in [Le99].

Let $\widehat{\mathbf{U}}$ be the completion of the $\mathbb{Q}(p,q)$-vector space $\mathbf{U}$ with respect to the following descending sequence of subspaces $\mathbf{U}^+\mathbf{U}^0\big(\sum_{\mathrm{ht}(\mu)\geq N}\mathbf{U}^-_\mu\big)$, for $N \geq 1$. Then we have the obvious embedding of $\mathbf{U}$ into $\widehat{\mathbf{U}}$. We let $\widehat{\mathbf{U}}^-$ be the closure of $\mathbf{U}^-$ in $\widehat{\mathbf{U}}$, and so $\widehat{\mathbf{U}}^- \subseteq \widehat{\mathbf{U}}$. By continuity the $\mathbb{Q}(p,q)$-algebra structure on $\mathbf{U}$ extends to a $\mathbb{Q}(p,q)$-algebra structure on $\widehat{\mathbf{U}}$. The bar involution ψ on $\mathbf{U}$ extends by continuity to an *anti-linear* (i.e., $p \mapsto p^{-1}, q \mapsto q^{-1}$) involution on $\widehat{\mathbf{U}}$, also denoted by ψ. Denote by $\mathbb{N}$ the set of nonnegative integers.

Proposition 2.3 ([BW13, Theorem 2.10])**.** *There is a unique family of elements $\Upsilon_\mu \in \mathbf{U}^-_{-\mu}$ for $\mu \in \mathbb{N}\Pi$ such that $\Upsilon_0 = 1$, and $\Upsilon = \sum_\mu \Upsilon_\mu \in \widehat{\mathbf{U}}^-$ intertwines the bar involution ψ on $\mathbf{U}$ and the bar involution $\psi_\imath$ on $\mathbf{U}^\imath$; that is, Υ satisfies the following identity:*

$$\psi_\imath(u)\cdot\Upsilon = \Upsilon\cdot\psi(u), \qquad \textit{for all } u \in \mathbf{U}^\imath.$$

Note that [BW13, Theorem 2.10] was proved in the setting of $p = q$ in (2.3), but the same proof works here. We shall call Υ the intertwiner.

Consider a $\mathbb{Q}(q)$-valued function ζ on Λ (which is independent of the parameter p), such that for all $\mu \in \Lambda$, $i \in \mathbb{I}^\imath$, we have

$$\begin{aligned} \zeta(\mu+\alpha_0) &= -q\zeta(\mu),\\ \zeta(\mu+\alpha_i) &= -q^{(\alpha_i-\alpha_{-i},\mu+\alpha_i)}\zeta(\mu),\\ \zeta(\mu+\alpha_{-i}) &= -q^{(\alpha_{-i},\mu+\alpha_{-i})-(\alpha_i,\mu)}\zeta(\mu). \end{aligned} \tag{2.4}$$

Such ζ clearly exists. For any weight $\mathbf{U}$-module M, we define a $\mathbb{Q}(p,q)$-linear map on M associated with ζ as follows:

$$\begin{aligned} &\widetilde{\zeta}: M \longrightarrow M,\\ &\widetilde{\zeta}(m) = \zeta(\mu)m, \quad \text{for all } m \in M_\mu. \end{aligned} \tag{2.5}$$

Let w_0 denote the longest element of the Weyl group $W_{A_{2r+1}}$ of type A_{2r+1} and T_{w_0} the associated braid group element. The following proposition is a 2-parameter variant of [BW13, Theorem 2.18], with the same proof.

Proposition 2.4. *For any finite-dimensional $\mathbf{U}$-module M, the composition map*

$$\mathcal{T} := \Upsilon\circ\widetilde{\zeta}\circ T_{w_0} : M \longrightarrow M$$

is a $\mathbf{U}^\imath$-module isomorphism.

2.2. **Hecke algebra of type B with two parameters.** We often write $q_i = q$ for $1 \leq i \leq m-1$ and $q_0 = p$. Let $\mathcal{H}_m$ be the Hecke algebra of type B_m with two parameters over $\mathbb{Q}(p,q)$ generated by $H_0, H_1, \ldots, H_{m-1}$ and subject to the following relations:

$$\begin{aligned} (H_i - q_i^{-1})(H_i + q_i) = 0, &\quad \text{for } i \geq 0,\\ H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}, &\quad \text{for } i \geq 1,\\ H_0H_1H_0H_1 = H_1H_0H_1H_0, \quad H_iH_j = H_jH_i, &\quad \text{for } |i-j| > 1. \end{aligned} \tag{2.6}$$

Let W_{B_m} be the Weyl group of type B_m with simple reflections $s_0, s_1, \ldots, s_{m-1}$. For each $w \in W_{B_m}$ with a reduced expression $w = s_{i_1}\cdots s_{i_r}$, the products $H_w = H_{i_1}\cdots H_{i_r}$ and $q_w = q_{i_1}\cdots q_{i_r}$ are independent of the choice of the reduced expressions.

2.3. **The $(\mathbf{U}^\imath, \mathcal{H}_m)$-duality.** Let $I = \mathbb{I} \pm \frac{1}{2}$. Let $\mathbb{V} = \bigoplus_{a\in I} \mathbb{Q}(p,q)u_a$ be the natural representation of $\mathbf{U}$. The $\mathbf{U}$-module structure of $\mathbb{V}$ can be visualized as follows:

$$u_{r+\frac{1}{2}} \underset{E_r}{\overset{F_r}{\rightleftarrows}} u_{r-\frac{1}{2}} \underset{E_{r-1}}{\overset{F_{r-1}}{\rightleftarrows}} \cdots \underset{E_1}{\overset{F_1}{\rightleftarrows}} u_{\frac{1}{2}} \underset{E_0}{\overset{F_0}{\rightleftarrows}} u_{-\frac{1}{2}} \underset{E_{-1}}{\overset{F_{-1}}{\rightleftarrows}} \cdots \underset{E_{-r}}{\overset{F_{-r}}{\rightleftarrows}} u_{-r-\frac{1}{2}}\ .$$

We denote by $\mathbb{V}^{\otimes m}$ the m-th tensor product of $\mathbb{V}$, which is naturally a $\mathbf{U}$-module via iterated comultiplication. Hence $\mathbb{V}^{\otimes m}$ is a $\mathbf{U}^\imath$-module by restriction.

