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# A LOCALLY QUASI-CONVEX ABELIAN GROUP WITHOUT A MACKEY GROUP TOPOLOGY

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Abstract. We give the first example of a locally quasi-convex (even countable reflexive and $k_\omega$) abelian group $G$ which does not admit the strongest compatible locally quasi-convex group topology. Our group $G$ is the Graev free abelian group $A_G(\mathbf{s})$ over a convergent sequence $\mathbf{s}$.

## 1. Introduction

Let $(E,\tau)$ be a locally convex space. A locally convex vector topology $\nu$ on $E$ is called *compatible with* $\tau$ if the spaces $(E,\tau)$ and $(E,\nu)$ have the same topological dual space. Using the terminology from [4], the famous Mackey–Arens theorem states the following

**Theorem 1.1** (Mackey–Arens)**.** *Let $(E,\tau)$ be a locally convex space. Then $(E,\tau)$ is a pre-Mackey locally convex space in the sense that there is the finest locally convex vector space topology $\mu$ on $E$ compatible with $\tau$. Moreover, the topology $\mu$ is the topology of uniform convergence on absolutely convex weakly* compact subsets of the topological dual space $E'$ of $E$.*

The topology $\mu$ is called the *Mackey topology* on $E$ associated with $\tau$, and if $\mu=\tau$, the space $E$ is called a *Mackey space*.

For an abelian topological group $(G,\tau)$ we denote by $\widehat{G}$ the group of all continuous characters of $(G,\tau)$. Two topologies $\mu$ and $\nu$ on an abelian group $G$ are said to be *compatible* if $\widehat{(G,\mu)}=\widehat{(G,\nu)}$. Being motivated by the Mackey–Arens Theorem 1.1 the following notion was introduced and studied in [4] (for all relevant definitions see the next section):

**Definition 1.2** ([4])**.** A locally quasi-convex abelian group $(G,\mu)$ is called a *Mackey group* if for every locally quasi-convex group topology $\nu$ on $G$ compatible with $\tau$ it follows that $\nu\le\mu$. In this case the topology $\mu$ is called a *Mackey group topology* on $G$. A locally quasi-convex abelian group $(G,\tau)$ is called a *pre-Mackey group* and $\tau$ is called a *pre-Mackey group topology on* $G$ if there is a Mackey group topology $\mu$ on $G$ compatible with $\tau$.

Not every Mackey locally convex space is a Mackey group. Indeed, answering a question posed in [5], we proved in [7] that the metrizable locally convex space $(\mathbb{R}^{(\mathbb{N})},\mathfrak{p}_0)$ of all finite sequences with the topology $\mathfrak{p}_0$ induced from the product

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space $\mathbb{R}^{\mathbb{N}}$ is not a Mackey group. In [8] we show that the space $C_p(X)$, which is a Mackey space for every Tychonoff space $X$, is a Mackey group if and only if it is barrelled.

A weaker notion than the notion of a Mackey group was introduced in [7]. Let $(G,\tau)$ be a locally quasi-convex abelian group. A locally quasi-convex group topology $\mu$ on $G$ is called *quasi-Mackey* if $\mu$ is compatible with $\tau$ and there is no locally quasi-convex group topology $\nu$ on $G$ compatible with $\tau$ such that $\mu < \nu$. The group $(G,\tau)$ is *quasi-Mackey* if $\tau$ is a quasi-Mackey group topology. Proposition 2.6 of [7] states that every locally quasi-convex abelian group has quasi-Mackey group topologies.

The Mackey–Arens theorem suggests the following general question posed in [4]: *Is every locally quasi-convex abelian group a pre-Mackey group*? In other words, if $(G,\tau)$ is a locally quasi-convex group, is there a Mackey group topology compatible with $\tau$? We answer this question in the negative as stated in Theorem 1.3, the main result of this paper.

Let $\mathbf{s} = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be the convergent sequence endowed with the topology induced from $\mathbb{R}$. Denote by $A_G(\mathbf{s})$ the Graev free abelian topological group over $\mathbf{s}$. Note that the group $A_G(\mathbf{s})$ is a countable reflexive group ([6]) and is a $k_\omega$-group ([10]). In Question 4.4 of [7] we ask: *Is it true that $A_G(\mathbf{s})$ is a Mackey group*? Below we answer this question negatively in a stronger form.

