

COTOTAL ENUMERATION DEGREES AND THEIR APPLICATIONS TO EFFECTIVE MATHEMATICS

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ABSTRACT. A set $A \subseteq \omega$ is *cototal* under enumeration reducibility if $A \leq_e \overline{A}$, that is, if the complement of A is total. We show that the e -degrees of cototal sets characterize the e -degrees of maximal anti-chain complements, the e -degrees of enumeration-pointed trees on $2^{<\omega}$, and the e -degrees of languages of minimal subshifts on 2^ω .

As a consequence, we obtain a characterization of the Turing degree spectra of nontrivial minimal subshifts: they are the enumeration cones of cototal sets. From the perspective of the Turing degrees, this provides a complete understanding of the computational power of minimal subshifts. We also obtain an application to computable structure theory, showing that the enumeration cones of cototal sets characterize those structure spectra which are Turing-upward closures of F_σ sets of reals.

1. INTRODUCTION

Questions concerning the effective or algorithmic properties of classical mathematical objects have had a significant impact on the development of modern mathematics. In the study of Diophantine sets, we have Matiyasevich's resolution of Hilbert's tenth problem; or on the topic of word problems for groups, the seminal work of Dehn in the early development of geometric group theory. A central goal of applied computability theory is to mathematically formalize this algorithmic perspective by precisely quantifying the computational content of classes of mathematical objects and relations.

The most widely studied measure of such computational content is that given by Turing reducibility, and one approach to quantifying the complexity of a class of objects is to calculate the *degree spectra* of the class, that is, the collection of Turing degrees obtained by the members of the class. Consider for example the characterization of the degrees of the block relations of computable linear orderings as precisely the Σ_3^0 degrees (see [9]), or the Novikov-Boone characterization of the degrees of word problems for finitely presented groups as the computably enumerable degrees (see [3]).

However, applications often must employ reducibilities other than Turing reduction. An early indication that one must consider other reducibilities comes from group theory. A result of Macintyre shows that for finitely generated groups G and H , that if G embeds as a subgroup of every algebraically closed extension of H , then the word problem for G is Turing reducible to the word problem for H

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[16]. Ziegler showed that the converse fails, but that the result becomes an equivalence if a stronger computability-theoretic reducibility, namely Ziegler reducibility, is substituted for Turing reducibility [27]. Because an algebraically closed group is determined by the finitely generated groups that embed in it, the computability theoretic structure not only provides a complete understanding of the algebraic one, but is in fact implicit within the algebraic structure (see also [2] and [11]).

Ziegler's reducibility (which he called **-reducibility*) was formulated as a strengthening of *enumeration reducibility*, a reducibility important in its own right and the primary reducibility we will use in this paper. Enumeration reducibility, along with its associated degree structure, measures the relative computational difficulty in producing enumerations of sets of natural numbers.

Enumeration reducibility also revealed its significance early on in the study of *structure spectra*, that is, degree spectra of isomorphism classes of structures. Richter used enumeration reducibility to give sufficient conditions on a first-order theory to ensure it has countable models whose structure spectra has no least element, so that the Turing degrees alone are not sufficient to capture the effective content of the structure considered up to isomorphism [21, 22].

Enumeration reducibility has also been important in applications to computable analysis. Miller, answering a question of Pour-El and Lempp, showed that the Turing degrees are similarly deficient for quantifying the complexity of continuous real-valued functions, introducing the *continuous degrees*, a subclass of the enumeration degrees which are able to capture the computational content of continuous functions [17]. Kihara and Pauly have extended this connection to associate degree structures to arbitrary quasi-Polish spaces, obtaining the enumeration degrees as the degree structure associated to the universal quasi-Polish space $\mathcal{O}(\mathbb{N})$ [15]. These connections have proven particularly fruitful, resulting in a solution to the general n 'th level Borel isomorphism problem.

In this paper, using a particular subclass of the enumeration degrees: the *cototal* enumeration degrees, we succeed in identifying several degree spectra which have been of particular interest in the fields of effective structure theory and symbolic dynamics. In Section 4, we see that the cototal enumeration degrees can be used to characterize which enumeration cones are obtainable as Turing upward closures of F_σ sets of reals, a question stemming from a result of Montalbán on structure spectra. Then in Section 5, we consider an example from symbolic dynamics, that of *subshifts*, and show that the cototal enumeration degrees provide a complete characterization of the Turing degree spectra of their building-blocks: the *minimal* subshifts.