For any $f = (f(1), \dots, f(m)) \in I^m$, we define

$$M_f = u_{f(1)} \otimes \cdots \otimes u_{f(m)}. \tag{2.7}$$

The Weyl group W_{B_m} acts on I^m on the right in the obvious way: for $j \geq 1$ and $i \in I$,

$$(f \cdot s_j)(i) = \begin{cases} f(j+1) & \text{if } i = j, \\ f(j) & \text{if } i = j+1, \\ f(i) & \text{otherwise;} \end{cases}$$
$$(f \cdot s_0)(i) = \begin{cases} -f(1) & \text{if } i = 1, \\ f(i) & \text{otherwise.} \end{cases}$$

The Hecke algebra $\mathcal{H}_m$ acts naturally on $\mathbb{V}^{\otimes m}$ on the right as follows:

$$\begin{aligned} M_f \cdot H_i &= \begin{cases} q^{-1} M_f, & \text{if } f(i) = f(i+1); \\ M_{f \cdot s_i}, & \text{if } f(i) < f(i+1); \\ M_{f\cdot s_i} + (q^{-1} - q) M_f, & \text{if } f(i) > f(i+1); \end{cases} \\ M_f \cdot H_0 &= \begin{cases} M_{f\cdot s_0}, & \text{if } f(1) > 0; \\ M_{f\cdot s_0} + (p^{-1} - p) M_f, & \text{if } f(1) < 0. \end{cases} \end{aligned} \tag{2.8}$$

We shall depict the actions of $\mathbf{U}^\imath$ and $\mathcal{H}_m$ on $\mathbb{V}^{\otimes m}$ as

$$\mathbf{U}^\imath \overset{\Phi}{\curvearrowright} \mathbb{V}^{\otimes m} \overset{\Psi}{\curvearrowleft} \mathcal{H}_m. \tag{2.9}$$

Now we fix ζ in (2.4) by setting $\zeta(\epsilon_{-r-\frac{1}{2}}) = 1$. Then, we have

$$\zeta(\epsilon_{-i-\frac{1}{2}}) = (-q)^{-r+i}, \qquad \forall i \in \{r, r-1, \dots, -r-1\}.$$

Lemma 2.5. *The actions of H_0 and $\mathcal{T}^{-1}$ on $\mathbb{V}$ coincide.*

Proof. Let

$$\mathbb{V}^+ = \bigoplus_{j\in\{0,\dots,r\}} \mathbb{Q}(p,q)\big(u_{-j-\frac{1}{2}} + p u_{j+\frac{1}{2}}\big),$$
$$\mathbb{V}^- = \bigoplus_{j\in\{0,\dots,r\}} \mathbb{Q}(p,q)\big(u_{-j-\frac{1}{2}} - p^{-1} u_{j+\frac{1}{2}}\big).$$

By direct calculations, we have (for $i \in \mathbb{I}^\imath$, $j \in \{0, \dots, r\}$)

$$\begin{aligned}
t \cdot \big(u_{-j-\frac{1}{2}} + pu_{j+\frac{1}{2}}\big) &= \frac{pq^{\delta_{j,0}} - p^{-1}q^{-\delta_{j,0}}}{q - q^{-1}} \big(u_{-j-\frac{1}{2}} + pu_{j+\frac{1}{2}}\big),\\
f_i \cdot \big(u_{-j-\frac{1}{2}} + pu_{j+\frac{1}{2}}\big) &= \delta_{i,j+1} \cdot \big(u_{-(j+1)-\frac{1}{2}} + pu_{(j+1)+\frac{1}{2}}\big),\\
e_i \cdot \big(u_{-j-\frac{1}{2}} + pu_{j+\frac{1}{2}}\big) &= \delta_{i,j} \cdot \big(u_{-(j-1)-\frac{1}{2}} + pu_{(j-1)+\frac{1}{2}}\big),\\
t \cdot \big(u_{-j-\frac{1}{2}} - p^{-1}u_{j+\frac{1}{2}}\big) &= \frac{pq^{-\delta_{j,0}} - p^{-1}q^{\delta_{j,0}}}{q - q^{-1}} \big(u_{-j-\frac{1}{2}} - p^{-1}u_{j+\frac{1}{2}}\big),\\
f_i \cdot \big(u_{-j-\frac{1}{2}} - p^{-1}u_{j+\frac{1}{2}}\big) &= \delta_{i,j+1} \cdot \big(u_{-(j+1)-\frac{1}{2}} - p^{-1}u_{(j+1)+\frac{1}{2}}\big),\\
e_i \cdot \big(u_{-j-\frac{1}{2}} - p^{-1}u_{j+\frac{1}{2}}\big) &= \delta_{i,j} \cdot \big(u_{-(j-1)-\frac{1}{2}} - p^{-1}u_{(j-1)+\frac{1}{2}}\big).
\end{aligned}$$

Hence, we have $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$ as a $\mathbf{U}^\imath$-module.

Since we have $T_{w_0}(u_{j+\frac{1}{2}}) = (-q)^{r-j}u_{-j-\frac{1}{2}}$, we obtain $\widetilde{\zeta} \circ T_{w_0}(u_{j+\frac{1}{2}}) = u_{-j-\frac{1}{2}}$. On the other hand, one computes the first terms of Υ as $\Upsilon_{\alpha_0} = (p - p^{-1})F_{\alpha_0}$, which is the only term with nontrivial action for weight reasons. Hence, we have that $\mathcal{T}(u_{-\frac{1}{2}}) = u_{\frac{1}{2}}, \mathcal{T}(u_{\frac{1}{2}}) = u_{-\frac{1}{2}} - (p^{-1} - p)u_{\frac{1}{2}}$, which implies the actions of $\mathcal{T}^{-1}$ on $\mathbb{V}^+$ and $\mathbb{V}^-$ are given by scalar multiplication by p^{-1} and $-p$, respectively. The lemma is proved. □

Now, we state our first main theorem.

