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# REFINED INTERLACING PROPERTIES FOR ZEROS OF PARAORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. The purpose of this note is to extend in a simple and unified way the known results on interlacing of zeros of paraorthogonal polynomials on the unit circle. These polynomials can be regarded as the characteristic polynomials of any matrix similar to a unitary upper Hessenberg matrix with positive subdiagonal elements.

## 1. Introduction and main result

The study of zeros of *orthogonal polynomials on the real line* (hereafter abbreviated by OPRL) can be regarded as an eigenvalue problem for Jacobi matrices.[1] This allows us to go back to one of the most important single books in the nineteenth century, *Cours d'analyse de l'École royale polytechnique* (1821) by Cauchy to deduce, at least in the weak sense, the zero interlacing property of consecutive OPRL from the simplest form of the nowadays called Cauchy interlacing theorem. The search of more refined eigenvalue interlacing properties was probably initiated by Cauchy himself in his work *Sur l' Équation à l' Aide de Laquelle on Détermine les Inegalitées Séculaires des Mouvements des Planètes* (1829) and later continued by Wilkinson [45], Kahan [29], Golub [20], Hill and Parlett [26], and Bar-On [6], among others. In the same spirit, this work recovers one of the earliest approaches used to study zeros of *paraorthogonal polynomials on the unit circle* (hereafter abbreviated by POPUC), which is based on an eigenvalue problem for certain unitary matrices which bear many similarities to Jacobi matrices (cf. [1, 3, 7, 9–11, 16, 23, 25, 30, 31, 35, 36, 38–40, 44]).

Without wishing to delve into a historical discussion,[2] as far as we know, the POPUC[3] were introduced (in a somewhat hidden form) and successfully developed in a series of papers by Delsarte and Genin at the end of the 1980s [13, 15, 16] when

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[1] A symmetric tridiagonal matrix whose next-to-diagonal elements are positive (cf. [27, p. 36]).

[2] The weakened orthogonality condition that POPUC satisfy appeared in [13, Equation 4.10] as far as we can tell, while it is true that in Geronimus' 1944 paper [18, Theorem IV] such polynomials were presented.

[3] In [13], Delsarte and Genin called these polynomials (symmetric) *predictor polynomials* and its weakened orthogonality property *quasi-orthogonality*. In [14], they refer to these polynomials as *quasi-orthogonal polynomials on the unit circle*. This determination could also be supported by the fact that in 1946 Geronimus in regard to these polynomials wrote that they *"...play the*

they were working in signal processing. In [16], the authors focus on the problem of computing the zeros of POPUC regarded as an eigenvalue problem for a unitary upper Hessenberg matrix with positive subdiagonal elements. Elegant and recent proofs of most interlacing properties of zeros of POPUC shared with OPRL are due to Simon [39] (cf. [40, Theorem 2.14.4]) where the theory of rank one perturbations plays a central role. However, before such work (and the references therein) the zeros of POPUC were studied by the Linear Algebra community based on ideas close to those of Simon but supported on more elementary facts. Further analysis of these ideas will allow us to easily extend the known results. Indeed, our main purpose is to prove and improve, in connection with the works of Delsarte and Genin on the subject, the known zero interlacing properties of POPUC, based on the development of the ideas discussed by Arbenz and Golub in [4, Section 6].[4]

Here and below, we mainly follow the notation of [35, 36, 40]. Denote by $\mathbb{D}$ the open unit disk and by $\mathbb{S}^1$ its boundary, i.e.,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \, , \quad \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} \, .$$

Let $(a_0, \ldots, a_{n-1}, b_n)$ with $a_j \in \mathbb{D}$ $(j = 0, 1, \ldots, n-1)$ and $b_n \in \mathbb{S}^1$. Set

$$\Theta_j := \Theta(a_j), \quad \Theta(a) := \begin{pmatrix} \overline{a} & r \\ r & -a \end{pmatrix} , \quad r := \left(1 - |a|^2\right)^{1/2} .$$

