PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 8, August 2018, Pages 3367–3380
http://dx.doi.org/10.1090/proc/14027
Article electronically published on May 4, 2018

STRONGER ROLLE'S THEOREM FOR COMPLEX POLYNOMIALS

BLAGOVEST SENDOV AND HRISTO SENDOV

(Communicated by Jeremy Tyson)

Abstract. The following Rolle's Theorem for complex polynomials is proved. If $p(z)$ is a complex polynomial of degree $n \geq 5$, satisfying $p(-i) = p(i)$, then there is at least one critical point of p in the union $D[-c; r] \cup D[c; r]$ of two closed disks with centres $-c, c$ and radius r, where

$$c = \cot(2\pi/n), \quad r = 1/\sin(2\pi/n).$$

If $n = 3$, then the closed disk $D[0; 1/\sqrt{3}]$ has this property; and if $n = 4$, then the union of the closed disks $D[-1/3; 2/3] \cup D[1/3; 2/3]$ has this property. In the last two cases, the domains are minimal, with respect to inclusion, having this property.

This theorem is stronger than any other known Rolle's Theorem for complex polynomials.

1. Introduction

Denote by $\mathcal{C}$ the complex plane and let $\mathcal{C}^* := \mathcal{C} \cup \{\infty\}$. In this paper, *domain* is a closed subset of the complex plane $\mathcal{C}$ with non-empty interior. By $D[c; r]$ we denote the closed disk with centre $c \in \mathcal{C}$ and radius r. A point $\zeta \in \mathcal{C}$ is called *critical* for the polynomial $p(z)$ if $p'(\zeta) = 0$.

Definition 1.1. A domain Θ_n is called a *Rolle's domain* if every complex polynomial p of degree n, satisfying $p(i) = p(-i)$, has at least one critical point in it. A Rolle's domain Θ_n^X is *stronger* than the Rolle's domain Θ_n^Y if $\Theta_n^X \subsetneqq \Theta_n^Y$. A Rolle's domain Θ_n^X is *sharp* if Θ_n^X is minimal with respect to inclusion.

A result asserting that Θ_n is a Rolle's domain is called a *Rolle's Theorem.* There are several known Rolle's Theorems for complex polynomials. The most famous one follows from the following; see [1, p. 126].

Theorem 1.2 (Grace-Heawood). *Let $p(z)$ be a polynomial of degree $n \geq 2$. If $z_1, z_2 \in \mathcal{C}$ are any two distinct points at which $p(z)$ takes the same value, then the disk*

$$\left\{ z \in \mathcal{C} : \left| z - \frac{z_1 + z_2}{2} \right| \leq \left| \frac{z_1 - z_2}{2} \right| \cot(\pi/n) \right\}$$

contains at least one critical point of $p(z)$.

In particular, when $z_1 = i$ and $z_2 = -i$, we have the following corollary.

Received by the editors July 5, 2016, and, in revised form, April 17, 2017.

2010 *Mathematics Subject Classification.* Primary 30C10; Secondary 30E20.

Key words and phrases. Zeros and critical points of polynomials, apolarity, locus of a polynomial, locus holder, Coincidence Theorem, Argument Coincidence Theorem, Grace-Heawood Theorem, complex Rolle's Theorem, Sector Theorem.

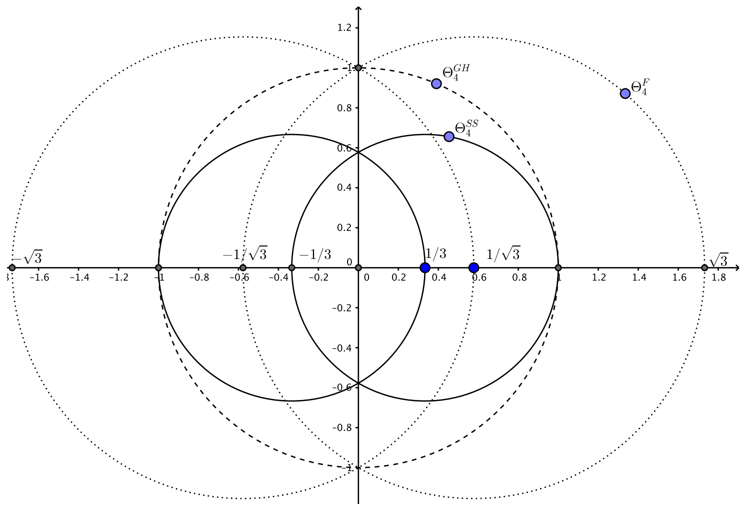

FIGURE 1. The Rolle's domains $\Theta_4^{GH}, \Theta_4^F$, and Θ_4^{SS}

Corollary 1.1 (Grace-Heawood)**.** *Let $p(z)$ be a polynomial of degree $n \geq 2$, satisfying $p(i) = p(-i)$. The disk*

$$\Theta_n^{GH} := D[0; \cot(\pi/n)] \tag{1.1}$$

is a Rolle's domain.

Another complex Rolle's Theorem, see [1, Theorem 4.3.4, p. 128], is the following.

Theorem 1.3. *Let $p(z)$ be a polynomial of degree $n \geq 3$, satisfying $p(i) = p(-i)$. The double disk $\Theta_n^F := D[-c; r] \cup D[c; r]$, where*

$$c = \cot(\pi/(n-1)), \quad r = 1/\sin(\pi/(n-1))$$

is a Rolle's domain.

Neither one of the above two domains is stronger than the other when $n \geq 5$. The main result of this paper is the following.

Theorem 1.4. *Let $p(z)$ be a polynomial of degree $n \geq 3$, satisfying $p(i) = p(-i)$.*

(a) *If $n = 3$, then $\Theta_3^{SS} := D[0; 1/\sqrt{3}]$ is a sharp Rolle's domain.*
(b) *If $n = 4$, then $\Theta_4^{SS} := D[-1/3; 2/3] \cup D[1/3; 2/3]$ is a sharp Rolle's domain.*
(c) *If $n \geq 5$, then $\Theta_n^{SS} := D[-c; r] \cup D[c; r]$, where*

$$c = \cot(2\pi/n), \quad r = 1/\sin(2\pi/n) \tag{1.2}$$

is a Rolle's domain.