Theorem 2.6 ($\imath$-Schur duality)**.** *The actions of* $\mathbf{U}^\imath$ *and* $\mathcal{H}_m$ *on* $\mathbb{V}^{\otimes m}$ (2.9) *commute and form double centralizers; that is,*

$$\Phi(\mathbf{U}^\imath) = End_{\mathcal{H}_m}(\mathbb{V}^{\otimes m}), \quad End_{\mathbf{U}^\imath}(\mathbb{V}^{\otimes m})^{\mathrm{op}} = \Psi(\mathcal{H}_m).$$

Proof. Recall that $\mathbf{U}^\imath$ acts as a subalgebra of $\mathbf{U}$ on $\mathbb{V}^{\otimes m}$ and the Hecke algebra $\mathcal{H}_{A_{m-1}}$ is a subalgebra of $\mathcal{H}_m$ generated by H_i, for $1 \le i \le m-1$. Hence it follows from the q-Schur duality of type A ([J86]) that the action of H_i, for $1 \le i \le m-1$, commutes with the action of $\mathbf{U}$, and hence of $\mathbf{U}^\imath$. Note that H_0 acts only on the first factor of $\mathbb{V}^{\otimes m}$. On the other hand, the commutativity of the actions of H_0 and $\mathbf{U}^\imath$ follows by Proposition 2.4, Lemma 2.5, and the fact that the $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$.

The double centralizer property is the multiparameter version of [BW13, Theorem 5.4], and it follows by a deformation argument in the same way as in [BW13, Theorem 5.4]. □

3. The L-bases and $\imath$-canonical bases

3.1. **Quantum symmetrizers.** The type B Hecke algebra $\mathcal{H}_m$ (2.6) has a unique $\mathbb{Q}$-algebra bar involution $^-\colon \mathcal{H}_m \to \mathcal{H}_m$ such that

$$\overline{q}_i = q_i^{-1}, \quad \overline{H_w} = H_{w^{-1}}^{-1}, \quad \text{for } i \ge 0, w \in W_{B_m}.$$

For any subset $J \subseteq \{0, 1, \dots, m-1\}$, let W_J be the parabolic subgroup of W generated by $\{s_j \mid j \in J\}$. Let $\mathcal{H}_J$ be the $\mathbb{Q}(p, q)$-subalgebra of $\mathcal{H}_m$ generated by $\{H_j \mid j \in J\}$, and ${}^JW$ be the set of minimal length coset representatives for $W_J \setminus W$. We define a quantum symmetrizer for the algebra $\mathcal{H}_J$:

$$\eta_J = q_{w_J} \sum_{x \in W_J} q_x^{-1} H_x \in \mathcal{H}_J, \tag{3.1}$$

where w_J denotes the longest element of W_J.

Lemma 3.1. *Let $J \subseteq \{0,1,\dots,m-1\}$. Then, the following hold:*

(1) $\eta_J H_j = q_j^{-1}\eta_J$, *for all* $j \in J$.

(2) *For* $w \in {}^J W$ *and* $j \in J$, *we have*

$$(\eta_J H_w)\cdot H_j = \begin{cases} q_j^{-1}\eta_J H_w & \text{if } ws_j \notin {}^J W, \\ \eta_J H_{ws_j} & \text{if } ws_j \in {}^J W \text{ and } w < ws_j, \\ \eta_J H_{ws_j} + (q_j^{-1} - q_j)\eta_J H_w & \text{if } ws_j \in {}^J W \text{ and } ws_j < w. \end{cases}$$

(3) $\overline{\eta_J} = \eta_J$.

Proof. Part (1) is proved by a direct calculation, and (2) follows from (1). Part (3) can be (essentially) found in [L03, §12]. □

3.2. **The L-bases.** Recall [L03, §3.1] that a map $\mathtt{L} : W_{B_m} \longrightarrow \mathbb{Z}$ is called a *weight function* if it satisfies

$$\mathtt{L}(yw) = \mathtt{L}(y) + \mathtt{L}(w)$$

for all $y, w \in W_{B_m}$ such that $\ell(yw) = \ell(y) + \ell(w)$. (Such a weight function is determined by values $\mathtt{L}(s_0)$ and $\mathtt{L}(s_1)$.)

We fix a weight function L. If $m = 1$, we adopt the convention that $\mathtt{L}(s_1) = 1$. Let v be an indeterminate. We consider a $\mathbb{Q}$-algebra homomorphism

$$v^{\mathtt{L}} : \mathbb{Q}(p,q) \longrightarrow \mathbb{Q}(v), \qquad p \mapsto v^{\mathtt{L}(s_0)},\ q \mapsto v^{\mathtt{L}(s_1)}.$$

We shall regard $\mathbb{Q}(v)$ as a $\mathbb{Q}(p,q)$-module via the $\mathbb{Q}$-algebra homomorphism $v^{\mathtt{L}}$. By a base change, we introduce the following algebras/spaces over $\mathbb{Q}(v)$:

$$\mathcal{H}_m^{\mathtt{L}} = \mathcal{H}_m \otimes_{\mathbb{Q}(p,q)} \mathbb{Q}(v), \quad \mathbf{U}^\imath_{\mathtt{L}} = \mathbf{U}^\imath \otimes_{\mathbb{Q}(p,q)} \mathbb{Q}(v), \quad \mathbb{V}_{\mathtt{L}} = \mathbb{V} \otimes_{\mathbb{Q}(p,q)} \mathbb{Q}(v). \tag{3.2}$$

We shall use the old notation of the generators of $\mathcal{H}_m$ for generators of $\mathcal{H}_m^{\mathtt{L}}$ (and similarly for $\mathbf{U}^\imath_{\mathtt{L}}$ and $\mathbb{V}_{\mathtt{L}}$). The bar involution on $\mathcal{H}_m$ (as well as on $\mathbf{U}^\imath$ and $\mathbb{V}$) induces a bar involution on $\mathcal{H}_m^{\mathtt{L}}$ (as well as on $\mathbf{U}^\imath_{\mathtt{L}}$ and $\mathbb{V}_{\mathtt{L}}$) such that $\overline{v} = v^{-1}$.

The following is a straightforward variant of Lusztig [L03, Theorem 5.2] (who treats the regular representation, i.e., the $J = \emptyset$ case).