Define the $(n+1)$-by-$(n+1)$ matrix

$$\mathcal{C} := \mathcal{L}\,\mathcal{M} \, , \tag{1}$$

where $\mathcal{L}$ and $\mathcal{M}$ are given explicitly by

$$\mathcal{L} := \begin{cases} \Theta_0 \oplus \Theta_2 \oplus \cdots \oplus \Theta_{n-2} \oplus \overline{b}_n & \text{if } n \text{ is even,} \\ \Theta_0 \oplus \Theta_2 \oplus \cdots \oplus \Theta_{n-1} & \text{if } n \text{ is odd} \, , \end{cases}$$

$$\mathcal{M} := \begin{cases} 1 \, \oplus \, \Theta_1 \oplus \Theta_3 \oplus \cdots \oplus \Theta_{n-1} & \text{if } n \text{ is even,} \\ 1 \, \oplus \, \Theta_1 \oplus \Theta_3 \oplus \cdots \oplus \Theta_{n-2} \oplus \overline{b}_n & \text{if } n \text{ is odd.} \end{cases}$$

Any unitary $(n+1)$-by-$(n+1)$ upper Hessenberg matrix with positive subdiagonal elements is uniquely parameterized by $2n+1$ real numbers that compose the parameters of the array $(a_0, \ldots, a_{n-1}, b_n)$ [22] (cf. [24] and [2, Proposition 1]). The resulting matrix after this process is referred to as the Schur parametric form of the original matrix. The factorization (1), which is unitarily similar to the Schur parametric form of an upper Hessenberg matrix with positive subdiagonal elements, was presented by Bunse-Gerstner and Elsner [9] (cf. [21, Section 12.2.10] and [7, Definition 3.3 and Lemma 3.4]). The explicit unitary pentadiagonal or double-staircase form of $\mathcal{C}$ (referred to as the *Doppel-Treppen-Matrix* in the original German source) was studied extensively by Bohnhorst [7]; see Figure 1 for an 8-by-8 example (cf. [7, Equation 3.9] and [30, Figure 1.1]). The matrix $\mathcal{C}$ becomes a very popular object in the Mathematical Physics and Orthogonal Polynomials communities after the work [11], in particular, after Simon's monographs [35, 36] where it was called (improper) CMV matrix (cf. [38, 40]).

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*same role here as the quasi-orthogonal polynomials of M. Riesz in the Hamburger problem."* (See [17, Remark I].) The expression POPUC was coined in [28].

[4]Such pioneering ideas were employed in the present context by Bohnhorst in her Ph.D. thesis [7] defended in 1993 at the Bielefeld University under the supervision of Elsner.

$$\begin{pmatrix}
\overline{a}_0 & r_0\overline{a}_1 & r_0r_1 & & & & & \\
r_0 & -a_0\overline{a}_1 & -a_0r_1 & & & & & \\
 & r_1\overline{a}_2 & -a_1\overline{a}_2 & r_2\overline{a}_3 & r_2r_3 & & & \\
 & r_1r_2 & -a_1r_2 & -a_2\overline{a}_3 & -a_2r_3 & & & \\
 & & & r_3\overline{a}_4 & -a_3\overline{a}_4 & r_4\overline{a}_5 & r_4r_5 & \\
 & & & r_3r_4 & -a_3r_4 & -a_4\overline{a}_5 & -a_4r_5 & \\
 & & & & & r_5\overline{a}_6 & -a_5\overline{a}_6 & r_6\overline{b}_7 \\
 & & & & & r_5r_6 & -a_5r_6 & -a_6\overline{b}_7
\end{pmatrix}$$

FIGURE 1. The matrix $\mathcal{C}$ for $n = 7$.

In order to make the notation more transparent, we write $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ instead of $\mathcal{C}$. We choose the representation (1) instead of their unitary similar upper Hessenberg matrix for a technical reason related to the manner in which Lemma 2.1 below is presented. In the next definition and subsequently, $\mathcal{I}$ denotes the identity matrix, whose order is made explicit or may be inferred from the context.

**Definition 1.1** (cf. [39, Proposition 3.2])**.** Let $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ be the matrix given by (1), where $a_j \in \mathbb{D}$ $(j = 0, 1, \dots, n-1)$ and $b_n \in \mathbb{S}^1$. The (monic) polynomial $P_{n+1}$ defined by

$$P_{n+1}(z) := \det\big(z\mathcal{I} - \mathcal{C}(a_0, \dots, a_{n-1}, b_n)\big)$$

is the POPUC of degree $n+1$ associated with the array $(a_0, \dots, a_{n-1}, b_n)$.