It is easy to see that when $n = 3$, we have $\Theta_3^{SS} = \Theta_3^{GH} = D[0; 1/\sqrt{3}] \subset \Theta_3^F = D[0; 1]$. When $n = 4$, we have $\Theta_4^{SS} \subset \Theta_4^{GH} \subset \Theta_4^F$ as illustrated in Figure 1. Finally, when $n \geq 5$, we have $\Theta_n^{SS} \subset \Theta_n^{GH} \cap \Theta_n^F$ as illustrated in Figure 2 for the case $n = 12$. In the figures, the Rolle's domain Θ_n^{GH} is the area bounded by the dashed circle; Θ_n^F is the area bounded by the two dotted circles; and Θ_n^{SS} is the area bounded by the two solid circles.

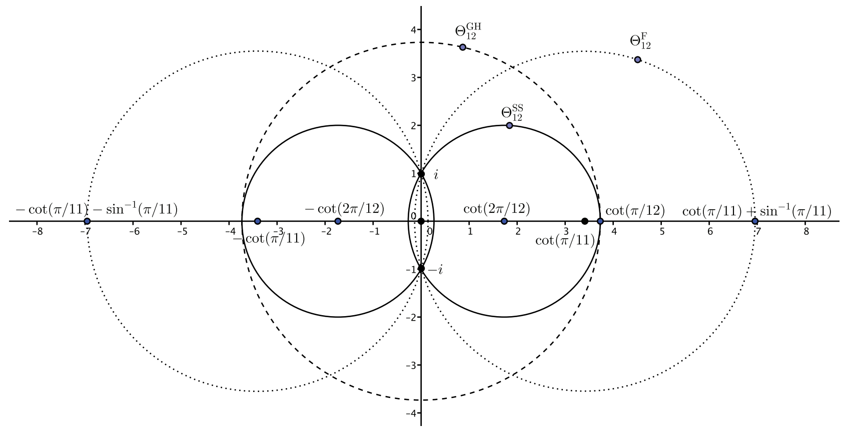

FIGURE 2. The Rolle's domains $\Theta_{12}^{GH}, \Theta_{12}^{F}$, and Θ_{12}^{SS}

The proof of Theorem 1.4 starts effectively with the statement of Theorem 5.2, where, in addition, we point out a whole family of sharp Rolle's domains in the cases $n = 3, 4$. The proof is based on three main ingredients: the notion of a *locus holder* introduced in [2]; on an analogue of the Grace-Walsh-Szegő Coincidence Theorem called the Argument Coincidence Theorem, proved in [4]; and on a non-convex analogue of the classical Gauss-Lucas Theorem, called the Sector Theorem, proved in [5]. In order to make the presentation as self-contained as possible, we present the necessary facts about locus holders and the above auxiliary results in Section 3.

We conclude this section with a restatement of the main result that matches that of the Grace-Heawood Theorem. Let $\mathcal{A}$ be a compact convex subset of $\mathcal{C}$. Denote by

$$P(\mathcal{A};\alpha) := \mathcal{A} \cup \left\{ z \in \mathcal{C} \setminus \mathcal{A} : \max_{a,b\in\mathcal{A}} \operatorname{Arg}\Big(\frac{a-z}{b-z}\Big) \geq \alpha \right\}$$

the *α-angular extension of $\mathcal{A}$*. For example, if $[z_1, z_2]$ denotes the segment between two complex numbers z_1 and z_2, then we have $\Theta_n^{SS} = P([i, -i]; 2\pi/n)$, whenever $n \geq 5$, and $\Theta_n^F = P([i, -i]; 2\pi/(n-1))$, while $\Theta_n^{GH} = P([\cot(\pi/n), -\cot(\pi/n)]; \pi/2)$. If $f(z) = az + b$, where $a \neq 0$, then $f\big(P([z_1, z_2]; \alpha)\big) = P(f([z_1, z_2]); \alpha)$; see Property (v) on page 122 in [1]. With this observation, the following corollary follows easily from Theorem 1.4.

Corollary 1.2. *Let $p(z)$ be a polynomial of degree $n \geq 3$. Let $z_1, z_2 \in \mathcal{C}$ be two distinct points at which $p(z)$ takes the same value.*

(a) *If $n = 3$ or 4, then $\mathcal{A}([z_1, z_2]; \pi/(n-1))$ contains at least one critical point of $p(z)$ and this is a minimal, by inclusion, set having this property.*
(b) *If $n \geq 5$, then $\mathcal{A}([z_1, z_2]; 2\pi/n)$ contains at least one critical point of $p(z)$.*

2. SOLUTIONS AND EXTENDED SOLUTIONS

Let P_n be the set of all complex polynomials of degree n:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \tag{2.1}$$

where $a_0, \ldots, a_n \in \mathcal{C}$ and $a_n \neq 0$. Let $\overline{\mathcal{P}}_n = \bigcup_{k=0}^n \mathcal{P}_k$ and for every $p(z) \in \overline{\mathcal{P}}_n$, consider its *polarization* or *symmetrization* with n variables:

$$P(z_1, z_2, \ldots, z_n) := \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, z_2, \ldots, z_n), \tag{2.2}$$

where

$$S_k(z_1, z_2, \ldots, z_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} z_{i_1} z_{i_2} \cdots z_{i_k}$$

is the elementary symmetric polynomial of degree $k = 1, 2, \ldots, n$, with

$$S_0(z_1, z_2, \ldots, z_n) := 1.$$

We say that an n-tuple $\{z_1, z_2, \ldots, z_n\}$ is a *solution* of $p(z)$ if $P(z_1, z_2, \ldots, z_n) = 0$. Clearly, an n-tuple $\{z, z, \ldots, z\}$ is a solution of p precisely when z is a zero of p.

Definition 2.1. A polynomial $q(z) \in \overline{\mathcal{P}}_n$, given by

$$q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

is called *apolar with* $p(z) \in \overline{\mathcal{P}}_n$ if

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = 0. \tag{2.3}$$

This definition allows the leading coefficients of $p(z)$ or $q(z)$ to be zero, unlike [1, Definition 3.3.1, p. 102]. It is easy to verify that the n-tuple $\{z_1, z_2, \ldots, z_n\} \subset \mathcal{C}$ is a solution of $p(z)$ if and only if the polynomial $q(z) = (z - z_1) \cdots (z - z_n)$ is apolar with $p(z)$. That observation allows us to extend the notion of a solution as follows.