The computational power of subshifts, and in particular of minimal subshifts, has generated interest for several years. From the perspective of computability theory, see Durand, Levin, and Shen [8], Cenzer, Dashti, and King [4], Cenzer, Dashti, Toska, and Wyman [5, 6], as well as Simpson [24], Jeandel and Vanier [14], Hochman and Vanier [12], and Jeandel [13]. In computer science, the study of the computational power of simple dynamical systems, especially subshifts, comprises an active body of recent research (see [7]). Minimal subshifts are also important in studying individual infinite sequences, because measures of sequence complexity that are important in characterizing the algebraic combinatorics of infinite sequences can be studied on the subshifts they generate (the well-studied Thue-Morse sequence, for example, generates a minimal subshift) [20].

We consider the results of Section 5 to constitute an extension of the connection between the enumeration degrees and applications of effective mathematics—already rich in application to group theory and analysis—now to the field of symbolic dynamics. Although these results mark the first use of the cototal enumeration degrees to identify a degree spectrum of independent interest, observations of Jeandel indicate intimate connections between cototality and both simple groups and maximal ideals of rings [13], so we anticipate further interest. Also, in the point-degree spectrum language of Kihara and Pauly, cototality results from topological tameness of the underlying represented spaces, so we believe a better understanding of the cototal degrees will be impactful with regard to those applications as well.

In addition to our applications, some of our results should be useful in pursuing a greater understanding of the cototal enumeration degrees themselves. For example, Theorem 3.2 provides a simple characterization of the cototal enumeration degrees as those e -degrees which contain complements of maximal anti-chains on $\omega^{<\omega}$, and Theorem 4.5 as the enumeration degrees which contain e -pointed trees. This second characterization is particularly useful because of the rich intro-enumerability properties of e -pointed trees. For example, Miller and Soskova have recently used Theorem 4.5 to prove that the cototal enumeration degrees are dense [18]. In the theory of the structure of the enumeration degrees, cototality corresponds to a combinatorial property of *good approximation* that has been essential in establishing structural properties of the enumeration degrees, so we anticipate that further study of the cototal degrees will be important in understanding the structure of the enumeration degrees, and in turn the c.e.-degrees and Turing degrees.

2. COTOTAL SETS AND DEGREES

Enumeration reducibility, introduced by Friedberg and Rogers in 1959, captures the relative difficulty of producing enumerations of sets of natural numbers. Alternatively, it can be thought of as a notion of computation between sets that uses only the positive portion of their set membership information.

An enumeration functional Γ is a computably enumerable (c.e.) set of pairs $\langle n, F \rangle$ with each $n \in \omega$ and F the canonical code of a finite subset of ω . We think of Γ as reading the positive membership information of X , and, for $\langle n, F \rangle \in \Gamma$, enumerating n upon seeing $F \subseteq X$. Given $X \subseteq \omega$, we define $\Gamma(X) = \{n : \exists F(\langle n, F \rangle \in \Gamma \text{ and } F \subseteq X)\}$.

For sets $A, B \subseteq \omega$, we say that $A \leq_e B$ if there exists an enumeration functional Γ with $\Gamma(B) = A$. Equivalently, $A \leq_e B$ if there is a single Turing functional which, given any enumeration of B , outputs an enumeration of A . The relation \leq_e defines a pre-order on 2^ω , the partial order it induces is called the *enumeration degrees*, or e -degrees, denoted \mathcal{D}_e .

Another way to characterize enumeration reducibility was given by Selman [23]. Given a set X , let $\mathcal{E}(X)$ denote the collection of all Turing degrees computing enumerations of X , called the *enumeration cone of X* . Then Selman showed:

Theorem 2.1 (Selman [23]). *$A \leq_e B$ if and only if $\mathcal{E}(B) \subseteq \mathcal{E}(A)$.*

A set X is *total* if its positive information already suffices to determine its negative information, or precisely: if $\overline{X} \leq_e X$. We call an enumeration degree *total* if it is the e -degree of a total set. The Turing degrees (which we denote by \mathcal{D}_T) embed in the enumeration degrees via the map induced set-wise by $X \mapsto X \oplus \overline{X}$. The image of this embedding is the total e -degrees.

The name “total” is evocative of the following fact: given a total function f , total in the sense of having full domain, the set $\text{graph}(f) = \{\langle n, f(n) \rangle : n \in \omega\}$ is total under enumeration reducibility. In fact, every total set is enumeration-equivalent to the graph of a total function, for example, its characteristic function.