It is not difficult to see that the eigenvalues of $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ are simple. This fact was observed in 1944 by Geronimus [18, Theorem IV] (cf. [17, Theorem III], [19, Theorem 9.1] and [5, Theorem 7.2.2]) using the connection between POPUC and *orthogonal polynomials on the unit circle* (hereafter abbreviated by OPUC). Note that if $b_n$ were in $\mathbb{D}$, then the corresponding characteristic polynomial would be an OPUC and their zeros would be in $\mathbb{D}$. A remarkable property of the eigenvectors of $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ is the fact that all their components are nonzero (cf. [35, Chapter 4] and the references therein). This property is clearly valid also for the corresponding unitarily similar Hessenberg matrix.

**Definition 1.2.** Two finite subsets $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ $(1 \le m \le n)$ of $\mathbb{S}^1$ interlace (respectively, strictly interlace) whenever there exist $n-m$ points $\zeta_{m+1}, \zeta_{m+2}, \dots, \zeta_n \in \mathbb{S}^1$ such that any closed arc (respectively, open arc) on $\mathbb{S}^1$ connecting two distinct elements of $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ contains at least one element of $\{\xi_1, \xi_2, \dots, \xi_n\}$, and vice versa.

We can now formulate our main result.

**Theorem 1.1.** *Let $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ be a matrix given by (1), where $a_j \in \mathbb{D}$ $(j = 0, 1, \dots, n-1)$ and $b_n \in \mathbb{S}^1$. The following sentences hold:*

(i) *Let $\beta \in \mathbb{S}^1 \setminus \{1\}$ and define $\mathcal{C}_m^\beta := \mathcal{C}(a_0, \dots, a_{m-1}, \beta a_m, \dots, \beta a_{n-1}, \beta b_n)$ $(0 \le m < n)$ and $\mathcal{C}_n^\beta := \mathcal{C}(a_0, \dots, a_{n-1}, \beta b_n)$. Then the eigenvalues of $\mathcal{C}(a_0, \dots, a_{n-1}, b_n)$ and $\mathcal{C}_m^\beta$ strictly interlace on $\mathbb{S}^1$ for each $0 \le m \le n$.*

(ii) *For each* $0 \leq m < n$, *let* $b_m \in \mathbb{S}^1$. *For each* $\zeta \in \mathbb{S}^1$, *define recursively the numbers*[5]

$$b_n(\zeta) := b_n, \quad b_j(\zeta) := \frac{\overline{\zeta}\; a_j + b_{j+1}(\zeta)}{\overline{a}_j b_{j+1}(\zeta) + \overline{\zeta}} \quad (j = n-1, \ldots, 1, 0)\,. \tag{2}$$

*Set*[6]

$$A := C \cap \sigma(\mathcal{C}(a_0, \ldots, a_{m-1}, b_m)), \quad B := \sigma(\mathcal{N}) \setminus A\,,$$

*where* $C := \{\zeta \in \mathbb{S}^1 : b_m(\zeta) = b_m\}$, $\mathcal{N} := \mathcal{C}(a_{m+1}, \ldots, a_{n-1}, b_n)\,\mathcal{D}$ *with* $\mathcal{D} := \operatorname{diag}(d_m, \mathcal{I})$,[7] *and*

$$d_m := \frac{a_m - b_m}{\overline{a}_m b_m - 1}\,. \tag{3}$$

*Then* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *and* $\mathcal{C}(a_0, \ldots, a_{m-1}, b_m)$ *have at most* $\min\{m+1, n-m\}$ *common eigenvalues. More precisely,* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *and* $\mathcal{C}(a_0, \ldots, a_{m-1}, b_m)$ *have* $A$ *as the set of common eigenvalues,* $A$ *being also given by the alternative expression*

$$A = \sigma(\mathcal{N}) \cap \sigma(\mathcal{C}(a_0, \ldots, a_{m-1}, b_m))\,.$$

*Furthermore, the elements of the sets* $\sigma\big(\mathcal{C}\ (a_0, \ldots,\ a_{n-1}, b_n)\big) \setminus A$ *and* $\sigma\big(\mathcal{C}(a_0,\ \ldots, a_{m-1},\ b_m)\big) \cup B$ *strictly interlace on* $\mathbb{S}^1$.