Definition 2.2. Let $1 \leq m \leq n$. An m-tuple $\{z_1, z_2, \ldots, z_m\}$ is an *extended solution* of $p(z) \in \overline{\mathcal{P}}_n$ if the polynomial $q(z) = (z - z_1) \cdots (z - z_m)$ is apolar with $p(z)$.

In other words, an m-tuple $\{z_1, z_2, \ldots, z_m\}$ is an extended solution of $p(z)$ if

$$\sum_{k=0}^{m} \frac{a_{n-(m-k)}}{\binom{n}{n-(m-k)}} S_k(z_1, \ldots, z_m) = 0. \tag{2.4}$$

Clearly, when $m = n$ an extended solution is a solution. It is convenient to formally complete an extended solution with $n - m$ infinities: $\{z_1, z_2, \ldots, z_m, \infty, \ldots, \infty\}$ and think of it as an n-tuple in $\mathcal{C}^*$.

Definition 2.3. A sequence $\{z_{1,m}, z_{2,m}, \ldots, z_{n,m}\}_{m=1}^{\infty}$ of n-tuples in $\mathcal{C}^*$ *converges* to the n-tuple $\{z_1, z_2, \ldots, z_n\}$ if it is possible to order the elements of the n-tuples in the sequence, so that $\lim_{m \to \infty} z_{k,m} = z_k$ for all $k = 1, \ldots, n$.

Since $\mathcal{C}^*$ is a compact, it is not difficult to see that every sequence of n-tuples in $\mathcal{C}^*$ has a convergent subsequence. In general, the limit of a convergent sequence of n-tuples may be the n-tuple consisting of n infinities. But if the convergent sequence consists of extended solutions, then at least one component of the limit has to be finite, as the next lemma shows; see [4, Lemma 2.4].

Lemma 2.4. *If a sequence $\{Z_k\}$ of extended solutions of $p(z) \in \mathcal{P}_n$ converges to Z, then Z is an extended solution of $p(z)$. That is, at least one component of Z is finite.*

Extended solutions of $p(z)$ are solutions of its derivative of appropriate order, as the next lemma shows.

Lemma 2.5. *Let $p(z) \in \overline{\mathcal{P}}_n$ and let $1 \le m \le n$. If $\{z_1, \dots, z_m\} \subset \mathcal{C}$ is an extended solution of $p(z)$, then it is an extended solution of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$.*

The proof of Lemma 2.5 is a straightforward calculation using (2.4); see [4, Lemma 2.6]. It is analogous to the proof of Lemma 2.7 below. The important point is that $p^{(n-m)}(z)$ is considered a polynomial in $\overline{\mathcal{P}}_m$. The lemma is not true if $p^{(n-m)}(z)$ is considered a polynomial in $\overline{\mathcal{P}}_n$, as the following example shows.

Example 2.6. Let $p(z) = z^2 + z + 1 \in \overline{\mathcal{P}}_2$. The polynomial $q(z) = z + 1/2$ is apolar with $p(z)$, hence $\{-1/2\}$ is an extended solution of $p(z)$. Now, $\{-1/2\}$ is a solution of $p'(z) = 2z + 1 \in \overline{\mathcal{P}}_1$, in fact a zero. But if we consider $p'(z)$ as a polynomial in $\overline{\mathcal{P}}_2$, that is, $p'(z) = 0z^2 + 2z + 1$, then $p'(z)$ and $q(z)$ are not apolar as polynomials in $\overline{\mathcal{P}}_2$.

Lemma 2.7. *Let $p(z) \in \overline{\mathcal{P}}_n$ and let $1 \le \ell \le m \le n$. If $\{z_1, \dots, z_\ell\}$ is an extended solution of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$, then it is an extended solution of $p(z)$.*

Proof. If $a_0, \dots, a_n$ are the coefficients of $p(z)$, then the coefficients of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$ are

$$b_{m-s} := (n-s)\cdots(n-s-(n-m)+1)a_{n-s} \text{ for } s = 0, \dots, m.$$

Apply (2.4) with n replaced by m and m replaced by ℓ, to get that $\{z_1, \dots, z_\ell\}$ is an extended solution of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$ if and only if

$$\begin{aligned} 0 &= \sum_{k=0}^{\ell} \frac{b_{m-(\ell-k)}}{\binom{m}{m-(\ell-k)}} S_k(z_1, \dots, z_\ell) \\ &= \sum_{k=0}^{\ell} \frac{(n-(\ell-k))\cdots(n-(\ell-k)-(n-m)+1)a_{n-(\ell-k)}}{\binom{m}{m-(\ell-k)}} S_k(z_1, \dots, z_\ell) \\ &= \sum_{k=0}^{\ell} \frac{n(n-1)\cdots(n-(n-m)+1)a_{n-(\ell-k)}}{\binom{n}{n-(\ell-k)}} S_k(z_1, \dots, z_\ell) \\ &= n(n-1)\cdots(n-(n-m)+1) \sum_{k=0}^{\ell} \frac{a_{n-(\ell-k)}}{\binom{n}{n-(\ell-k)}} S_k(z_1, \dots, z_\ell). \end{aligned}$$

The last sum is just (2.4) applied to $\{z_1, \dots, z_\ell\}$ and $p(z) \in \overline{\mathcal{P}}_n$. □

3. Background and preliminary results

Definition 3.1. Let Ω be a closed subset of $\mathcal{C}^*$. We say that Ω is a *locus holder* of $p(z) \in \mathcal{P}_n$ if Ω contains at least one point from every solution of $p(z)$. A minimal by inclusion locus holder Ω is called a *locus* of $p(z)$.

Every locus holder contains a locus; see [2, Lemma 1.5].