Proposition 3.2 ([L03, Deo87]). *Let $J \subseteq \{0,1,\dots,m-1\}$. Then, for each $w \in {}^J W$, there exists a unique element $C_w^J \in \eta_J \mathcal{H}_m^L$ such that*

(1) $\overline{C_w^J} = C_w^J$,

(2) $C_w^J \in \eta_J\big(H_w + \sum_{y \in {}^J W,\, y<w} v\mathbb{Z}[v] H_y\big)$.

Moreover, the elements $\{C_w^J \mid w \in {}^J W\}$ form a $\mathbb{Q}(v)$-basis of $\eta_J \mathcal{H}_m^L$ (called the L*-basis).*

Proof. Thanks to Lemma 3.1, the proof is the same as for the usual KL-setting [Deo87, Propositions 3.1, 3.2]. □

Remark 3.3. We write $C_w^J = C_w^{\mathtt{L},J}$ to emphasize the dependence on the weight function L. By replacing H_0 and H_i $(i \geq 1)$ by e_0H_0 and e_1H_i, respectively, where $e_0, e_1 \in \{1,-1\}$, one obtains an isomorphism $\mathcal{H}_m^{\mathtt{L}} \cong \mathcal{H}_m^{\mathtt{L}'}$, where $\mathtt{L}'$ is the weight function determined by $\mathtt{L}'(s_0) = e_0\mathtt{L}(s_0)$ and $\mathtt{L}'(s_1) = e_1\mathtt{L}(s_1)$. Moreover, one checks that the isomorphism is compatible with bar involutions, and it sends the L-basis to the $\mathtt{L}'$-basis up to sign, that is, $C_w^{\mathtt{L},J} \mapsto (-1)^{\ell(w)} C_w^{\mathtt{L}',J}$. This observation is valid in the general setting of [L03]. Therefore we may assume that a weight function is nonnegative integer valued if needed.

3.3. **The $\imath$-canonical bases.** It is well known that there exists a bar-involution ψ on the tensor product of several simple finite-dimensional $\mathbf{U}_{\mathtt{L}}$-modules, such as $\mathbb{V}_{\mathtt{L}}^{\otimes m}$, using the quasi-$R$-matrix Θ ([L94, Chap. 4]). Following [BW13, Proposition 3.10] we can define another anti-linear (i.e., $v \mapsto v^{-1}$) involution on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ as (recall the definition of Υ in Proposition 2.3)

$$\psi_\imath = \Upsilon \circ \psi. \tag{3.3}$$

By construction, $\psi_\imath$ is well defined on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ and fixes all M_f such that $0 < f(1) \leq f(2) \leq \cdots \leq f(m)$.

Remark 3.4. We use the same notation ψ (as in [L94]) for both the anti-linear involution on $\mathbf{U}$ (as well as the specialization $\mathbf{U}_{\mathtt{L}}$) and the anti-linear involution on the $\mathbf{U}$-module $\mathbb{V}^{\otimes m}$ (as well as the $\mathbf{U}_{\mathtt{L}}$-module $\mathbb{V}_{\mathtt{L}}^{\otimes m}$), since they are compatible. Similarly we use the same notation $\psi_\imath$ in a multiple of settings.

To develop a theory of $\imath$-canonical basis, besides the new bar involution $\psi_\imath$, we also need to establish the integrality of the intertwiner Υ. The following is an $\mathtt{L}$-variant of [BW13, Theorem 4.18].

Proposition 3.5. *Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and ${}_{\mathcal{A}}\mathbf{U}$ be the $\mathcal{A}$-form of $\mathbf{U}$. Then, we have $\Upsilon_\mu \in {}_{\mathcal{A}}\mathbf{U}$ for any $\mu \in \mathbb{N}[\mathbb{I}]$.*

Proof. Following the strategy of the proof of [BW13, Theorem 4.18], the proof of the integrality of Υ is reduced to verifying that the intertwiner is integral for the case $\mathbb{I} = \{0\}$ (which is the counterpart of [BW13, Lemma 4.6]).

We write $\Upsilon_c = \Upsilon_{c\alpha_0} = \gamma_c E^{(c)}$ for $c \geq 0$. Note that $\gamma_0 = 1$ by definition. The same computation as [BW13, Lemma 4.6] shows that

$$\begin{aligned}\gamma_{c+1} &= -\big(v^{\mathtt{L}(s_1)} - v^{-\mathtt{L}(s_1)}\big)v^{-c\mathtt{L}(s_1)}\Big(v^{\mathtt{L}(s_1)}[c]_{v^{\mathtt{L}(s_1)}}\gamma_{c-1} + \frac{v^{\mathtt{L}(s_0)} - v^{-\mathtt{L}(s_0)}}{v^{\mathtt{L}(s_1)} - v^{-\mathtt{L}(s_1)}}\gamma_c\Big)\\ &= -\big(v^{\mathtt{L}(s_1)} - v^{-\mathtt{L}(s_1)}\big)v^{-c\mathtt{L}(s_1)}v^{\mathtt{L}(s_1)}[c]_{v^{\mathtt{L}(s_1)}}\gamma_{c-1} - v^{-c\mathtt{L}(s_1)}\big(v^{\mathtt{L}(s_0)} - v^{-\mathtt{L}(s_0)}\big)\gamma_c,\end{aligned}$$

where

$$[c]_{v^{\mathtt{L}(s_1)}} = \frac{v^{c\mathtt{L}(s_1)} - v^{-c\mathtt{L}(s_1)}}{v^{\mathtt{L}(s_1)} - v^{-\mathtt{L}(s_1)}} \in \mathbb{Z}[v, v^{-1}].$$

Hence the proposition follows by induction on c. □

Following [BW13, Theorem 4.26] (or [BW16] for more general quantum symmetric pairs with parameters), we obtain the $\imath$-canonical bases on finite-dimensional simple $\mathbf{U}$-modules and their tensor products. Let us just formulate a special case which we need later in our general weight function $\mathtt{L}$ setting.