Let $P_{n+1}$ be the POPUC of degree $n+1$ associated to the array $(0, \ldots, 0, 1)$. Since $\mathcal{C}(0, \ldots, 0, 1)$ is a permutation matrix, it follows that $P_{n+1}(z) = z^{n+1} - 1$. The sequence $(P_j)_{j \geq 1}$ (all of whose zeros are roots of unity) produce, by geometric intuition, illuminating examples that fall within Theorem 1.1.

**Example 1.1.** Let $P_3$ and $P_6$ be the POPUC associated to the arrays $(0,0,1)$ and $(0,0,0,0,0,1)$, respectively. In this situation,

$$\mathcal{C}(0,0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{C}(0,0,0,0,0,1) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and, therefore,

$$\sigma(\mathcal{C}(0,0,1)) = \left\{1, e^{\pm i2\pi/3}\right\}, \quad \sigma(\mathcal{C}(0,0,0,0,0,1)) = \left\{\pm 1, e^{\pm i2\pi/3}, e^{\pm i\pi/3}\right\}\,.$$

In the notation of Theorem 1.1 we have $n = 5$, $m = 2$, $b_j(\zeta) = \zeta^{5-j}$ $(0 \leq j \leq 5)$, $A = C = \sigma(\mathcal{C}(0,0,1))$, and $B = \emptyset$, where $A$ is obtained by using any of the expressions outlined in Theorem 1.1. Clearly, $\mathcal{C}(0,0,0,0,0,1)$ and $\mathcal{C}(0,0,1)$ have $A$ as the set of common eigenvalues and the elements of the sets $\sigma(\mathcal{C}(0,0,0,0,0,1)) \setminus A$ and $\sigma(\mathcal{C}(0,0,1))$ strictly interlace on $\mathbb{S}^1$, in concordance with sentence (ii) of Theorem 1.1.

[5]In [15] (cf. [16, Equation 2.6]), Delsarte and Genin have shown that if the $b_j(\zeta)$'s (known as *pseudo-reflection coefficients*) are given by (2), then the corresponding POPUC satisfy a three-term recurrence relation (cf. [12]). Bunse-Gerstner and He [10] have provided an illuminating discussion of the works of Delsarte and Genin on POPUC in matrix terms.

[6]$\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$.

[7]$\operatorname{diag}(d_m, \mathcal{I})$ denotes the block diagonal matrix $d_m \bigoplus \mathcal{I}$.

Regarding Theorem 1.1, as far as we know, sentence (i) for $m = 0$ was proved by Ammar, Gragg and Reichel [1, Proposition 4.2], although the particular case $\beta = -1$ is known since Geronimus' work [18, Theorem IV] (cf. [17, Theorem III]). The sentence (i) for $m = n$ was proved by Bohnhorst in [7, Theorem 3.19] (cf. [8, Theorem 3.5]). In [39, Theorem 3.4], Simon proved a weaker version of sentence (ii) that reads as follows: *Strictly between any pair of eigenvalues of* $\mathcal{C}(a_0, \ldots, a_{m-1}, b_m)$ *there is at least one eigenvalue of* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$.

**Corollary 1.1.** *Let* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *be a matrix given by* (1), *where* $a_j \in \mathbb{D}$ ($j = 0, 1, \ldots, n-1$) *and* $b_n \in \mathbb{S}^1$. *Let* $b_{n-1} \in \mathbb{S}^1$ *and define* $d_{n-1}$ *as in* (3) *for* $m = n-1$. *Then* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *and* $\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})$ *have at most one common eigenvalue. More precisely, either* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *and* $\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})$ *have* $\overline{b}_n d_{n-1}$ *as* (*only*) *common eigenvalue and the elements of* $\sigma\big(\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)\big) \setminus \{\overline{b}_n d_{n-1}\}$ *and* $\sigma\big(\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})\big)$ *strictly interlace on* $\mathbb{S}^1$, *or else* $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ *and* $\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})$ *have no common eigenvalues, and in such case* $\overline{b}_n d_{n-1}$ *is not an eigenvalue of either, and the elements of the sets* $\sigma\big(\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)\big)$ *and* $\sigma\big(\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})\big) \cup \{\overline{b}_n d_{n-1}\}$ *strictly interlace on* $\mathbb{S}^1$.