Let $\mathcal{P}_n^+$ (resp., $\mathcal{P}_n^{++}$) be the set of all polynomials of degree n with non-negative (resp., positive) coefficients. Let $\mathcal{P}_n^+(\varphi)$ (resp., $\mathcal{P}_n^{++}(\varphi)$) be the set of all polynomials from $\mathcal{P}_n^+$ (resp., $\mathcal{P}_n^{++}$) with zeros in the sector

$$S(\varphi) := \{z : |\arg(z)| \ge \varphi, z \in \mathcal{C}\},$$

where $\varphi \in [0, \pi]$. Let $S^c(\varphi) := \mathcal{C} \setminus S(\varphi)$ and denote by $\bar{S}^c(\varphi)$ the closure of $S^c(\varphi)$. The following theorem is the main result in [5].

Theorem 3.2 (Sector Theorem). *If $p(z) \in \mathcal{P}_n^+(\varphi)$, then $p'(z) \in \mathcal{P}_{n-1}^+(\varphi)$.*

For $\varphi \in [\pi/2, \pi]$, Theorem 3.2 is a trivial application of the Gauss-Lucas Theorem, as in this case $S(\varphi)$ is a convex set. The first corollary of Theorem 3.2 is trivial.

Corollary 3.1. *If $p(z) \in \mathcal{P}_n^+(\varphi)$, then $p^{(k)}(z) \in \mathcal{P}_{n-k}^+(\varphi)$ for all $k = 0, 1, \dots, n-1$.*

In order to formulate the next corollary, and for further use, we need the notion of a polar derivative.

Definition 3.3. For a polynomial $p(z)$ of degree n, the linear operator

$$\mathcal{D}_u(p; z) := np(z) - (z - u)p'(z)$$

is called the *polar derivative of p with pole u.*

It is obvious that

$$\lim_{u \to \infty} \frac{1}{u} \mathcal{D}_u(p; z) = p'(z),$$

so one extends the notation to $\mathcal{D}_\infty(p; z) := p'(z)$. The polar derivative of order ℓ is defined recursively:

$$\mathcal{D}_{u_1,\dots,u_{\ell-1},u_\ell}(p; z) := \mathcal{D}_{u_\ell}\big(\mathcal{D}_{u_1,\dots,u_{\ell-1}}(p; z)\big). \tag{3.1}$$

If all the poles are equal and repeated k times we denote

$$\mathcal{D}_u^{(k)}(p; z) := \mathcal{D}_{u,\dots,u}(p; z).$$

Corollary 3.2. *If $p(z) \in \mathcal{P}_n^{++}(\varphi)$, then $\mathcal{D}_0^{(k)}(p; z) \in \mathcal{P}_{n-k}^{++}(\varphi)$.*

Proof. For a polynomial $p(z)$ of degree n, define the operation $\mathcal{L}_n[p(z)] := z^n p(1/z)$. A simple calculation shows that

$$\mathcal{L}_{n-1}\Big[\frac{d}{dz}\mathcal{L}_n[p(z)]\Big] = np(z) - zp'(z) = \mathcal{D}_0(p; z). \tag{3.2}$$

It is clear that $p(z) \in \mathcal{P}_n^{++}(\varphi)$ implies that $\mathcal{L}_n[p(z)] \in \mathcal{P}_n^{++}(\varphi)$. Then, by Theorem 3.2, $\frac{d}{dz}\mathcal{L}_n[p(z)] \in \mathcal{P}_{n-1}^{++}(\varphi)$. This together with (3.2) and an inductive argument leads to the result. $\square$

The next example shows that the degree of the polar derivative may drop more than expected and that is important for the theorem that follows.

Example 3.4. The actual degree of a polar derivative depends on the choice of the poles. Indeed, for the polynomial $p(z) = z^3 + z^2 + z + 1$ and a pole $u \in \mathcal{C}$, we have

$$D_u(p; z) = (3u + 1)z^2 + 2(u + 1)z + u + 3$$

and thus for the second polar derivative, we find:

$$D_{v,u}(p; z) = \begin{cases} 2(3uv + u + v + 1)z + 2(uv + u + v + 3) & \text{if } u \neq -1/3, \\ \frac{8}{3} + \frac{4}{3}v & \text{if } u = -1/3. \end{cases}$$

We need Theorem 5.2 from [3], that we now state.

Theorem 3.5. *Let p be a polynomial of degree n and let $u_1, \dots, u_k \in \mathcal{C}$. If the degree of $\mathcal{D}_{u_1,\dots,u_{k-1}}(p; z)$ is $n-(k-1)$, then*

$$\mathcal{D}_{u_1,\dots,u_k}(p; z) = \frac{n!}{(n-k)!} P(u_1, \dots, u_k, z, \dots, z), \tag{3.3}$$

where P is the symmetrization of p with n variables and $k \in \{1, \dots, n\}$.

If two multi-affine symmetric polynomials $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ satisfy $P(z, \dots, z) = Q(z, \dots, z)$ for all $z \in \mathcal{C}$, then they are equal. Theorem 3.5 says that, under its conditions, the symmetrization of $\mathcal{D}_{u_1,\dots,u_k}(p; z)$ with $n-k$ variables is

$$\frac{n!}{(n-k)!} P(u_1, \dots, u_k, z_1, \dots, z_{n-k}).$$

Applying inductively (3.2) shows that if $p(z) \in \mathcal{P}_n^{++}$, then the degree of $\mathcal{D}_0^{(k)}(p; z)$ is $n-k$. Denote by $P^{(m)}$ the symmetrization of $p^{(m)}$ with $n-m$ variables.

Lemma 3.6. *For any $p(z) \in \mathcal{P}_n$, we have*

$$\frac{\partial}{\partial z_{n-m}} P^{(m)}(z_1, \dots, z_{n-m}) = \frac{1}{n-m} P^{(m+1)}(z_1, \dots, z_{n-m-1}).$$

Proof. We prove the result for $m = 0$ only, since the general case is analogous:

$$\frac{\partial}{\partial z_n} P(z_1, \dots, z_n) = \frac{\partial}{\partial z_n} \Big(\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, \dots, z_n) \Big) = \sum_{k=1}^{n} \frac{a_k}{\binom{n}{k}} S_{k-1}(z_1, \dots, z_{n-1}).$$

Letting $z_1 = z_2 = \cdots = z_{n-1} =: z$, the last expression becomes

$$\sum_{k=1}^{n} \frac{a_k}{\binom{n}{k}} \binom{n-1}{k-1} z^{k-1} = \sum_{k=1}^{n} k \frac{a_k}{n} z^{k-1} = \frac{1}{n} \sum_{k=1}^{n} k a_k z^{k-1} = \frac{1}{n} p'(z).$$

The result follows from here. □

Finally, we need the main results from [4]. For $\alpha, \beta \in [-\pi, \pi]$, with $\alpha \le \beta$, and $u \in \mathcal{C}$, define the sector

$$S_u(\alpha, \beta) := \big\{ u + re^{i\varphi} : r \ge 0,\ \alpha \le \varphi \le \beta \big\}.$$

Definition 3.7. Sector $S_u(\alpha, \beta)$ is called a *zero-free sector* for the polynomial $p(z) \in \mathcal{P}_n$ if it does not contain a zero of $p^{(k)}(z)$ for all $k \in \{0, 1, \dots, n-1\}$.