For $f \in I^m$, define a weight $\mathtt{wt}(f) = \sum_{1 \leq i \leq m} \varepsilon_{f(i)} \in \Lambda$. Let θ be the involution of the weight lattice Λ such that

$$\theta(\varepsilon_{i-\frac{1}{2}}) = -\varepsilon_{-i+\frac{1}{2}}, \quad \text{for all } i \in \mathbb{I}.$$

We say two weights $\lambda, \mu \in \Lambda$ have *identical $\imath$-weight* (and denote $\lambda \equiv_\imath \mu$) if $\lambda - \mu$ is fixed by θ. Define a partial ordering $\preceq$ on the set I^m as follows (cf. [BW13, proof of Theorem 5.8]): for $g, f \in I^m$, we let

$$g \preceq f \quad \Leftrightarrow \quad \mathtt{wt}(g) \equiv_\imath \mathtt{wt}(f) \ \text{ and } \ \mathtt{wt}(f) - \mathtt{wt}(g) \in \mathbb{N}\Pi. \tag{3.4}$$

We say $g \prec f$ if $g \preceq f$ and $g \neq f$.

Proposition 3.6. *The* $\mathbf{U}^\imath_L$*-module* $\mathbb{V}_L^{\otimes m}$ *admits a unique basis* $\{b^\imath_f \mid f \in I^m\}$ *such that* $b^\imath_f$ *is* $\psi_\imath$*-invariant and* $b^\imath_f \in M_f + \sum_{g \prec f} v\mathbb{Z}[v] M_g$.

Proof. This is a straightforward $\mathtt{L}$-generalization of [BW13], and we outline the proof for the convenience of the reader. By Proposition 3.5, Υ and hence also the bar involution $\psi_\imath$ (3.3) preserve the integral form of $\mathbb{V}_{\mathtt{L}}^{\otimes m}$, i.e., the $\mathbb{Z}[v, v^{-1}]$-span of $\{M_f | f \in I^m\}$. The existence of a $\psi_\imath$-invariant basis $\{b^\imath_f \mid f \in I^m\}$ in the $\mathbb{Z}[v]$-span of Lusztig's canonical basis on $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ follows by applying [L94, Lemma 24.2.1] (as we showed in [BW13, Theorem 4.25] in a general based $\mathbf{U}$-module setting). The partial order as stated in the proposition follows from arguments in [BW13, proof of Theorem 5.8]. ☐

Definition 3.7. We call the basis $\{b^\imath_f \mid f \in I^m\}$ constructed in Proposition 3.6 the $\imath$-canonical basis of $\mathbb{V}_{\mathtt{L}}^{\otimes m}$.

Remark 3.8. For each $f \in I^m$, $b^\imath_f$ is the unique element in $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ which is $\psi_\imath$-invariant such that $b^\imath_f \in M_f + \sum_g v\mathbb{Z}[v] M_g$ (without the partial ordering condition on g).

3.4. **The $\imath$-canonical bases and $\mathtt{L}$-bases.** The double centralizer property in Theorem 2.6 specializes to the following double centralizing actions:

$$\mathbf{U}^\imath_{\mathtt{L}} \overset{\Phi}{\curvearrowright} \mathbb{V}_{\mathtt{L}}^{\otimes m} \overset{\Psi}{\curvearrowleft} \mathcal{H}^{\mathtt{L}}_m.$$

The following is an $\mathtt{L}$-variant of [BW13, Theorem 5.8] (where $\mathtt{L}(s_0) = \mathtt{L}(s_1) = 1$). The original proof, which uses Lemma 2.5, works here.

Proposition 3.9. *The anti-linear bar involution* $\psi_\imath : \mathbb{V}_L^{\otimes m} \to \mathbb{V}_L^{\otimes m}$ *is compatible with both the bar involution of* $\mathcal{H}^L_{B_m}$ *and the bar involution of* $\mathbf{U}^\imath_L$*; that is, for all* $u \in \mathbb{V}_L^{\otimes m}$*,* $h \in \mathcal{H}_m$*, and* $x \in \mathbf{U}^\imath_L$*, we have* $\psi_\imath(xuh) = \psi_\imath(x)\, \psi_\imath(u)\overline{h}$.

Let

$$I^m_+ := \{f \in I^m \mid 0 \le f(1) \le f(2) \le \cdots \le f(m)\}. \tag{3.5}$$

Then, as a right $\mathcal{H}^{\mathtt{L}}_m$-module, $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ is decomposed as

$$\begin{aligned} &\mathbb{V}_{\mathtt{L}}^{\otimes m} = \bigoplus_{f \in I^m_+} \Big(\bigoplus_{w \in {}^{J(f)}W_{B_m}} \mathbb{Q}(v) M_{f\cdot w} \Big), \\ &\omega_f : \bigoplus_{w \in {}^{J(f)}W_{B_m}} \mathbb{Q}(v) M_{f\cdot w} \stackrel{\simeq}{\longrightarrow} \eta_{_{J(f)}} \mathcal{H}^{\mathtt{L}}_m, \quad M_f \mapsto \eta_{_{J(f)}}, \end{aligned} \tag{3.6}$$

where

$$J(f) = \{j \mid 0 \le j \le m-1,\ f \cdot s_j = f\}.$$

It follows by Lemma 3.1 and the Hecke algebra action (2.8) that

$$\omega_f(M_{f\cdot w}) = \eta_{_{J(f)}} H_w, \qquad \text{for } f \in I^m_+,\ w \in {}^{J(f)}W_{B_m}. \tag{3.7}$$

By Lemma 3.1(3) each $\eta_{_{J(f)}}\mathcal{H}^{\mathtt{L}}_m$ is preserved by the involution $^-$. Thanks to Proposition 3.2 and the identification (3.6), the space $\mathbb{V}_{\mathtt{L}}^{\otimes m}$ admits an $\mathtt{L}$-basis

$$\Big\{ c_{f\cdot w} := \omega_f^{-1}(C^{J(f)}_w) \mid f \in I^m_+,\ w \in {}^{J(f)}W_{B_m} \Big\}.$$

We have the following main theorem of this section.