*Proof.* Take $m = n-1$ in Theorem 1.1. Hence, (2) and (3) yield $C = \{\overline{b}_n d_{n-1}\}$ which, in turn, is equal to $\sigma(\mathcal{N})$. Then either $A = C$ and $B = \emptyset$ if $\overline{b}_n d_{n-1} \in \sigma\big(\mathcal{C}(a_0, \ldots, a_{n-2}, b_{n-1})\big)$, or else $A = \emptyset$ and $B = C$ otherwise. The result follows immediately from sentence (ii) of Theorem 1.1. $\square$

Corollary 1.1 was proved by Bohnhorst [7, p. 48] (cf. [8, p. 819]) and rediscovered by Simon [39, Theorem 1.4]. It is worth noting that in view of Corollary 1.1 and besides the several and well-known practical consequences, POPUC answered the following open-ended question proposed by Turán as far back as 1974 [42, Problem LXVI, p. 60]: *"It is known that the zeros of the* $n$*th orthogonal polynomial* (*with respect to a Lebesgue-integral function on an interval*) *separate the zeros of the* $(n+1)$*th polynomial. What corresponds to this fact on the unit circle?"*.[8]

## 2. Proof of Theorem 1.1

### 2.1. Some preliminary lemmas.

Theorem 1.1 will be proved through the following sequence of lemmas.

**Lemma 2.1.** *Let* $\mathcal{U}$ *and* $\mathcal{S}$ *be unitary matrices of the same order and suppose that* $\operatorname{rank}(\mathcal{I} - \mathcal{S}) = 1$. *Then* $\mathcal{U}$ *and* $\mathcal{U}\mathcal{S}$ *have interlacing eigenvalues on* $\mathbb{S}^1$. *Moreover, assume that* $\mathcal{U}\mathcal{S}$ *admits a decomposition* $\mathcal{U}\mathcal{S} = \mathcal{U}_1 \oplus \mathcal{U}_2$, *and let* $\mathcal{U}$ *be partitioned as*

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix},$$

$\mathcal{U}_{11}$ *and* $\mathcal{U}_1$ *being of the same order. Set* $U_1 := \sigma(\mathcal{U}_1)$, $U_2 := \sigma(\mathcal{U}_2)$, *and* $U := \sigma(\mathcal{U})$. *Assume further that the eigenvalues of* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are simple and* $\sigma(\mathcal{U}_{11}) \cap U_1 = \sigma(\mathcal{U}_{22}) \cap U_2 = \emptyset$. *Then the elements of the sets* $U \setminus \big(U_1 \cap U_2\big)$ *and* $U_1 \cup \big(U_2 \setminus (U_1 \cap U_2)\big)$ *strictly interlace on* $\mathbb{S}^1$.

[8] We quote the English translation provided by Szüsz [43, Problem LXVI].

*Proof.* The first sentence of the lemma is the simplest form of a result due to Arbenz and Golub [4, Section 6] (cf. [7, Theorem 2.9] and [8, Theorem 2.7]).[9] In order to deduce the second one, we first claim that

$$U_1 \cap U_2 = U_1 \cap U \; = U_2 \cap U\,. \tag{4}$$

Indeed, since $\operatorname{rank}(\mathcal{US}-\mathcal{U})=1$, there exist nonzero vectors $u,v\in\mathbb{C}^n$ ($n$ being the common order of $\mathcal{U}$ and $\mathcal{S}$) such that $\mathcal{US}=\mathcal{U}+uv^T$. Using the formula for the determinant of a rank one perturbation (cf. [34, Proposition 3.21]), we may write for each $\zeta\in\mathbb{C}$ [10]

$$\chi_{\mathcal{U}}(\zeta)=\chi_{\mathcal{US}}(\zeta)+v^T\operatorname{Adj}(\zeta\mathcal{I}-\mathcal{US})u\,. \tag{5}$$