Theorem 3.8 (Argument Coincidence)**.** *Let $S_u(\alpha, \beta)$ be a zero-free sector for $p(z) \in \mathcal{P}_n$ with $\beta - \alpha < \pi$. Suppose there is a solution of $p(z)$ in $S_u(\alpha, \beta)$. Then, there exists an extended solution of $p(z)$ of the form*

$$\big\{ u + s_1 e^{i\psi}, u + s_2 e^{i\psi}, \dots, u + s_m e^{i\psi} \big\} \tag{3.4}$$

for some $\psi \in [\alpha, \beta]$, where $s_k \ge 0$ for all $k = 1, 2, \dots, m$, $2 \le m \le n$, and at least one of $\{s_1, s_2, \dots, s_m\}$ is strictly positive.

Theorem 3.8 is reminiscent of the classical Grace-Walsh-Szegő Coincidence Theorem; see [1]. A circular domain, open or closed, is the interior or exterior of a circle, or a half-plane determined by a line in the complex plane.

Theorem 3.9 (Grace-Walsh-Szegő Coincidence). *Let $P(z_1, \dots, z_n)$ be a multi-affine symmetric polynomial. If the degree of P is n, then every circular domain containing the points $z_1, \dots, z_n$ contains at least one point z such that $P(z_1, \dots, z_n) = P(z, \dots, z)$. If the degree of P is less than n, then the same conclusion holds, provided the circular domain is convex.*

Finally, an application of the Grace-Walsh-Szegő Coincidence Theorem leads to the next needed result; see [2, Theorem 1.7].

Theorem 3.10. *Let p be a polynomial of degree n with at least two distinct zeros. If all zeros of p are on the boundary of a closed circular domain B, then B is a locus of p.*

4. The main result

This section is devoted to the proof of the following result, of which the stronger Rolle's Theorem is a corollary.

Theorem 4.1. *Suppose $\varphi \in [0, \pi/2)$. If $p(z) \in P_n^{++}(\varphi)$, then $S(\varphi)$ is a locus holder for $p(z)$.*

Proof. The theorem is trivial if $\varphi = 0$, so assume $\varphi > 0$. The proof is by induction on n. It is trivially true for $n = 1$ since the only solution of $az + b = 0$ is on the negative real axis. As part of the base case we also consider the case $n = 2$.

Claim 1. The theorem holds for $n = 2$.

Proof. The polynomial $p(z)$ either has one pair of complex conjugate zeros, say $re^{\pm i\theta}$, in the left-half of the complex plane, or has two real negative zeros. In the first case, consider the disk with $re^{\pm i\theta}$ on its boundary and contained in the sector $S(\theta)$ (that is, the disk tangent to the boundary of the sector). By Theorem 3.10, the disk is a locus, making $S(\theta)$ a locus holder of $p(z)$.

In the case when $p(z)$ has two negative zeros, a similar argument shows that the left-half of the complex plane, $S(\pi/2)$, is a locus holder. (If the two zeros coincide, then that point is itself a locus.)

Suppose $n \geq 3$ and the theorem holds for polynomials of degree up to $n - 1$. Suppose $S(\varphi)$ is not a locus holder for $p(z)$, that is, there is a solution $\{z_1, \dots, z_n\} \subset S^c(\varphi)$. Since $S^c(\varphi)$ is an open set, we may assume that $\{z_1, \dots, z_n\} \subset S^c(\varphi - \epsilon)$ for some $\epsilon > 0$. The next three claims are needed in a wider generality. So, suppose

$$\{z_1, \dots, z_m\} \subset S^c(\varphi - \epsilon) \tag{4.1}$$

is an extended solution of $p(z)$, where $1 \leq m \leq n$. By Lemma 2.5, (4.1) is an extended solution of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$ and since the degree of $p^{(n-m)}(z)$ is m, is its solution.

Claim 2. Extended solution (4.1) satisfies $z_\ell \neq 0$ for all $\ell = 1, 2, \dots, m$.

Proof. If $z_\ell = 0$ for all $\ell = 1, \dots, m$, then the polynomial z^m is apolar with $p(z)$ which is impossible, since $p(z)$ has positive coefficients. Suppose that $z_\ell = 0$ for $\ell = 1, \dots, k$ and $z_\ell > 0$ for $\ell = k + 1, \dots, m$. Applying (3.3), we obtain

$$\mathcal{D}_0^{(k)}(p^{(n-m)}; z) = \frac{m!}{(m-k)!} P^{(n-m)}(0, \dots, 0, z, \dots z), \tag{4.2}$$

where 0 is repeated k times and z is repeated $m-k$ times in the argument of $P^{(n-m)}$. This shows that $\{z_{k+1}, \ldots, z_m\}$ is a solution of $\mathcal{D}_0^{(k)}(p^{(n-m)}; z)$. Corollary 3.2 shows that $\mathcal{D}_0^{(k)}(p^{(n-m)}; z) \in P_{m-k}^{++}(\varphi)$ and by the induction hypothesis $S(\varphi)$ is a locus holder for $\mathcal{D}_0^{(k)}(p^{(n-m)}; z)$, so one of $\{z_{k+1}, \ldots, z_m\}$ is in $S(\varphi)$, a contradiction. □

Claim 3. Extended solution (4.1) contains at least two distinct (finite) points.