Theorem 3.10. *The* $\imath$*-canonical basis and the* $\mathtt{L}$*-basis on* $\mathbb{V}_L^{\otimes m}$ *coincide.*

Proof. By Proposition 3.2 and (3.7), we have $c_{f\cdot w} \in M_{f\cdot w} + \sum_{\sigma \in {}^{J(f)}W_{B_m}} v\mathbb{Z}[v] M_{f\cdot\sigma}$, for $f \in I^m_+$, $w \in {}^{J(f)}W_{B_m}$. By Proposition 3.9, $b^\imath_{f\cdot w}$ is $\psi_\imath$-invariant. By the existence of the $\imath$-canonical basis in Proposition 3.6 and the uniqueness in Remark 3.8, we must have $c_{f\cdot w} = b^\imath_{f\cdot w}$. The theorem is proved. □

Example 3.11. The $\imath$-canonical basis on $\mathbb{V}_{\mathtt{L}}$ is given as follows: for $i \in I$ and $i > 0$,

$$
\begin{aligned}
&u_i, \quad u_{i\cdot s_0}, && \text{if } \mathtt{L}(s_0) = 0;\\
&u_i, \quad u_{i\cdot s_0} + v^{\mathtt{L}(s_0)} u_i, && \text{if } \mathtt{L}(s_0) > 0;\\
&u_i, \quad u_{i\cdot s_0} - v^{-\mathtt{L}(s_0)} u_i, && \text{if } \mathtt{L}(s_0) < 0.
\end{aligned}
$$

The above $\imath$-canonical basis on $\mathbb{V}$ coincides with Lusztig's example for the $\mathtt{L}$-basis in [L03, §5.5].

4. The $\jmath$-Schur duality with two parameters

In this section, we establish the duality between a coideal subalgebra $\mathbf{U}^\jmath$ of the quantum group for $\mathfrak{sl}_{2r+1}$ and the Hecke algebra of type B. This section is parallel to sections 2–3, and we shall omit many redundant details to avoid much repetition. We sometimes use the same notation in similar circumstances, as both cases are special cases for $\mathfrak{sl}_k$, with $k = 2r+1$ here (and $k = 2r+2$ in sections 2–3).

4.1. **The quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$.** Let r be a positive integer. We set

$$
\mathbb{I} = \Big\{ -r + \frac{1}{2}, -r + \frac{3}{2}, \dots, r - \frac{1}{2} \Big\}, \qquad \mathbb{I}^\jmath = \Big\{ \frac{1}{2}, \frac{3}{2}, \dots, r - \frac{1}{2} \Big\}. \tag{4.1}
$$

Let $\Pi = \big\{ \alpha_i = \varepsilon_{i-\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}} \mid i \in \mathbb{I} \big\}$ be the simple system of type A_{2r}, and Φ the associated root system. Denote the weight lattice by

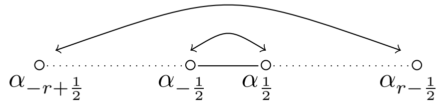
$$
\Lambda = \bigoplus_{i=-r}^{r} \mathbb{Z}\varepsilon_i.
$$

Let

$$
\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_{2r+1}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(p,q)
$$

be the quantum group of type A_{2r} over $\mathbb{Q}(p,q)$ with the standard generators E_i, F_i and $K_i^{\pm 1}$ for $i \in \mathbb{I}$. We denote by $\psi : \mathbf{U} \to \mathbf{U}$ and $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ the bar involution and comultiplication on $\mathbf{U}$ given by the same formulas as in (2.1) and (2.2).

Let $(\mathbf{U}, \mathbf{U}^\jmath)$ be the quantum symmetric pair (cf. [Le99]) over $\mathbb{Q}(p,q)$ with the following Satake diagram:

A_{2r} :


The $\mathbb{Q}(p,q)$-algebra $\mathbf{U}^\jmath$ is the $\mathbb{Q}(p,q)$-subalgebra of $\mathbf{U}$ generated by (for $i \in \mathbb{I}^\jmath$)

$$
\begin{aligned}
&k_i = K_i K_{-i}^{-1}, \quad e_i = E_i + F_{-i} K_i^{-1} \ (i \neq \tfrac{1}{2}), \quad f_i = E_{-i} + K_{-i}^{-1} F_i \ (i \neq \tfrac{1}{2}),\\
&e_{\frac{1}{2}} = E_{\frac{1}{2}} + p^{-1} F_{-\frac{1}{2}} K_{\frac{1}{2}}^{-1}, \quad f_{\frac{1}{2}} = E_{-\frac{1}{2}} + p K_{-\frac{1}{2}}^{-1} F_{\frac{1}{2}}.
\end{aligned} \tag{4.2}
$$

The $\mathbb{Q}(p,q)$-algebra $\mathbf{U}^\jmath$ has the following presentation: it is generated by e_i, f_i, $k_i^{\pm 1}$ (for $i \in \mathbb{I}^\jmath$), subject to the following relations (for $i, j \in \mathbb{I}^\jmath$):

$$
\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,\\
k_i e_j k_i^{-1} &= q^{(\alpha_i - \alpha_{-i}, \alpha_j)} e_j, \quad k_i f_j k_i^{-1} = q^{-(\alpha_i - \alpha_{-i}, \alpha_j)} f_j,\\
e_i f_j - f_j e_i &= \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}},\\
e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i \qquad \text{if } |i-j| > 1,\\
e_i^2 e_j + e_j e_i^2 &= (q+q^{-1}) e_i e_j e_i, \quad f_i^2 f_j + f_j f_i^2 = (q+q^{-1}) f_i f_j f_i \qquad \text{if } |i-j| = 1,\\
e_{\frac12}^2 f_{\frac12} + f_{\frac12} e_{\frac12}^2 &= (q+q^{-1}) \Big(e_{\frac12} f_{\frac12} e_{\frac12} - e_{\frac12} \big(pq k_{\frac12} + p^{-1} q^{-1} k_{\frac12}^{-1} \big) \Big),\\
f_{\frac12}^2 e_{\frac12} + e_{\frac12} f_{\frac12}^2 &= (q+q^{-1}) \Big(f_{\frac12} e_{\frac12} f_{\frac12} - \big(pq k_{\frac12} + p^{-1} q^{-1} k_{\frac12}^{-1} \big) f_{\frac12} \Big).
\end{aligned}
$$

In contrast to the $\mathbf{U}^\imath$ case, the presentation of $\mathbf{U}^\jmath$ depends on the parameter p.

The following counterpart of Lemma 2.1 follows from the above presentation.