A closely related notion is that of cototality. A set A is *cototal* if $A \leq_e \bar{A}$, that is, if the complement of A is total as a set, and we call an enumeration degree *cototal* if it is the e -degree of a cototal set.

For example, the complements of graphs of total functions are cototal as sets. The e -degrees of such sets are called *graph cototal*. The class of graph cototal degrees has been studied by Solon in [25] and [26]. Several other natural classes of cototal sets were brought to attention by Jeandel in [13], including examples from symbolic dynamics and algebra.

The cototal degrees were studied recently by Andrews et al. in [1]. In addition to showing that the cototal enumeration degrees are a proper subclass of the enumeration degrees (that not every enumeration degree is cototal), they also separate the class of cototal degrees from the class of graph cototal degrees. That is: not every cototal set is enumeration equivalent to a complement of the graph of a total function. It is natural then to look for classes of objects that *do* capture cototality in the enumeration degrees. Andrews et al. show that the complements of maximally independent sets in $\omega^{<\omega}$ (with $\omega^{<\omega}$ considered as an undirected graph) form one such class.

In Section 3 we identify another simple class of objects characterizing cototality, showing that the cototal degrees are the degrees of complements of maximal anti-chains on $\omega^{<\omega}$. In Section 4, we show that the e -degrees of enumeration pointed trees are the same as those of the maximal anti-chain complements, providing yet another class of objects whose e -degrees are the cototal degrees. Although we use this section as a stepping stone to approach the class considered in Section 5, e -pointed trees are interesting in their own right and provide us with applications to computable structure theory.

Section 5 focuses on a particular example of a class of cototal objects identified by Jeandel in [13], namely, the languages of minimal subshifts. Jeandel and Vanier in [14] prove that the Turing degree spectra of a nontrivial minimal subshift is the enumeration cone of its language. We show that the enumeration degrees of languages of minimal subshifts are the same as the enumeration degrees of enumeration pointed trees, providing a characterization of the Turing degree spectra of nontrivial minimal subshifts as precisely the enumeration cones of cototal sets.

3. MAXIMAL ANTI-CHAIN COMPLEMENTS

Theorem 3.1. *If A is a maximal anti-chain on $\omega^{<\omega}$, then \bar{A} is cototal.*

Proof. To determine if a string $\sigma \in \omega^{<\omega}$ is in \bar{A} , we wait for some element comparable but not equal to σ to enter A . Since A is an anti-chain, we only enumerate elements of \bar{A} in this way. And by maximality, if $\sigma \in \bar{A}$, then something comparable but not equal to σ must be in A , so our procedure enumerates all of \bar{A} . \square

Note that every total set is enumeration-equivalent to a maximal anti-chain. Given a total set A , consider C given by $\{n : n \in A\} \cup \{n \hat{\ } k \dots : n \in \bar{A}, k \in \omega\}$. Then $A \equiv_e C$. However, it may be that $\bar{A} \not\equiv_e \bar{C}$, i.e., when \bar{A} is not total itself.

Nonetheless, the degrees of cototal sets are, in fact, exactly the degrees of the complements of maximal anti-chains:

Theorem 3.2. *If A is cototal, then $A \equiv_e \overline{C}$ for some C a maximal anti-chain on $\omega^{<\omega}$.*

Contrast this result with the case for function graphs: every total set A is enumeration-equivalent to the graph of a total function, for example, the graph of its characteristic function χ_A , but not every cototal degree contains a set of the form $\overline{\text{graph}(f)}$ [1].

Proof of Theorem 3.2. Let $A \leq_e \overline{A}$ via the enumeration operator Γ . Fix a computable listing of Γ to work with. We construct a subset \overline{C} of $\omega^{<\omega}$ as follows:

First, put $\lambda \in \overline{C}$. For the first layer of \overline{C} as a subset of $\omega^{<\omega}$, we enumerate A . That is, $n \in \overline{C} \iff n \in A$.

For each node $\alpha \in \omega^{<\omega} \setminus \{\lambda\}$, we attach some finite set, which we call the *claim* of α . We think of α as claiming this finite set is a subset of \overline{A} . A node is filled in, that is, enumerated into \overline{C} , when we witness its claim to be false. For the first-level nodes $\alpha = n$, we set their claims to $\{n\}$. So for nodes $\alpha = n$, $\alpha \in \overline{C} \iff \alpha \in A$, that is, if and only if α is wrong about its claim.