Let $\mathcal{US}=\mathcal{Z}\Lambda\mathcal{Z}^*$ be the spectral decomposition of $\mathcal{US}$ in which $\Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ and $\mathcal{Z}=(z_1\ldots z_n)$. Thompson-McEnteggert's formula for the adjugate [41] (cf. [33, Theorem 2.1]) gives

$$\operatorname{Adj}(\lambda_j\mathcal{I}-\mathcal{US})=\chi'_{\mathcal{US}}(\lambda_j)z_jz_j^*\,, \tag{6}$$

where the prime denotes the derivative. Combining (5) with (6) yields[11]

$$\chi_{\mathcal{U}}(\lambda_j)=\Big(\chi'_{\mathcal{U}_1}(\lambda_j)\chi_{\mathcal{U}_2}(\lambda_j)+\chi_{\mathcal{U}_1}(\lambda_j)\chi'_{\mathcal{U}_2}(\lambda_j)\Big)\;z_j^*uv^Tz_j\,. \tag{7}$$

We next claim that if $\lambda_j\in(U_1-U_2)\cup(U_2-U_1)$,[12] then $z_j^*uv^Tz_j\neq 0$. We only prove that $\lambda_j\in U_1-U_2$ implies $v^Tz_j\neq 0$. (To prove that $\lambda_j\in U_1-U_2$ implies $z_j^*u\neq 0$, we proceed similarly, as well as for proving that $\lambda_j\in U_2-U_1$ implies $z_j^*uv^Tz_j\neq 0$.) Indeed, suppose that $\lambda_j\in U_1-U_2$ and $v^Tz_j=0$. Since there is a normalized eigenvector $v_j$ of $\mathcal{U}_1$ associated with $\lambda_j$ such that $z_j=(v_j^T,0,\ldots,0)^T$, we deduce

$$\mathcal{U}_{11}\,v_j=\lambda_jv_j\,,$$

hence $\lambda_j\in\sigma(\mathcal{U}_{11})\cap U_1$, contrary to $\sigma(\mathcal{U}_{11})\cap U_1=\emptyset$. Consequently, (4) follows from (7). Finally, it follows from (4) that the sets $U\setminus\big(U_1\cap U_2\big)$ and $U_1\cup\big(U_2\setminus(U_1\cap U_2)\big)$ have no common elements, thus the second sentence of the lemma follows from the first one. $\square$

**Lemma 2.2.** *Let $\mathcal{U}$ be a unitary matrix and for a fixed $k$ let $\mathcal{S}$ be the diagonal matrix obtained from the identity matrix by replacing the $(k,k)$ entry with a number on $\mathbb{S}^1\setminus\{1\}$. Assume that $\mathcal{U}$ and $\mathcal{S}$ have the same order. Assume further that the eigenvalues of $\mathcal{U}$ are simple and all its eigenvectors have a nonzero component at the position $k$. Then $\mathcal{U}$ and $\mathcal{US}$ have strictly interlacing eigenvalues on $\mathbb{S}^1$.*

*Proof.* Without loss of generality we can assume that $k=1$, and so $\mathcal{S}=\operatorname{diag}(\beta,\mathcal{I})$ with $\beta\in\mathbb{S}^1\setminus\{1\}$. Let $\mathcal{U}=\mathcal{Z}\Lambda\mathcal{Z}^*$ be the spectral decomposition of $\mathcal{U}$ in which $\Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ and $\mathcal{Z}=(z_1\ldots z_n)$. Arguing as in the proof of Lemma 2.1 we have

$$\chi_{\mathcal{US}}(\lambda_j)=\chi'_{\mathcal{U}}(\lambda_j)\,z_j^*uv^Tz_j\,. \tag{8}$$

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[9] It can be deduced directly using [32, p. 222] and [27, Corollary 4.3.9].

[10] $\chi_{\mathcal{A}}$ denotes the characteristic polynomial of $\mathcal{A}$.