**Theorem 1.3.** *The group $A_G(\mathbf{s})$ is neither a pre-Mackey group nor a quasi-Mackey group.*

This result gives the first example of a locally quasi-convex group which is not pre-Mackey and additionally shows an essential difference between the classes of locally quasi-convex groups and of locally convex spaces. For historical remarks and references on Mackey topology on locally quasi-convex groups see [11].

## 2. Proof of Theorem 1.3

Set $\mathbb{N} := \{1,2,\dots\}$. Denote by $\mathbb{S}$ the unit circle group and set $\mathbb{S}_+ := \{z \in \mathbb{S} : \operatorname{Re}(z) \geq 0\}$. Let $G$ be an abelian topological group. A character $\chi \in \widehat{G}$ is a continuous homomorphism from $G$ into $\mathbb{S}$. A subset $A$ of $G$ is called *quasi-convex* if for every $g \in G \setminus A$ there exists $\chi \in \widehat{G}$ such that $\chi(x) \notin \mathbb{S}_+$ and $\chi(A) \subseteq \mathbb{S}_+$. An abelian topological group $G$ is called *locally quasi-convex* if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. It is well known that the class of locally quasi-convex abelian groups is closed under taking products and subgroups. The dual group $\widehat{G}$ of $G$ endowed with the compact-open topology is denoted by $G^\wedge$. The homomorphism $\alpha_G : G \to G^{\wedge\wedge}$, $g \mapsto (\chi \mapsto \chi(g))$, is called *the canonical homomorphism.* If $\alpha_G$ is a topological isomorphism the group $G$ is called *reflexive.* Any reflexive group is locally quasi-convex; see for instance Proposition 1 of [3] and the comments after.

Let $X$ be a Tychonoff space with a distinguished point $e$. Following [10], an abelian topological group $A_G(X)$ is called *the Graev free abelian topological group* over $X$ if $A_G(X)$ satisfies the following conditions:

(i) $X$ is a subspace of $A_G(X)$;
(ii) any continuous map $f$ from $X$ into any abelian topological group $H$, sending $e$ to the identity of $H$, extends uniquely to a continuous homomorphism $\bar{f} : A_G(X) \to H$.

For every Tychonoff space $X$, the Graev free abelian topological group $A_G(X)$ exists, is unique up to isomorphism of abelian topological groups, and is independent of the choice of $e$ in $X$; see [10]. Further, $A_G(X)$ is algebraically the free abelian group on $X \setminus \{e\}$.

We denote by $\tau$ the topology of the group $A_G(\mathbf{s})$. For every $n \in \mathbb{N}$, set

$$e_n := (0, \dots, 0, 1, 0, \dots) \in \mathbb{Z}^{(\mathbb{N})},$$

where 1 is placed in position $n$ and $\mathbb{Z}^{(\mathbb{N})}$ is the direct sum $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Now the map $i(1/n) := e_n$, $n \in \mathbb{N}$, defines an algebraic isomorphism of $A_G(\mathbf{s})$ onto $\mathbb{Z}^{(\mathbb{N})}$. So we can algebraically identify $A_G(\mathbf{s})$ and $\mathbb{Z}^{(\mathbb{N})}$.

Let $g_n$ be a sequence in $A_G(\mathbf{s})$ of the form

$$g_n = (0, \dots, 0, r^n_{i_n}, r^n_{i_n+1}, r^n_{i_n+2}, \dots),$$

where $i_n \to \infty$ and there is a $C > 0$ such that $\sum_j |r^n_j| \le C$ for every $n \in \mathbb{N}$. Since $e_n \to 0$ in $\tau$ we obtain