Layer by layer, we attach a claim to each node α as follows: each node α of length $|\alpha| = k$ looks at all the $k - 1$ nodes below it (apart from λ), and their claims, and chooses one element from each claim to attempt to prove wrong. For each chosen element n , α picks axioms $\langle n, F \rangle$ from Γ . We choose claims in such a way that directly above each node, we attach all possible claims from all possible choices of axioms from Γ that could prove the claims below them wrong.

To do this assignment computably we assume, without loss, that Γ has the following property: for all $n \in \omega$, $\exists F$ such that $\langle n, F \rangle \in \Gamma$. To achieve this we can, for example, add the axioms $\langle n, \{a\} \rangle$ to Γ for some fixed $a \in A$.

We define the claim of α to be the union of the F 's from those axioms chosen. A node α is put into \overline{C} when our enumeration of A proves its claim wrong. That is, when some element in the claim of α is enumerated into A .

So \overline{C} is an enumeration below A via the construction, and also A is an enumeration below \overline{C} because A can be read out explicitly in the first layer of \overline{C} .

Claim. C is a maximal anti-chain.

We visualize the nodes of $\omega^{<\omega}$ as open circles, which are filled in as they are enumerated into \overline{C} .

If some $\alpha \in C$, then its claim is never proven wrong. But its claim was chosen such that, if it were never proven wrong, all the nodes below α must be filled in, enumerated into \overline{C} . Moreover, all nodes above α were chosen such that, if they are in C , then the nodes below them must all be in \overline{C} . But α is not in \overline{C} , so by contrapositive no node above it is in C .

Now suppose C were not maximal as an anti-chain. That is, suppose we could add some node α to C . Then under our construction, the node α is filled in, and also all nodes below it are filled in. But then α and all of its ancestors are wrong about their claims. So there is some choice of axioms from Γ proving these claims wrong. So there is some descendent of α , say $\alpha \hat{\ } n$ whose claim is the union of those axioms proving the ancestor's claims wrong. But then the claim of $\alpha \hat{\ } n$ is never proven wrong, so $\alpha \hat{\ } n$ is in C . So we cannot add α to C and still obtain an anti-chain. □

4. ENUMERATION-POINTED TREES

Definition 4.1. An *e-pointed tree* is a tree $T \subseteq 2^{<\omega}$ with no dead-ends, such that $X \geq_e T$ for all $X \in [T]$.

These objects were encountered in work of Antonio Montalbán in computable structure theory:

Theorem 4.2 (Montalbán [19]). *The Turing upward closure of an F_σ set of reals in ω^ω cannot be the degree spectra of a structure unless it is an enumeration cone. In fact, for $X \subseteq \mathcal{D}_T$, the following are equivalent:*

- (1) *X is the degree spectra of a structure and the Turing upward closure of an F_σ set of reals in ω^ω .*
- (2) *X is the enumeration cone of an e-pointed tree.*

Recall that the enumeration cone of a set $A \subseteq \omega$ is the collection $\mathcal{E}(A)$ of all Turing degrees computing enumerations of A . In the case that A has total enumeration degree, $\mathcal{E}(A)$ coincides with the Turing cone of A : the upward closure of $\mathbf{d}(A)$ in the Turing degrees. Montalbán notes that there are structure spectra as in Theorem 4.2 which are not Turing cones. However, it was not known precisely which enumeration cones were possible, for example, whether every enumeration cone is realized as the Turing upward closure of an F_σ set of reals. We provide an answer by showing that the enumeration degrees of e-pointed trees are exactly the cototal degrees. In particular, not every enumeration cone is realized as a structure spectra as in Theorem 4.2.

We first show that e-pointed trees appear in every cototal degree. In fact, every cototal degree contains an e-pointed tree of a particular form.

Definition 4.3. A *uniformly e-pointed tree* is a tree $T \subseteq 2^{<\omega}$ with no dead-ends for which there exists an enumeration functional W so that $W(X) = T$ for all $X \in [T]$.

Uniformly e-pointed trees have a useful intro-enumerability property:

Theorem 4.4. *If T is a uniformly e-pointed tree, then for each n , there exists an m so that $T \upharpoonright n \subseteq W_{|\sigma|}(\sigma)$ for all $\sigma \in T$ with $|\sigma| \geq m$.*

When working with a uniformly e-pointed tree T and functional W , we denote the function taking n to the first such m by $s(n) = m$.