Proof. If $z_1 = \cdots = z_m$, then this common value is a zero of $p^{(n-m)}(z)$. But by Corollary 3.1 all zeros of $p^{(n-m)}(z)$ are in $S(\varphi)$, a contradiction. The same argument shows that $m \geq 2$. □

Consider the Möbius transformation $v = T(w)$, defined by

$$P^{(n-m)}(z_1, \ldots, z_{m-2}, v, w) = 0$$

after fixing the points $z_1, \ldots, z_{m-2}$.

Claim 4. The Möbius transformation T is non-degenerate.

Proof. Write $P^{(n-m)}(z_1, \ldots, z_{m-2}, v, w) = 0$ in the form

$$awv + b(w + v) + c = 0, \tag{4.3}$$

where the coefficients a, b, c are symmetric multi-affine functions of $z_1, \ldots, z_{m-2}$. Suppose the Möbius transformation T is degenerate: $b^2 - ac = 0$. By Lemma 3.6:

$$a = \frac{\partial^2}{\partial z_m \partial z_{m-1}} P^{(n-m)}(z_1, \ldots, z_m) = \frac{1}{m(m-1)} P^{(n-m+2)}(z_1, \ldots, z_{m-2}).$$

So, if $a = 0$, then $\{z_1, \ldots, z_{m-2}\}$ is a solution of $p^{(n-m+2)}(z)$. By Corollary 3.1, $p^{(n-m+2)}(z) \in P_{m-2}^{++}(\varphi)$ and by the induction hypothesis $S(\varphi)$ is a locus holder for it. Thus, one of $\{z_1, \ldots, z_{m-2}\}$ is in $S(\varphi)$, a contradiction.

Thus, we have $a \neq 0$ and equation (4.3) holds with $v = -b/a$ and any $w \in \mathcal{C}$. But $av + b$ is the derivative of (4.3) with respect to w:

$$av + b = \frac{\partial}{\partial z_m} P^{(n-m)}(z_1, \ldots, z_m) = \frac{1}{n} P^{(n-m+1)}(z_1, \ldots, z_{m-1}),$$

where Lemma 3.6 was used. That is, $\{z_1, \ldots, z_{m-1}\}$ is a solution of $p^{(n-m+1)}(z)$. But $p^{(n-m+1)}(z) \in P_{m-1}^{++}(\varphi)$, so by the induction hypothesis, $S(\varphi)$ is a locus holder for it. Hence, one of $\{z_1, \ldots, z_{m-1}\}$ is in $S(\varphi)$, a contradiction.

Since $p(z) \in P_n^{++}(\varphi)$, Corollary 3.1 asserts that $\bar{S}^c(\varphi - \epsilon)$ is a zero-free sector. Since $\{z_1, \ldots, z_n\} \subset S^c(\varphi - \epsilon)$ is a solution of $p(z)$, Theorem 3.8 asserts the existence of an extended solution of $p(z)$ of the form

$$\left\{s_1 e^{i\psi}, s_2 e^{i\psi}, \ldots, s_m e^{i\psi}\right\} \subset S^c(\varphi - \epsilon) \tag{4.4}$$

for some $\psi \in [-(\varphi - \epsilon), (\varphi - \epsilon)]$, where $s_k \geq 0$ for all $k = 1, 2, \ldots, m$, $2 \leq m \leq n$, and at least one of $\{s_1, s_2, \ldots, s_m\}$ is strictly positive.

Claim 5. The set of all $\psi \in [-(\varphi - \epsilon), (\varphi - \epsilon)]$ for which an extended solution of $p(z)$ of the form (4.4) exists, is closed.

Proof. Take a sequence of arguments $\{\psi_\ell\} \subset [-(\varphi - \epsilon), (\varphi - \epsilon)]$ converging to ψ such that

$$\left\{s_1^\ell e^{i\psi_\ell}, s_2^\ell e^{i\psi_\ell}, \ldots, s_{m_\ell}^\ell e^{i\psi_\ell}\right\} \tag{4.5}$$

is an extended solution of $p(z)$, where $s_k^n \geq 0$ for all $k = 1, 2, \ldots, m_\ell$, $2 \leq m_\ell \leq n$, and at least one of $\{s_1^\ell, s_2^\ell, \ldots, s_{m_\ell}^\ell\}$ is strictly positive. By Lemma 2.4 and the comments before it, a subsequence of these extended solutions converges to an extended solution $\{s_1 e^{i\psi}, s_2 e^{i\psi}, \ldots, s_m e^{i\psi}\}$, where $m \geq 1$. Claims 2 and 3 show that $m \geq 2$ and that all of $s_1, s_2, \ldots, s_m$ are strictly positive. □

Choose an extended solution of the form (4.4) with the smallest possible non-negative argument ψ. Again, Claims 2 and 3 assert that $m \geq 2$ and that all of $s_1, \ldots, s_m$ are strictly positive.

If $\psi = 0$, then we reach a contradiction with the fact that $\{s_1, s_2, \ldots, s_m\}$ is a solution of $p^{(n-m)}(z)$, that is, $P^{(n-m)}(s_1, s_2, \ldots, s_m) = 0$. The latter is impossible since $p^{(n-m)}(z)$ has positive coefficients.

Suppose then that $\psi > 0$ and let $z_i := s_i e^{i\psi}$ for $i = 1, 2, \ldots, m$. Since at least two of the points in the solution (4.4) are distinct, assume $z_1 \neq z_2$. By Claim 2, z_1 and z_2 are not 0. Consider the Möbius transformation $v = T(w)$, defined by $P^{(n-m)}(v, w, z_3, \ldots, z_m) = 0$ after fixing the points $z_3, \ldots, z_\ell$. By Claim 4, T is non-degenerate, so let ζ_1, ζ_2 be its (distinct) fixed points.

Let C be the unique circle, called the *joint circle* of the pair z_1, z_2, that passes through the points $z_1, z_2, \zeta_1, \zeta_2$. Such a circle exists, since T is an involution, that is, $T(T(v)) = v$ for all $v \in \mathcal{C}$. The joint circle has the property that it is invariant under T and when v moves over C, $w = T(v)$ moves over C in the opposite direction until they meet over one of the fixed points ζ_1 or ζ_2. The fixed points ζ_1 and ζ_2 are on different arcs of C defined by v and w. Finally, if v is inside C, then $w = T(v)$ is outside of C, and vice versa.