$$g_n \to 0 \quad \text{in } \tau. \tag{2.1}$$

The following group plays an essential role in the proof of Theorem 1.3. Set

$$c_0(\mathbb{S}) := \{(z_n) \in \mathbb{S}^{\mathbb{N}} : \ z_n \to 1\},$$

and denote by $\mathfrak{F}_0(\mathbb{S})$ the group $c_0(\mathbb{S})$ endowed with the metric $d\big((z^1_n), (z^2_n)\big) = \sup\{|z^1_n - z^2_n|, n \in \mathbb{N}\}$. Then $\mathfrak{F}_0(\mathbb{S})$ is a Polish group, and the sets of the form $V^{\mathbb{N}} \cap c_0(\mathbb{S})$, where $V$ is an open neighborhood at the identity 1 of $\mathbb{S}$, form a base at 1 in $\mathfrak{F}_0(\mathbb{S})$. Actually $\mathfrak{F}_0(\mathbb{S})$ is isomorphic to $c_0/\mathbb{Z}^{(\mathbb{N})}$ (see [6]). In [6] we proved that the group $\mathfrak{F}_0(\mathbb{S})$ is reflexive and $\mathfrak{F}_0(\mathbb{S})^\wedge = A_G(\mathbf{s})$.

If $g$ is an element of an abelian group $G$, we denote by $\langle g \rangle$ the subgroup of $G$ generated by $g$. We need the following lemma.

**Lemma 2.1.** *Let $z, w \in \mathbb{S}$ and let $z$ have infinite order. Let $V$ be a neighborhood of 1 in $\mathbb{S}$. If $w^l = 1$ for every $l \in \mathbb{N}$ such that $z^l \in V$, then $w = 1$.*

*Proof.* The main result of [2] applied to $\langle z \rangle$ states the following: there exists a sequence $A = \{a_n\}_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that if $v \in \mathbb{S}$, then

$$\lim_n v^{a_n} = 1 \ \text{ if and only if } \ v \in \langle z \rangle.$$

Now suppose for a contradiction that $w \ne 1$. Since $\langle z \rangle$ is dense in $\mathbb{S}$, there is an $l \in \mathbb{N}$ such that $z^l \in V$. So $w$ has finite order, say $q$. Observe that $w \notin \langle z \rangle$. Then, by assumption, for every $l \in \mathbb{N}$ such that $z^l \in V$ we have $w^l = 1$, and hence there is a $c(l) \in \mathbb{N}$ such that $l = c(l) \cdot q$. Since $\lim_n z^{a_n} = 1$, there exists an $N \in \mathbb{N}$ such that $z^{a_n} \in V$ for every $n > N$. So $a_n = c(a_n) \cdot q$ for every $n > N$. But in this case we trivially have $\lim_n w^{a_n} = 1$ which contradicts the choice of the sequence $A$ since $w \notin \langle z \rangle$. Thus $w = 1$. □

In the proof of Theorem 1.3 we use the following result; see Proposition 3.11 of [4] or Theorem 2.7 of [7].

**Theorem 2.2** ([4,7])**.** *For a locally quasi-convex abelian group $(G, \tau)$ the following assertions are equivalent:*

(i) *the group $(G, \tau)$ is pre-Mackey;*
(ii) *$\tau_1 \vee \tau_2$ is compatible with $\tau$ for every locally quasi-convex group topologies $\tau_1$ and $\tau_2$ on $G$ compatible with $\tau$.*

Now we are ready to prove Theorem 1.3.

*Proof of Theorem* 1.3. First we construct a family

$$\{\mathcal{T}_z : z \in \mathbb{S} \text{ has infinite order}\}$$

of locally quasi-convex group topologies on $\mathbb{Z}^{(\mathbb{N})}$ compatible with the topology $\tau$ of $A_G(\mathbf{s})$. To this end, we use the idea described in Proposition 4.1 of [7].

Let $z \in \mathbb{S}$ be of infinite order. For every $i \in \mathbb{N}$, set

$$\chi_i := (1, \dots, 1, z, 1, \dots) \in \mathfrak{F}_0(\mathbb{S}) = A_G(\mathbf{s})^\wedge,$$

where $z$ is placed in position $i$. For every $(n_k) \in A_G(\mathbf{s})$, it is clear that $\chi_i\big((n_k)\big) = 1$ for all sufficiently large $i \in \mathbb{N}$ (i.e., $\chi_i \to 1$ in the pointwise topology on $\mathfrak{F}_0(\mathbb{S})$). So we can define the following algebraic monomorphism $T_z : \mathbb{Z}^{(\mathbb{N})} \to A_G(\mathbf{s}) \times \mathfrak{F}_0(\mathbb{S})$ by

$$T_z\big((n_k)\big) := \Big((n_k), \big(\chi_i\big((n_k)\big)\big)\Big) = \Big((n_k), \big(z^{n_k}\big)\Big) \quad \forall\ (n_k) \in \mathbb{Z}^{(\mathbb{N})}. \tag{2.2}$$

Denote by $\mathcal{T}_z$ the topology on $\mathbb{Z}^{(\mathbb{N})}$ which is the inverse image under the mapping $T_z$ of the product topology of $A_G(\mathbf{s}) \times \mathfrak{F}_0(\mathbb{S})$. It is a locally quasi-convex group topology.