Proof of Theorem 4.4. Since every path through T enumerates $T \upharpoonright n$, the collection of basic clopen sets $\{[\sigma] : \sigma \in T \text{ and } W_{|\sigma|}(\sigma) \supseteq T \upharpoonright n\} \cup \{[\sigma] : \sigma \notin T\}$ covers 2^ω . By compactness of 2^ω , finitely many $[\sigma]$ suffice, so by some finite level every path in T has enumerated $T \upharpoonright n$. □

Theorem 4.5. *If A is cototal, then $A \equiv_e T$ for some uniformly e-pointed tree T .*

Proof. Fix C a maximal anti-chain on $\omega^{<\omega}$ with $\overline{C} \equiv_e A$. Each nonzero level of $2^{<\omega}$ will be associated to a pair of comparable, unequal strings in $\omega^{<\omega}$.

Put λ in T . If level n is associated with (σ, τ) , then every node on level n of T branches left if $\sigma \in \overline{C}$ and right if $\tau \in \overline{C}$.

Since C is an anti-chain, at most one of σ, τ belong to C , so T has no dead-ends. By construction, $\overline{C} \geq_e T$.

To see that $T \geq_e \overline{C}$, we enumerate τ when we branch left at the level associated to (τ, σ) and enumerate σ when we branch right. But we have more: we claim that by the same functional, each path in T uniformly enumerates \overline{C} .

Let $\sigma \in \overline{C}$. Then by maximality of C , there must be some element τ comparable to σ with $\tau \in C$. Then the level of $2^{<\omega}$ associated to (τ, σ) can only branch to the right in T , so whichever path we take in T , we must enumerate σ .

Therefore, we have one functional W' witnessing $X \geq_e \overline{C}$ for each $X \in [T]$ uniformly. Composing this with the reduction $\overline{C} \geq_e T$, we obtain a functional W witnessing that T is a uniformly e -pointed tree. \square

It is not hard to see, by an application of compactness of 2^ω , that uniformly e -pointed trees are cototal. It is not as easy to see that e -pointed trees themselves are all of cototal degree. To prove this, we first pass through a superclass of uniformly e -pointed trees.

Definition 4.6. A *uniformly e -pointed tree with dead-ends* is a tree $T \subseteq 2^{<\omega}$, possibly with dead-ends, for which there exists an enumeration functional W so that $W(X) = T$ for all $X \in [T]$.

In particular, a uniformly e -pointed tree is a uniformly e -pointed tree with dead-ends.

Theorem 4.7. *If T is a uniformly e -pointed tree with dead ends, then T is cototal.*

Proof. Let W be the enumeration functional witnessing T has the e -pointed property uniformly.

Let $n \in T$ and consider the following cover of 2^ω by clopen sets:

$$\{[\sigma] : \sigma \notin T\} \cup \{[\sigma] : W(\sigma) \ni n\}.$$

Since 2^ω is compact, finitely many σ suffice to cover the space. But then after enumerating finitely many $\sigma \in \overline{T}$, for any $n \in T$ we witness at some finite stage that every path remaining enumerates n , so we can safely enumerate n into T . This procedure is uniform, so $\overline{T} \geq_e T$. \square

Lemma 4.8. *If T is a (nonuniformly) e -pointed tree, then there exists some uniformly e -pointed tree T' with dead-ends such that $T \geq_e T'$ and $\overline{T'} \geq_e T$.*

Proof. Given T , we attempt to build a sequence of subsets T_i diagonalizing against enumeration functionals W_i as follows:

$$T_0 = T.$$

To define T_{n+1} , consider the enumeration functional W_n .

If there exists $\sigma \in T_n$ with $[\sigma] \cap [T_n] \neq \emptyset$ and $W_n(\sigma) \not\subseteq T$, then set $T_{n+1} = T_n \cap \llbracket \sigma \rrbracket$. Otherwise, assuming W_n does not enumerate T uniformly on $[T_n]$, there must be some path $X \in [T_n]$ so that $W_n(X) \not\supseteq \tau_n$ for some $\tau_n \in T$. Then define T_{n+1} by removing any node $\sigma \in T_n$ for which $W_{n,|\sigma|}(\sigma) \ni \tau_n$.

If this procedure continues indefinitely, we have nested sets T_i with $[T_i]$ all nonempty. A nested sequence of compact nonempty sets has nonempty intersection, so we obtain a path $X \in T$ on which no W_n enumerates T .