[11] The eigenvalue interlacing already stated implies $U_1\cap U_2\subseteq U$, and so $U_1\cap U_2\subseteq U_1\cap U$ and $U_1\cap U_2\subseteq U_2\cap U$.

[12] Given a set $E$ and $F,G\subseteq E$, we define $F-G:=F\cap(E\setminus G)$; if $G\subseteq F$, then $F-G=F\setminus G$.

Let $z_{j,1} \neq 0$ be the first component of the vector $z_j$. Then

$$z_j^* u v^T z_j = z_j^* \mathcal{U}(\mathcal{I} - \mathcal{S}) z_j = \lambda_j (1-\beta) \left| z_{j,1} \right|^2 \neq 0 .$$

Thus the result follows from (8) and the first sentence of Lemma 2.1. □

**Lemma 2.3.** *Let $a_j \in \mathbb{D}$ $(j = 0, 1, \ldots, n-1)$ and $b_n \in \mathbb{S}^1$. The following sentences hold:*

(i) *Let $\mathcal{S}$ be a diagonal matrix obtained from the $(n+1)$-by-$(n+1)$ identity matrix by replacing one of its diagonal entries with a number on $\mathbb{S}^1 \setminus \{1\}$. Then $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ and $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)\mathcal{S}$ have strictly interlacing eigenvalues on $\mathbb{S}^1$.*

(ii) *Let $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ be partitioned as*

$$\mathcal{C}(a_0, \ldots, a_{n-1}, b_n) = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} , \tag{9}$$

*$\mathcal{C}_{11}$ being the $(m+1)$-by-$(m+1)$ leading principal submatrix of $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$. Then, for each $0 \leq m < n$, $\mathcal{C}_{22}$ has no eigenvalues on $\mathbb{S}^1$.*

*Proof.* (i) The result follows directly from Lemma 2.2 and the fact that all the components of the eigenvectors of $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ are nonzero.

(ii) Assume that $m$ is even. Note that $\mathcal{C}_{22}$ is the $(n-m)$-by-$(n-m)$ trailing principal submatrix of each of the matrices $\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)$ and $\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)\,\mathcal{S}$, where $\mathcal{S} := \operatorname{diag}(\beta, \mathcal{I})$. Suppose the assertion (ii) is false. Since $\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)$ and $\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)\,\mathcal{S}$ are unitary matrices, these matrices share all the eigenvalues of $\mathcal{C}_{22}$ on $\mathbb{S}^1$, which contradicts sentence (i). If $m$ is odd, we argue in the same way noting that $\mathcal{C}_{22}^T$ is the $(n-m)$-by-$(n-m)$ trailing principal submatrix of each of the matrices $\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)$ and $\mathcal{S}\,\mathcal{C}(a_m, \ldots, a_{n-1}, b_n)$. □

**Lemma 2.4.** *Let $a_j \in \mathbb{D}$ $(j = 0, 1, \ldots, n-1)$ and $b_n \in \mathbb{S}^1$. Let $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ be partitioned as in (9), where $0 \leq m < n$. Let $b_m \in \mathbb{S}^1$ and define $b_m(\zeta)$ via (2) for each $\zeta \in \mathbb{S}^1$. Then $\mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ and $\mathcal{C}(a_0, \ldots, a_{m-1}, b_m)$ have at most* $\min\{m+1, n-m\}$ *common eigenvalues, which consist of the set of different solutions $\zeta$ of the equation $b_m(\zeta) = b_m$ on $\sigma(\mathcal{C}(a_0, \ldots, a_{m-1}, b_m))$.*

*Proof.* We begin by noting that

$$\det\big(\zeta\mathcal{I} - \mathcal{C}_n\big) = \det\big(\zeta\mathcal{I} - \mathcal{C}(a_0, \ldots, a_{m-1}, b_m(\zeta))\big) \det\big(\zeta\mathcal{I} - \mathcal{C}_{22}\big) \tag{10}$$

for each $\zeta \in \mathbb{S}^1$. Indeed, by sentence (ii) of Lemma 2.3, $\zeta\mathcal{I} - \mathcal{C}_{22}$ is nonsingular, hence (10) follows from the equality (cf. [7, Equation 3.41])