We consider two cases.

Case 1. Suppose that the joint circle of some pair of distinct points in the solution (4.4), say z_1 and z_2, is a proper circle (i.e., not a straight line). Necessarily, T has one fixed point on each side of the line through the origin with argument ψ; see Figure 3. Fixing the points $z_3, \ldots, z_m$, move slightly z_1 along the circle C,

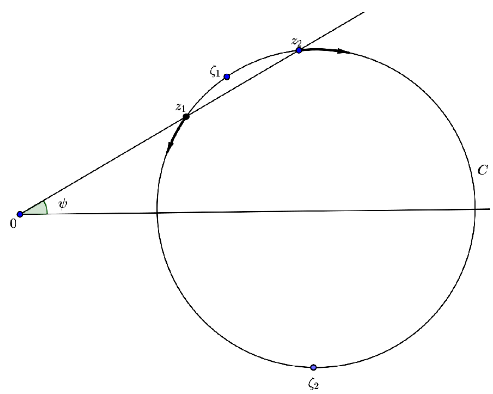

FIGURE 3. Illustrating the last part of the proof of Theorem 4.1

diminishing its argument, causing the point z_2 to move along C in the opposite direction, also diminishing its argument.

Next, fix point z_2 and consider the pair z_1, z_3 and the Möbius transformation defined by $P^{(n-m)}(v, z_2, w, z_4, \ldots, z_m) = 0$. Since z_1 is in the interior of the sector

$S_0(0,\psi)$, we can perturb it, while keeping it in the interior, so that z_3 also enters the interior of that sector. Doing so with all points of the extended solution (4.4), we produce a new one inside the sector $S_0(0,\psi-\delta)$ for some $\delta>0$. Sector $S_0(0,\psi-\delta)$ is a zero-free sector for $p^{(n-m)}(z)$ and the new extended solution is in fact a solution of $p^{(n-m)}(z)$. According to Theorem 3.8, there is an extended solution of $p^{(n-m)}(z)$ of the form

$$\{s_1e^{i\psi_\circ}, s_2e^{i\psi_\circ}, \dots, s_\ell e^{i\psi_\circ}\} \subset S_0(0,\psi-\delta), \tag{4.6}$$

where $s_k \geq 0$ for all $k=1,2,\dots,\ell$, $2\leq \ell \leq m$, and at least one of $\{s_1,s_2,\dots,s_\ell\}$ is strictly positive. By Lemma 2.7, (4.6) is an extended solution of $p(z)$. This contradicts the minimal choice of ψ and completes the proof in this case.

Case 2. Suppose that the joint circle of any pair of distinct points in the solution (4.4) is a straight line, necessarily the line $\{te^{i\psi} : t\in\mathbb{R}\}$. In fact, we may assume that every extended solution of the form (4.4) has this property, or else we apply Case 1. Take any two distinct points, say z_i and z_j, from (4.4). One of their corresponding fixed points, say ζ, is strictly between them, that is, on the open segment (z_i,z_j). Thus, in the solution (4.4) we may replace the pair $\{z_i,z_j\}$ with the pair $\{\zeta,\zeta\}$ and keep the property that the m-tuple is an extended solution. Continuing in this way, trying to minimize the diameter of the solution $\max\{|z_i-z_j| : 1\leq i<j\leq m\}$, one can see that we end up with a solution of the form (4.4) in which all points coincide. This contradicts Claim 3 and completes the proof in this case. □

5. Stronger Rolle's domain

Let p be a complex polynomial of degree n satisfying $p(i)=p(-i)$. Let

$$p'(z) = (z-z_1)(z-z_2)\cdots(z-z_{n-1}) = \sum_{k=0}^{n-1}(-1)^{n-1-k}S_{n-1-k}(z_1,\dots,z_{n-1})z^k.$$

Integrating the last expression, one sees that condition $p(i)=p(-i)$ is equivalent to

$$\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^{n-1-k}}{2k+1}S_{n-1-2k}(z_1,\dots,z_{n-1})=0.$$

Observe that the left-hand side of the last equality is the symmetrization of the polynomial

$$\begin{aligned}\kappa_n(z) &:= \sum_{k=0}^{[(n-1)/2]}\frac{(-1)^{n-1-k}}{2k+1}\binom{n-1}{2k}z^{n-1-2k}\\ &= \frac{(-1)^{n-1}}{n}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}z^{n-(2k+1)}\\ &= \frac{(-1)^n}{2ni}\big((z-i)^n-(z+i)^n\big).\end{aligned}$$

Note that the degree of $\kappa_n(z)$ is $n-1$. Hence, $\{z_1,\dots,z_{n-1}\}$ is a solution of κ_n if and only if $z_1,\dots,z_{n-1}$ are the zeros of the derivative of a polynomial p of degree n satisfying $p(i)=p(-i)$. We point out the known fact, that $\kappa_n(z)$ has only real zeros $\{\cot(k\pi/n) : k=1,\dots,n-1\}$.

We mention in passing that combining this fact with the classical theorem of Grace leads to a proof of Corollary 1.1, where the classical Grace's Theorem (see [1, p. 107]) states the following.

Theorem 5.1 (Grace's Theorem). *Let p and q be apolar polynomials of degree n. Then every circular domain containing all the zeros of one of them contains at least one zero of the other.*

Indeed, in our terminology, Theorem 5.1 says that every circular domain containing all the zeros of p is a locus holder for p. That is, the disk $D[0; \cot(\pi/n)]$ is a locus holder for $\kappa_n(z)$, hence a Rolle's domain for complex polynomials of degree n. Thus, Definition 3.1 implies a trivial refinement of the Grace-Heawood's Theorem.

Theorem 5.2. *Every locus of κ_n is a sharp Rolle's domain for complex polynomials of degree n.*

Since $\kappa_n(z)$ has only real zeros, Theorem 3.10 implies that the closed upper (resp., lower) half-plane is a locus of $\kappa_n(z)$ and so it is a sharp Rolle's domain but trivial and unbounded. Of interest are bounded loci of $\kappa_n(z)$ especially those symmetric with respect to the real axis.