So this procedure must stop at some finite stage, that is, after intersecting T with some finitely many $\llbracket \sigma_k \rrbracket$ and removing all nodes σ such that $W_{n,|\sigma|}(\sigma) \ni \tau_k$ for finitely many k, τ_k , we have that W_n enumerates T uniformly on $[T_{n+1}]$.

Let $T' = T_{n+1}$, $W' = W_n$. Notice that since we intersect with finitely many $\llbracket \sigma_k \rrbracket$ and remove only nodes σ such that $W_{k,|\sigma|}(\sigma) \ni \tau_k$ for finitely many k, τ_k , we have that $T \geq_e T'$: as we enumerate T we allow only those nodes lying above or below the finitely many σ_k , and before enumerating a node σ check whether $W_{k,|\sigma|}(\sigma) \ni \tau_k$ for the finitely many k, τ_k .

Claim. $\overline{T'} \geq_e T$.

Again by a similar compactness argument as in Theorem 4.7, for $n \in T$ we have that

$$\{[\sigma] : \sigma \notin T'\} \cup \{[\sigma] : W'(\sigma) \ni n\}$$

is an open cover of 2^ω . By compactness, in enumerating $\overline{T'}$, by some finite stage we will have enumerated enough of $\overline{T'}$ to see, checking finitely many other σ , that all remaining paths enumerate n . \square

Theorem 4.9. *If T is an e -pointed tree, then T has cototal degree.*

Proof. Let $X = T \oplus T'$, with T' as in Lemma 4.8. Clearly $X \geq_e T$, and since $T \geq_e T'$, we see $T \geq_e X$. So $X \equiv_e T$. And since $\overline{T'} \geq_e T \geq_e T'$, we see that $\overline{X} \geq_e X$, so X is cototal. \square

As a corollary:

Corollary 4.9.1. *The following are equivalent of an e -degree \mathbf{e} :*

1. \mathbf{e} contains a uniformly e -pointed tree.
2. \mathbf{e} contains an e -pointed tree.
3. \mathbf{e} contains a uniformly e -pointed tree with dead-ends.
4. \mathbf{e} is cototal.

We obtain a corollary to Montalbán’s Theorem 4.2:

Corollary 4.9.2. *A degree spectrum is the Turing upward closure of an F_σ set of reals in ω^ω if and only if it is the enumeration cone of a cototal set. In fact, for $X \subseteq \mathcal{D}_T$ the following are equivalent:*

- (1) X is the degree spectrum of a structure and the Turing upward closure of an F_σ set of reals.
- (2) X is the enumeration cone of a cototal set.

In particular, not every enumeration cone may be simultaneously obtained both as the degree spectrum of a structure and as the Turing upward closure of an F_σ set of reals.

5. MINIMAL SUBSHIFTS

In [13], Emmanuel Jeandel gave several examples of classes of algebraic and combinatorial objects exhibiting cototality. We will consider one such class, the languages of minimal subshifts. More on minimal subshifts can be found in [12]. The *shift operator* on 2^ω is the map taking a real $\alpha \in 2^\omega$ to the unique $\beta \in 2^\omega$ such that $\alpha = n \hat{\ } \beta$ for some $n \in 2$, that is, the operator which erases the first bit of a real. In functional notation, it is the operator $\alpha(n) \mapsto \alpha(n + 1)$.

Definition 5.1. A *subshift* is a closed, shift-invariant subspace of 2^ω .

The trivial example of 2^ω itself is called the *full binary shift*. Binary subshifts are thought of as describing the evolution of a symbolic dynamical system taking states in $\{0, 1\}$. More generally, subshifts on n^ω can be defined for any finite set of n states, with elements of the subshift describing a possible sequence of states taken over some evolution of the system.

Definition 5.2. A subshift X is *minimal* if it satisfies one of the following equivalent conditions:

1. X contains no proper subshifts.
2. X is the shift-invariant closure of any of its points.
3. Every point of X contains the same subwords.

Definition 5.3. The *language* of a subshift X , denoted $\mathcal{L}(X)$, is the collection of all subwords appearing in any of its points. The set $\overline{\mathcal{L}(X)}$ is called the *set of forbidden words*.

The closure condition guarantees that a subshift is characterized by its language, or equivalently by its collection of forbidden words. Conversely, designating any collection of words as *forbidden* determines a unique subshift consisting of all infinite strings which avoid the designated forbidden words (i.e., which do not contain any forbidden word as a subword).

Definition 5.4. The Turing degree *spectrum* of a subshift X is the collection of Turing degrees of its points.