*Claim* 1. The topology $\mathcal{T}_z$ is compatible with $\tau$.

Indeed, set $G := \big(\mathbb{Z}^{(\mathbb{N})}, \mathcal{T}_z\big)$. We must prove that $\widehat{G} = c_0(\mathbb{S})$. Since $\mathcal{T}_z$ is weaker than the discrete topology $\tau_d$ on $\mathbb{Z}^{(\mathbb{N})}$, we obtain $\widehat{G} \subseteq \widehat{\big(\mathbb{Z}^{(\mathbb{N})}, \tau_d\big)} = \mathbb{S}^{\mathbb{N}}$. Fix arbitrarily $\chi = (y_n) \in \widehat{G}$. To prove the claim we have to show that $y_n \to 1$.

Suppose for a contradiction that $y_n \not\to 1$. As $\mathbb{S}$ is compact we can find a sequence $0 < m_1 < m_2 < \dots$ of indices such that $y_{m_i} \to w \neq 1$ at $i \to \infty$. Since $\chi$ is $\mathcal{T}_z$-continuous, there exists a standard neighborhood $W = T_z^{-1}\big(U \times V^{\mathbb{N}}\big)$ of zero in $G$, where $U$ is a neighborhood of zero in $A_G(\mathbf{s})$ and $V$ is a neighborhood of 1 in $\mathbb{S}$, such that $\chi(W) \subseteq \mathbb{S}_+$. Observe that, by (2.2), $(n_k) \in W$ if and only if

$$(n_k) \in U \text{ and } z^{n_k} \in V \text{ for every } k \in \mathbb{N}, \tag{2.3}$$

and, the inclusion $\chi(W) \subseteq \mathbb{S}_+$ means that

$$\chi\big((n_k)\big) = \prod_k y_k^{n_k} \in \mathbb{S}_+ \text{ for every } (n_k) \in W. \tag{2.4}$$

We assume additionally that $w \notin V$. Set $L := \{l \in \mathbb{N} : z^l \in V\}$. Since $\langle z \rangle$ is dense in $\mathbb{S}$, the set $L$ is not empty. We distinguish between two cases.

*Case A* 1. Assume that $w^l = 1$ for every $l \in L$.

Then Lemma 2.1 implies $w = 1$. Since $w \neq 1$ we obtain that this case is impossible.

*Case B* 1. There is an $l_0 \in L$ such that $w^{l_0} \neq 1$.

Then there exists a $t \in \mathbb{N}$ such that $w^{l_0 t} \notin \mathbb{S}_+$. By (2.1), there is an $N(t) \in \mathbb{N}$ such that every $\mathbf{x}_i \in \mathbb{Z}^{(\mathbb{N})}$ of the form

$$\mathbf{x}_i = (0, \dots, 0, \underbrace{l_0}_{m_{i+1}}, 0, \dots, 0, \underbrace{l_0}_{m_{i+2}}, 0, \dots, 0, \underbrace{l_0}_{m_{i+t}}, 0, \dots) \tag{2.5}$$

belongs to $W$ for every $i \geq N(t)$. For every $\mathbf{x}_i \in W$ of the form (2.5), (2.4) implies

$$\chi\big(\mathbf{x}_i\big) = \big(y_{m_{i+1}} \cdots y_{m_{i+t}}\big)^{l_0} \to w^{l_0 t} \notin \mathbb{S}_+ \quad \text{at } i \to \infty. \tag{2.6}$$

Therefore, by (2.6), $\chi(W) \nsubseteq \mathbb{S}_+$, a contradiction.