For example, when $n = 3$, we have $\kappa_3(z) = (z - 1/\sqrt{3})(z + 1/\sqrt{3})$. By Theorem 3.10 any closed circular domain having $1/\sqrt{3}$ and $-1/\sqrt{3}$ on its boundary is a sharp Rolle's domain. The domain $\Theta_3^{GH} = D[0; 1/\sqrt{3}]$ is such and is, symmetric with respect to the real axis.

When $n = 4$, we have $\kappa_4(z) = -z(z^2 - 1)$. A family of loci of this polynomial is described in Section 6.2 of [2]. The representative of this family with the smallest area, that is, symmetric with respect to both coordinate axes, is the locus $D[-1/3; 2/3] \cup D[1/3; 2/3]$. Another curious locus of $\kappa_4(z)$, also contained in the unit disk but this time symmetric only with respect to the imaginary axis, is given in [2, Theorem 5.1].

Finding a bounded locus of $\kappa_n(z)$ for $n \geq 5$, especially one symmetric with respect to the real axis, appears to be difficult. Instead, we find a locus holder of $\kappa_n(z)$ leading to a Rolle domain stronger than the known ones.

We recall a few final facts. Let

$$T(z) = (az + b)/(cz + d) \quad \text{with } ad - bc \neq 0 \tag{5.1}$$

be a non-degenerate Möbius transformation. For every polynomial $p \in \overline{\mathcal{P}}_n$ define

$$T[p](z) := (cz + d)^n p(T(z)).$$

If $U(z) = (ez + f)/(gz + h)$ is another Möbius transformation, then for every polynomial $p \in \overline{\mathcal{P}}_n$ we have

$$U[T[p]](z) = (gz + h)^n (cU(z) + d)^n p(T(U(z))) = (T \circ U)[p](z).$$

Theorem 2.5 in [2] explains the relationship between the loci of p and $T[p]$.

Theorem 5.3. *Suppose that p and $T[p]$ are polynomials of degree n with at least two distinct zeros. Then Ω is a locus of p if and only if $T^{-1}(\Omega)$ is a locus of $T[p]$.*

Since every locus holder contains a locus (see Lemma 1.5 in [2]), it is easy to see that Theorem 5.3 remains true if the word 'locus' is replaced by 'locus holder'.

Proof of Theorem 1.4. The cases $n = 3, 4$ were discussed above, so assume $n \geq 5$. Consider the Möbius transformation

$$T(z) = i\frac{z+1}{z-1} \quad \text{with} \quad T^{-1}(z) = \frac{z+i}{z-i}.$$

We have

$$\begin{aligned} T[\kappa_n](z) &= \frac{(-1)^n}{2ni}(z-1)^{n-1}\Big(\Big(i\frac{z+1}{z-1} - i\Big)^n - \Big(i\frac{z+1}{z-1} + i\Big)^n\Big) \\ &= \frac{(-1)^n}{2ni}(z-1)^{n-1}\Big(\Big(\frac{2i}{z-1}\Big)^n - \Big(\frac{2iz}{z-1}\Big)^n\Big) \\ &= \frac{(-2i)^{n-1}}{n}\Big(\frac{z^n-1}{z-1}\Big) = \frac{(-2i)^{n-1}}{n} q_n(z), \end{aligned}$$

where

$$q_n(z) := z^{n-1} + z^n + \cdots + 1. \tag{5.2}$$

It is easy to see that

$$T(e^{2\varphi i}) = \cot(\varphi), \;\; T(e^{-2\varphi i}) = -\cot(\varphi), \;\; T(0) = -i, \;\; T(\infty) = i. \tag{5.3}$$

It is easy to see that $q_n(z) \in \mathcal{P}_n^{++}(2\pi/n)$. Since $n \geq 5$, we have $2\pi/n < \pi/2$ and can apply Theorem 4.1 to conclude that sector $S(2\pi/n)$ is a locus holder of $q_n(z)$. Theorem 5.3 then implies that $T(S(2\pi/n))$ is a locus holder of $\kappa_n(z)$. Since

$$T(e^{2\pi i/n}) = \cot(\pi/n), \;\; T(e^{-2\pi i/n}) = -\cot(\pi/n), \;\; T(0) = -i, \;\; T(\infty) = i, \tag{5.4}$$

in addition to $T(-1) = 0$, $T(z)$ maps sector $S(2\pi/n)$ onto the double disk $DD[c; r] = D[-c; r] \cup D[c; r]$, where the boundary of the disk $D[c; r]$ passes through the points $i, -i$, and $\cot(\pi/n)$. From here, one readily calculates the values of c and r given in (1.2). □

Acknowledgments

We are grateful to an anonymous referee for a thoughtful reading of this work and for pointing out the missing Case 2 at the end of the proof of Theorem 4.1.

The first author was partly supported by the Bulgarian National Science Fund under project FNI I 20/20 "Eficient Parallel Algorithms for Large-Scale Computational Problems". The second author was partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

[1] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press, Oxford University Press, Oxford, 2002. MR1954841

[2] Blagovest Sendov and Hristo Sendov, *Loci of complex polynomials, part I*, Trans. Amer. Math. Soc. **366** (2014), no. 10, 5155–5184. MR3240921

[3] Blagovest Sendov and Hristo Sendov, *Loci of complex polynomials, part II: polar derivatives*, Math. Proc. Cambridge Philos. Soc. **159** (2015), no. 2, 253–273. MR3395371

[4] Blagovest Sendov and Hristo Sendov, *Two Walsh-type theorems for the solutions of multi-affine symmetric polynomials*, Progress in approximation theory and applicable complex analysis, Springer Optim. Appl., vol. 117, Springer, Cham, 2017, pp. 145–162. MR3644743

[5] Blagovest Sendov and Hristo Sendov, *On the zeros and critical points of polynomials with nonnegative coefficients: a nonconvex analogue of the Gauss-Lucas theorem*, Constr. Approx. **46** (2017), no. 2, 305–317. MR3691231

Bulgarian Academy of Sciences, Institute of Information and Communication Technologies, Acad. G. Bonchev Str., bl. 25A, 1113 Sofia, Bulgaria
Email address: acad@sendov.com

Department of Statistical and Actuarial Sciences, Western University, 1151 Richmond Str., London, Ontario, N6A 5B7 Canada
Email address: hssendov@stats.uwo.ca