Lemma 4.1. *There exists a unique $\mathbb{Q}$-algebra bar involution $\psi_\jmath$ on the algebra $\mathbf{U}^\jmath$ such that $p \mapsto p^{-1}$, $q \mapsto q^{-1}$, $k_i \mapsto k_i^{-1}$, $e_i \mapsto e_i$, and $f_i \mapsto f_i$, for $i \in \mathbb{I}^\jmath$.*

Just as Proposition 2.3 for $\mathbf{U}^\imath$, we have the intertwiner $\Upsilon \in \widehat{\mathbf{U}}$ between the involution ψ on $\mathbf{U}$ and the involution $\psi_\jmath$ on $\mathbf{U}^\jmath$ such that

$$\psi_\jmath(u) \cdot \Upsilon = \Upsilon \cdot \psi(u), \qquad \forall u \in \mathbf{U}^\jmath.$$

This is a straightforward multiparameter variant of [BW13, Theorem 6.4] (cf. [BK15, BW16]).

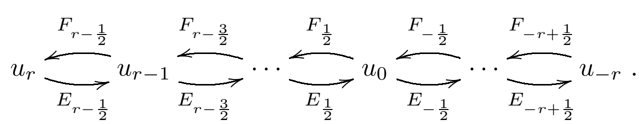
Consider a $\mathbb{Q}(p,q)$-valued function ζ on Λ such that ($\forall \mu \in \Lambda,\ i \in \left\{\frac12, \dots, r - \frac12\right\}$)

$$
\begin{aligned}
\zeta(\mu + \alpha_i) &= -q^{(\alpha_i - \alpha_{-i}, \mu + \alpha_i)} \zeta(\mu),\\
\zeta(\mu + \alpha_{-i}) &= -q^{(\alpha_{-i}, \mu + \alpha_{-i}) - (\alpha_i, \mu)} \zeta(\mu),\\
\zeta(\mu + \alpha_{\frac12}) &= -pq^{(\alpha_{\frac12} - \alpha_{-\frac12}, \mu + \alpha_{\frac12}) - 1} \zeta(\mu),\\
\zeta(\mu + \alpha_{-\frac12}) &= -p^{-1} q^{(\alpha_{-\frac12}, \mu + \alpha_{-\frac12}) - (\alpha_{\frac12}, \mu) + 1} \zeta(\mu).
\end{aligned}
\tag{4.3}
$$

Such ζ clearly exists. For any weight $\mathbf{U}$-module M, we obtain a $\mathbb{Q}(p,q)$-linear map $\widetilde{\zeta} : M \to M$ as in (2.5). Let w_0 be the longest element of the Weyl group $W_{A_{2r}}$ and T_{w_0} the associated braid group element. The following multiparameter variant of [BW13, Theorem 6.6] holds by the same proof.

Proposition 4.2. *For any finite-dimensional $\mathbf{U}$-module M, the composition map $\mathcal{T} := \Upsilon \circ \widetilde{\zeta} \circ T_{w_0} : M \longrightarrow M$ is a $\mathbf{U}^\jmath$-module isomorphism.*

4.2. **The $(\mathbf{U}^\jmath, \mathcal{H}_m)$-duality.** Let $I = \mathbb{I} \pm \frac12$. Let $\mathbb{V} = \bigoplus_{a \in I} \mathbb{Q}(p,q) u_a$ be the natural representation of $\mathbf{U}$. The $\mathbf{U}$-module structure of $\mathbb{V}$ can be visualized as follows:



We regard the $\mathbf{U}$-module $\mathbb{V}^{\otimes m}$ as a $\mathbf{U}^\jmath$-module by restriction.

Recall from (2.7) the element $M_f \in \mathbb{V}^{\otimes m}$, for any $f \in I^m$ (except that I here is understood as in (4.1)). The Weyl group W_{B_m} acts on I^m in the obvious way. The Hecke algebra $\mathcal{H}_m$ acts on $\mathbb{V}^{\otimes m}$ as follows:

$$
\begin{aligned}
M_f \cdot H_i &= \begin{cases} q^{-1} M_f, & \text{if } f(i) = f(i+1); \\ M_{f \cdot s_i}, & \text{if } f(i) < f(i+1); \\ M_{f \cdot s_i} + (q^{-1} - q) M_f, & \text{if } f(i) > f(i+1); \end{cases} \\
M_f \cdot H_0 &= \begin{cases} p^{-1} M_f, & \text{if } f(1) = 0; \\ M_{f \cdot s_0}, & \text{if } f(1) > 0; \\ M_{f \cdot s_0} + (p^{-1} - p) M_f, & \text{if } f(1) < 0. \end{cases}
\end{aligned}
\tag{4.4}
$$

Summarizing, we shall depict the actions of $\mathbf{U}^\jmath$ and $\mathcal{H}_m$ on $\mathbb{V}^{\otimes m}$ as

$$
\mathbf{U}^\jmath \overset{\Phi}{\curvearrowright} \mathbb{V}^{\otimes m} \overset{\Psi}{\curvearrowleft} \mathcal{H}_m. \tag{4.5}
$$

We fix ζ in (4.3) such that $\zeta(\epsilon_{-r}) = 1$. Then, we have

$$
\zeta(\epsilon_{-i}) = \begin{cases} (-q)^{-r+i} & \text{if } i \neq 0, \\ (-q)^r p & \text{if } i = 0, \end{cases}
$$

for all $i \in \{-r, -r+1, \dots, r\}$.