Cases A and B show that our assumption that $y_n \not\to 1$ is wrong. Therefore, $y_n \to 1$ and $\widehat{G} \subseteq c_0(\mathbb{S})$. In order to prove the equality, observe that $\tau \le \mathcal{T}_z$. In fact, if $U$ is a neighborhood of zero in $A_G(\mathbf{s})$, we have $T_z^{-1}\big(U \times \mathfrak{F}_0(\mathbb{S})\big) = U$. So $U$ is also a zero neighborhood in $\mathcal{T}_z$. Therefore, $c_0(\mathbb{S}) \subseteq \widehat{G}$. Thus $\widehat{G} = c_0(\mathbb{S})$ and $\mathcal{T}_z$ is compatible with $\tau$.

*Claim* 2. For every element $a \in \mathbb{S} \setminus \{1\}$ of finite order, the topology $\mathcal{T}_z \vee \mathcal{T}_{az}$ is not compatible with $\tau$.

Indeed, let $r$ be the order of $a$ and set

$$D_r := r\mathbb{Z}^{(\mathbb{N})} = \left\{ (s_k \cdot r) \in \mathbb{Z}^{(\mathbb{N})} : (s_k) \in \mathbb{Z}^{(\mathbb{N})} \right\}.$$

Consider standard neighborhoods of zero

$$W_z = T_z^{-1}\big(U \times V^{\mathbb{N}}\big) \quad \text{and} \quad W_{az} = T_{az}^{-1}\big(U \times V^{\mathbb{N}}\big)$$

in $\mathcal{T}_z$ and $\mathcal{T}_{az}$, respectively, where $U \in \tau$ and $V$ is a symmetric neighborhood of 1 in $\mathbb{S}$ such that $V \cdot V \cap \langle a \rangle = \{1\}$. Then, by (2.3), we have

$$W_z \cap W_{az} = \left\{ (n_k) \in \mathbb{Z}^{(\mathbb{N})} : (n_k) \in U \text{ and } z^{n_k}, (az)^{n_k} \in V \text{ for every } k \in \mathbb{N} \right\}.$$

We show that $W_z \cap W_{az} \subseteq D_r$. Indeed, if $(n_k) \in W_z \cap W_{az}$, then $a^{n_k} \in V \cdot V$, and hence $a^{n_k} = 1$ for every $k \in \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$, there is an $s_k \in \mathbb{N}$ such that $n_k = s_k \cdot r$. Thus $W_z \cap W_{az} \subseteq D_r$.

Set $\eta := (a, a, \dots) \in \mathbb{S}^{\mathbb{N}}$. Then $\eta(W_z \cap W_{az}) \subseteq \eta(D_r) = \{1\}$. As $W_z \cap W_{az} \in \mathcal{T}_z \vee \mathcal{T}_{az}$ it follows that $\eta$ is $\mathcal{T}_z \vee \mathcal{T}_{az}$-continuous. Since $\eta \notin c_0(\mathbb{S})$ we obtain that $\mathcal{T}_z \vee \mathcal{T}_{az}$ is not compatible with $\tau$.

*Claim* 3. $\tau < \mathcal{T}_z$, so $\tau$ is not quasi-Mackey.

By (2.2), it is clear that $\tau \le \mathcal{T}_z$. To show that $\tau \ne \mathcal{T}_z$, suppose for a contradiction that $\mathcal{T}_z = \tau$. Then, by Claim 1, $\mathcal{T}_z \vee \mathcal{T}_{az} = \tau \vee \mathcal{T}_{az} = \mathcal{T}_{az}$ is compatible with $\tau$. But this contradicts Claim 2.

*Claim* 4. The group $A_G(\mathbf{s})$ is not pre-Mackey.

This immediately follows from Claim 2 and Theorem 2.2. □

We finish with the following question.

**Question 2.3.** Does there exist a locally convex space without a Mackey group topology? Is the free locally convex space $L(\mathbf{s})$ over $\mathbf{s}$ a pre-Mackey group?

Note that the space $L(\mathbf{s})$ is not a Mackey space; see [9].

*Remark* 2.4. Just before submission of the paper, Professor Lydia Außenhofer informed the author that she had also solved the problem posed: namely, whether $A_G(\mathbf{s})$ is a Mackey group and had proved Theorem 1.3; see [1]. It is worth mentioning that the author's proof totally differs from hers, being much simpler and shorter.

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