Definition 5.5. A subshift is *trivial* (or *periodic*) if it is the shift-invariant closure of a point of the form $X = w^\infty$ for some finite word $w \in 2^{<\omega}$.

The language of a subshift is particularly relevant to us because of the following theorem:

Theorem 5.6 (Jeandel, Vanier [14]). *If X is a minimal subshift which is not trivial, the Turing degree spectrum of X is the enumeration cone $\mathcal{E}(\mathcal{L}(X))$.*

Note by Theorem 2.1, the set $\mathcal{E}(\mathcal{L}(X))$ of Turing degrees which compute enumerations of $\mathcal{L}(X)$ is characterized by the enumeration degree of $\mathcal{L}(X)$. So understanding the possible Turing degree spectra of minimal subshifts X reduces to understanding what enumeration degrees lie at the base of these enumeration cones $\mathcal{E}(\mathcal{L}(X))$. The cototal degrees enter the picture here:

Theorem 5.7 (Jeandel [13]). *If X is a minimal subshift, then $\mathcal{L}(X)$ is cototal.*

Degreewise then, an enumeration degree must be cototal to be the enumeration degree of the language of a minimal subshift. We show that this condition is sufficient. That is: each cototal degree contains the language of a minimal subshift.

Our construction is similar to the main construction in [12], in that we build a minimal subshift X as a nested intersection of subshifts X_n generated by languages L_n , with each L_{n+1} built up from concatenations of words in L_n . In [12], minimality is ensured by requiring that each word in L_{n+1} contains all of L_n as subwords. Our main insight is that it is enough that for each n there exists an $m > n$ so each word in L_m contains all of L_n as subwords. This relaxed condition allows us to exploit the intro-enumerability property of e -pointed trees given in Theorem 4.4.

Theorem 5.8. *If A is cototal, then $A \equiv_e \mathcal{L}(X)$ for some X a minimal subshift on 2^ω .*

Proof. Given A cototal, let $T \in \text{deg}_e(A)$ be a uniformly e -pointed tree with functional W .

For each string $\sigma \in 2^{<\omega}$, the set $W_{|\sigma|}(\sigma)$ defines a subtree of $2^{\leq|\sigma|}$ given by the downward closure of $2^{\leq|\sigma|} \cap W_{|\sigma|}(\sigma)$, which we will denote by W^σ . Note that W^σ

are increasing in σ , i.e., $\sigma \succ \tau \Rightarrow W^\sigma \supseteq W^\tau$, and that without loss of generality the W^σ are closed downward, i.e., $\tau \in W^\sigma \Rightarrow W^\tau \subseteq W^\sigma$. We define levels L_i^σ inductively in i and σ .

For each σ , define $L_0^\sigma = \{0, 1\}$. Then L_{n+1}^σ consists of words of the form

$$AAAB(AB)^k AA(CDE\dots Z)B,$$

where $\{A, B, \dots, Z\} = \bigcup_{\tau \in W^\sigma} L_n^\tau$ with A, B, \dots, Z distinct, and k , thought of as an element of $2^{<\omega}$, is both in W^σ and has length $n + 1$.

We let

$$L_n = \bigcup_{\sigma \in T} L_n^\sigma$$

and define X_n to be the subshift generated by concatenations of the words in L_n . Let $X = \bigcap_{n < \omega} X_n$. This ends the construction. We now verify this X satisfies our conditions:

Claim. X is a subshift.

Each word in L_{n+1} is made up of concatenations of words in L_n , so we have that $X_{n+1} \subseteq X_n$. Hence X , being a nested intersection of closed, shift-invariant subsets of 2^ω , is itself closed and shift-invariant.

Notice that since $W^\sigma \subseteq 2^{\leq|\sigma|}$, and words are put in L_m^σ only to code for nodes in W^σ of length m , we have $L_m^\sigma = \emptyset$ for $m > |\sigma|$. Hence we can also write:

$$L_n = \bigcup_{\sigma \in T, |\sigma| \geq n} L_n^\sigma.$$

Claim. $T \geq_e \mathcal{L}(X)$.

By the construction, we have that $T \geq_e \bigcup L_n$. Then T enumerates the collection of all subwords of $\bigcup L_n$, which we denote by L . We claim that $L = \mathcal{L}(X)$.