Lemma 4.3. *The actions of H_0 and $\mathcal{T}^{-1}$ on $\mathbb{V}$ coincide.*

Proof. We define

$$
\begin{aligned}
\mathbb{V}^+ &= \bigoplus_{j \in \mathbb{I}^\jmath} \mathbb{Q}(p,q)(u_{-j-\frac12} + p u_{j+\frac12}) \bigoplus \mathbb{Q}(p,q) u_0, \\
\mathbb{V}^- &= \bigoplus_{j \in \mathbb{I}^\jmath} \mathbb{Q}(p,q)(u_{-j-\frac12} - p^{-1} u_{j+\frac12}).
\end{aligned}
$$

By direct calculations, we have, for $j \in \mathbb{I}^\jmath$,

$$
\begin{aligned}
f_{\alpha_{\frac12}} \cdot u_0 &= u_{-1} + p u_1, \\
f_i \cdot \big(u_{-j-\frac12} + p u_{j+\frac12}\big) &= \delta_{i,j+1} \cdot \big(u_{-(j+1)-\frac12} + p u_{(j+1)+\frac12}\big), \\
e_i \cdot \big(u_{-j-\frac12} + p u_{j+\frac12}\big) &= \delta_{i,j} \cdot \big(p^{-\delta_{\frac12,i}} u_{-(j-1)-\frac12} + p u_{(j-1)+\frac12}\big), \\
f_i \cdot \big(u_{-j-\frac12} - p^{-1} u_{j+\frac12}\big) &= \delta_{i,j+1} \cdot \big(u_{-(j+1)-\frac12} - p^{-1} u_{(j+1)+\frac12}\big), \\
e_i \cdot \big(u_{-j-\frac12} - p^{-1} u_{j+\frac12}\big) &= \delta_{i,j} \cdot \big(p^{-\delta_{\frac12,i}} u_{-(j-1)-\frac12} - p^{-1} u_{(j-1)+\frac12}\big).
\end{aligned}
$$

Hence, $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$ as a $\mathbf{U}^\jmath$-module. Furthermore, H_0 acts as the scalar multiplication by p^{-1} (resp., $-p$) on $\mathbb{V}^+$ (resp., $\mathbb{V}^-$).

Since we have $T_{w_0}(u_j) = (-q)^{r-j} \cdot u_{-j}$, we obtain

$$
\widetilde{\zeta} \circ T_{w_0}(u_j) = \begin{cases} u_{-j} & \text{if } j \neq 0, \\ p \cdot u_0 & \text{if } j = 0. \end{cases}
$$

On the other hand, one computes the first term of Υ as

$$
\Upsilon_{\alpha_{\frac12} + \alpha_{-\frac12}} = (p - p^{-1}) F_{\alpha_{\frac12}} F_{\alpha_{-\frac12}}.
$$

Hence, we have $\mathcal{T}(u_0) = p u_0, \mathcal{T}(u_1) = u_{-1} - (p^{-1} - p) u_1$, and $\mathcal{T}(u_{-1}) = u_1$, which imply that the action of $\mathcal{T}^{-1}$ on $\mathbb{V}^+$ and $\mathbb{V}^-$ are given by scalar multiplication by p^{-1} and $-p$, respectively. The lemma follows. $\square$

Now with the help of Lemma 4.3, we obtain the following counterpart of Theorem 2.6 by the same argument.

Theorem 4.4 ($\jmath$-Schur duality)**.** *The actions of* $\mathbf{U}^\jmath$ *and* $\mathcal{H}_m$ *on* $\mathbb{V}^{\otimes m}$ (4.5) *commute and form double centralizers; that is,*

$$\Phi(\mathbf{U}^\jmath) = End_{\mathcal{H}_m}(\mathbb{V}^{\otimes m}), \quad End_{\mathbf{U}^\jmath}(\mathbb{V}^{\otimes m})^{\rm op} = \Psi(\mathcal{H}_m).$$

4.3. **The $\jmath$-canonical basis and L-basis.** All the results in Sections 3.3–3.4 admit natural counterparts in the setting of $\mathbf{U}^\jmath$. The proofs are similar or easier in the $\mathbf{U}^\jmath$ setting (e.g., the integrality of the intertwiner Υ is completely the same as in [BW13]). So we shall be brief.

Given a weight function $\mathtt{L} : W_{B_m} \to \mathbb{Z}$, by a base change we have a $\mathbb{Q}(v)$-algebra

$$\mathbf{U}^\jmath_{\mathtt{L}} = \mathbf{U}^\jmath \otimes_{\mathbb{Q}(p,q)} \mathbb{Q}(v).$$

Recall that the $\mathbf{U}_{\mathtt{L}}$-module $\mathbb{V}^{\otimes m}$ admits a bar involution ψ using the quasi-R-matrix Θ ([L94, Chap. 4]). We define another anti-linear bar involution on the $\mathbf{U}^\jmath_{\mathtt{L}}$-module $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ as

$$\psi_\jmath = \Upsilon \circ \psi.$$

Entirely similarly to [BW13], we can establish the $\jmath$-canonical bases on finite-dimensional simple $\mathbf{U}$-modules and their tensor products. In particular $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ admits a $\jmath$-canonical basis (similar to Proposition 3.6). As in Proposition 3.9, we have compatible bar maps in the following sense: for all $u \in \mathbb{V}^{\otimes m}_{\mathtt{L}}$, $h \in \mathcal{H}^{\mathtt{L}}_m$, and $x \in \mathbf{U}^\jmath_{\mathtt{L}}$, we have

$$\psi_\jmath(xuh) = \psi_\jmath(x)\, \psi_\jmath(u)\overline{h}.$$

We still define I^m_+ as in (3.5) and the decomposition of $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ as a right $\mathcal{H}^{\mathtt{L}}_m$-module as in (3.6). Then $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ admits a bar involution and an L-basis (inherited from $\mathcal{H}^{\mathtt{L}}_m$). Keep in mind again that I^m_+ and $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ are slightly different from those in section 3.4, because I here is understood as in (4.1). We have the following counterpart of Theorem 3.10.

Theorem 4.5. *The $\jmath$-canonical basis of* $\mathbb{V}^{\otimes m}_{\mathtt{L}}$ *is identical to the* L*-basis of* $\mathbb{V}^{\otimes m}_{\mathtt{L}}$*.*

Example 4.6. We have the following $\jmath$-canonical basis for $\mathbb{V}_{\mathtt{L}}$ (for $1 \le i \le r$):

$$\begin{aligned} u_0, \quad u_i, \quad u_{i\cdot s_0}, &\quad \text{for } \mathtt{L}(s_0) = 0;\\ u_0, \quad u_i, \quad u_{i\cdot s_0} + v^{\mathtt{L}(s_0)} u_i, &\quad \text{for } \mathtt{L}(s_0) > 0;\\ u_0, \quad u_i, \quad u_{i\cdot s_0} - v^{-\mathtt{L}(s_0)} u_i &\quad \text{for } \mathtt{L}(s_0) < 0. \end{aligned}$$

Again this example coincides with Lusztig's example in [L03, §5.5].

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