$$\mathcal{C}(a_0, \ldots, a_{m-1}, b_m(\zeta)) = \mathcal{C}_{11} - \mathcal{C}_{12}(\mathcal{C}_{22} - \zeta\mathcal{I})^{-1}\mathcal{C}_{21} ,$$

after taking into account the Schur complement of $\zeta\mathcal{I} - \mathcal{C}_{22}$ in $\zeta\mathcal{I} - \mathcal{C}(a_0, \ldots, a_{n-1}, b_n)$ is $\zeta\mathcal{I} - \big(\mathcal{C}_{11} - \mathcal{C}_{12}(\mathcal{C}_{22} - \zeta\mathcal{I})^{-1}\mathcal{C}_{21}\big)$. The result follows from (10) and the fact that for $\nu, \zeta \in \mathbb{S}^1$, with $\nu \neq \zeta$, $\mathcal{C}(a_0, \ldots, a_{m-1}, \nu)$ and $\mathcal{C}(a_0, \ldots, a_{m-1}, \zeta)$ have no common eigenvalues (see, e.g., [40, Theorem 2.14.4]; alternatively, apply sentence (i) of Lemma 2.3). □

2.2. **Proof of Theorem 1.1.**

(i) Let $\mathcal{S} := \operatorname{diag}(\mathcal{I}_m, \overline{\beta}, \mathcal{I}_{n-m})$, $\mathcal{D} := \operatorname{diag}(\mathcal{I}_m,\, \mathcal{J}^{\beta}_{n-m+1})$, and $\mathcal{V} := \operatorname{diag}(\mathcal{I}_{m+1},\, \mathcal{J}^{\beta}_{n-m})$, where $\mathcal{J}^{\beta}_{k} := \operatorname{diag}(\beta, 1, \beta, 1, \dots)$ is a $k$-by-$k$ diagonal matrix. Note that

$$\begin{pmatrix} \overline{\beta} & 0 \\ 0 & 1 \end{pmatrix} \Theta(a) \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \Theta(\beta a)\,. \tag{11}$$

Using (11) it is easily seen that[13]

$$\mathcal{D}^* \,\mathcal{C}(a_0, \dots, a_{n-1}, b_n)\, \mathcal{D}\, \mathcal{S} = \left(\mathcal{D}^* \,\mathcal{L}\, \mathcal{V}\right) \left(\mathcal{V}^* \,\mathcal{M}\, \mathcal{D}\, \mathcal{S}\right) = \mathcal{C}^{\beta}_{m}\,,$$

when $m$ is even. Similarly, the transpose of (11) leads to

$$\mathcal{S}\, \mathcal{D}\, \mathcal{C}(a_0, \dots, a_{n-1}, b_n)\, \mathcal{D}^* = \left(\mathcal{S}\, \mathcal{D}\, \mathcal{L}\, \mathcal{V}^*\right) \left(\mathcal{V}\, \mathcal{M}\, \mathcal{D}^*\right) = \mathcal{C}^{\beta}_{m}\,,$$

when $m$ is odd. The result follows from sentence (i) of Lemma 2.3.

(ii) Define the block diagonal matrix $S := \operatorname{diag}(\mathcal{I}_m, \mathcal{Z}, \mathcal{I}_{n-m-1})$, where

$$\mathcal{Z} = \Theta^*_m \begin{pmatrix} \overline{b}_m & 0 \\ 0 & d_m \end{pmatrix}.$$

Hence

$$\mathcal{C}(a_0, \dots, a_{m-1}, b_m) \,\oplus\, \mathcal{N} = \mathcal{C}(a_0, \dots, a_{n-1}, b_n)\, \mathcal{S}\,,$$

when $m$ is odd, and

$$\mathcal{C}(a_0, \dots, a_{m-1}, b_m) \,\oplus\, \mathcal{N}^T = \mathcal{S}^T\, \mathcal{C}(a_0, \dots, a_{n-1}, b_n)\,,$$

when $m$ is even. Note that $\mathcal{N}$ has simple eigenvalues (on $\mathbb{S}^1$) by sentence (i) of Lemma 2.3. The result follows from Lemma 2.1, sentence (ii) of Lemma 2.3, and Lemma 2.4.

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