First, $\mathcal{L}(X) \subseteq L$: suppose w is a subword of some point in X . Pick n large enough so that the length of w is less than any word in L_n . Then since w appears in X_n it appears in some concatenation of words in L_n , and by choice of n , w must then appear in a concatenation of at most two words in L_n , say AB . Let $m = s(n)$, where $s(n)$ is the function from Theorem 4.4. Any word in L_m we know is taken from some L_m^σ with $|\sigma| \geq m$. But since $|\sigma| \geq m$, by choice of m we have $W^\sigma \supseteq T \upharpoonright n$, so that $\bigcup_{\tau \in W^\sigma} L_n^\tau \supseteq L_n$. In particular, $\bigcup_{\tau \in W^\sigma} L_n^\tau$ contains A and B , so for example w appears in the word $AAAB(AB)^k AA(CDE\dots Z)B$ in L_m .

Secondly, $L \subseteq \mathcal{L}(X)$: given a subword w of some word in L_n^σ for $|\sigma| \geq n$, let $m = s(|\sigma|)$. Then every word in L_m contains w since $\sigma \in W^\tau$ for every $\tau \in T$ with $|\tau| \geq m$. Hence w appears in every point of X_m , so certainly in X .

Claim. $\mathcal{L}(X) \geq_e T$:

To see that $\mathcal{L}(X) \geq_e T$, we decode layers L_n inductively. Recall that $\mathcal{L}(X) = L$.

We describe one step in the decoding: given an enumeration of L , we know already that $L_0 = \{0, 1\}$ so we can identify words in L_1 by looking for subwords of the form $0001(01)^k 00(1)$ and $1110(10)^k 11()0$ in the words enumerated in L . Then we know to output the corresponding k into our enumeration of T , and moreover we eventually will know all the words in L_1 , so we can continue decoding L_2 in the same way.

That is, we proceed by induction. Our hypothesis is that by inductively searching for subwords of the appropriate form we will eventually enumerate all k at level n

of T and will have listed all of L_n . Then since words in L_{n+1} consist of words of the form $AAAB(AB)^k AA(CDE\dots Z)B$, for some elements $\{A, B, \dots, Z\} \subseteq L_n$, we can search for subwords of this form and thereby list all of L_{n+1} , and since all $k \in T$ at level $n + 1$ are enumerated into a word of this form in L_{n+1} , we will enumerate all of T up to level $n + 1$.

In this procedure, we must also see that we only enumerate elements that are actually in T and words that have actually been put in L_n . This also we verify by induction: since $\mathcal{L}(X) = L$, any w of the appropriate form that we do find is a subword of some concatenation of words in L_j . Let j be smallest such: i.e., w does not appear strictly within any of the words A, B, C, \dots, Z in L_j . Then to have seen w and accepted it as in L_j , we must have seen a subword made up of these letters and of the form $AAAB(AB)^k AA(CDE\dots Z)B$. In particular, we found w while searching for subwords of concatenations of L_j that begin with a word in L_j repeated three times. But there is no way in L_j to concatenate words of the correct form to obtain any new word of this form: the only way to even obtain a new subword of the form AAA is to concatenate a word ending in A , say $PPPA(PA)^k(QR..Z)A$ with a word starting in A , say $AAAV(AV)^j(WX..Z)V$ but then the only new sequence of three A s is followed by another A , not some distinct letter B , so it is not of the correct form either.

Claim. X is minimal.

Suppose p appears in some point of X . Then p appears in some word $w \in L_n$ so in some $w \in L_n^\sigma$ for some $\sigma \in T$.

Let $m = s(|\sigma|)$. Then for $|\tau| \geq m$, since $W^\tau \supseteq T \upharpoonright |\sigma| \ni \sigma$, we know w appears in L_{n+1}^τ . So taking $k \geq \max(m, n + 1)$ we see that w appears in every word in L_k . So all the points in X , being points in X_k , contain p .

Since p was arbitrary, all points of X contain the same subwords. \square

Theorem 5.6 makes our result of particular interest, as it allows us to find examples of degree spectra of minimal subshifts using known examples of cototal sets. We close with one such application:

Definition 5.9. The *co-spectrum* of a minimal subshift X is the collection of all lower-bounds of the degree spectra of X .

Gutteridge shows that there is a quasi-minimal cototal degree \mathbf{q} [10], that is, a cototal degree \mathbf{q} which is nonzero, and bounds no nonzero total e -degree. Taking X a minimal subshift with $\deg_e(\mathcal{L}(X)) = \mathbf{q}$, we obtain the following:

Theorem 5.10. *There exists a minimal subshift with no computable points, but whose co-spectrum is $\{\mathbf{0}\}$.*

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