

## INTERNAL CHARACTERIZATIONS OF PRODUCTIVELY LINDELÖF SPACES

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ABSTRACT. We present an internal characterization for the productively Lindelöf property, thus answering a long-standing problem attributed to Tamano. We also present some results about the relation “Alster spaces” vs. “productively Lindelöf spaces”.

### 1. INTRODUCTION

We say that a topological space is Lindelöf if every open covering for it has a countable subcovering. We say that a Lindelöf space  $X$  is productively Lindelöf if  $X \times Y$  is Lindelöf for every Lindelöf space  $Y$ . This is a class that contains all  $\sigma$ -compact spaces, but as of yet we do not know much more about which other spaces it contains. For example, it is not known if the space of the irrationals is productively Lindelöf (although consistently it is not); see [5] and the references therein. This is the famous Michael’s problem.

As with the case of the irrationals, most of the known results about productively Lindelöf spaces use some kind of combinatorial hypothesis beyond ZFC (see, e.g., [1, 5]). So this property seems to have a set-theoretic nature.

The main result of the second section is Theorem 2.13 which gives an internal characterization of the class of productively Lindelöf spaces and thus solves a problem attributed to H. Tamano in [6]. The formulation of the property is combinatorial in the sense that it looks like a diagonalization property. Basically, it says that for a regular space  $X$ ,  $X$  is productively Lindelöf if, and only if, for every collection  $\mathcal{V}$  of open coverings of  $X$  that is “small enough”, there is a countable collection  $\mathcal{C}$  of open sets such that  $V \cap \mathcal{C}$  is still a covering for  $X$  for every  $V \in \mathcal{V}$ . Here it is important to stress that we are talking about an arbitrary collection of open coverings: It is not enough to use only open coverings made by elements of a fixed open base since then the conclusion in Theorem 2.13 for second countable spaces is simply trivial. However, following the proof presented here, one could use

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only sets of the form “basic open set minus two points”. So far, we were not able to use this characterization to solve the Michael’s problem.

In the third section we present some new results about the relation “Alster spaces” (defined below) vs. “productively Lindelöf spaces”. In particular, we obtain the original Alster’s result ([2]) as a corollary. This gives a simplification compared to the original proof. Alster’s result is probably the best known one regarding productively Lindelöf spaces in general. The result presented here has the advantage that the set-theoretic assumption in it is much weaker than CH.

Finally, in the fourth section, we present a new property which is (formally) in between Alster and productively Lindelöf ones. One of the applications of this property is that it makes clearer the relation between the properties of Alster and Hurewicz.

## 2. AN INTERNAL CHARACTERIZATION FOR PRODUCTIVELY LINDELÖF SPACES

The internal formulation (pL) is presented in the first subsection, where it is also showed that if the space is productively Lindelöf, then pL holds. Then, in the second subsection, we show that pL, together with a certain technical property, is enough to prove that a space is productively Lindelöf. After that, in the third subsection, we finally show that every regular space has that technical property, concluding the main result (Theorem 2.13). Finally, in the last subsection we discuss some cases when this characterization can be extended for spaces that are not regular.

**2.1. A topology over open coverings.** Let  $(X, \tau)$  be a topological space. We denote by  $\mathcal{O}$  the collection of all open coverings of  $X$ . Given a finite  $F \subset \tau$ , we use the following notation:

$$F^* = \{\mathcal{U} \in \mathcal{O} : F \subset \mathcal{U}\}.$$

Note that, since  $F^* \cap G^* = (F \cup G)^*$ , the collection  $\{F^* : F \subset \tau \text{ is finite}\}$  is a base for a topology over  $\mathcal{O}$ . Also, note that for any  $\mathcal{U} \in \mathcal{O}$ ,  $\{F^* : F \subset \mathcal{U} \text{ is finite}\}$  is a local base for  $\mathcal{U}$ . Therefore, the following definition is just a translation of the Lindelöfness property of subsets of  $\mathcal{O}$ .

**Definition 2.1.** Let  $(X, \tau)$  be a topological space. We say that  $\mathcal{Y} \subset \mathcal{O}$  is a **Lindelöf collection** if, for every  $f : \mathcal{Y} \rightarrow [\bigcup \mathcal{Y}]^{<\omega}$  such that  $f(\mathcal{U}) \subset \mathcal{U}$  for all  $\mathcal{U} \in \mathcal{Y}$ , there is a sequence  $(\mathcal{U}_n)_{n \in \omega}$  of coverings in  $\mathcal{Y}$  such that for every  $\mathcal{U} \in \mathcal{Y}$ , there is an  $n \in \omega$  such that  $f(\mathcal{U}_n) \subset \mathcal{U}$ .

If we cover  $X \times \mathcal{Y}$  in such a way that each  $(x, \mathcal{U}) \in A \times \{A\}^*$  for some  $A \in \mathcal{U}$  with  $x \in A$ , then the Lindelöf property of  $X \times \mathcal{Y}$  would imply that there is a sequence  $(A_n)_{n \in \omega}$  that is enough to cover the whole of  $X \times \mathcal{Y}$ . This motivates the following definition.

**Definition 2.2.** Let  $(X, \tau)$  be a topological space. We say that  $X$  has the **pL property** if, for every Lindelöf collection  $\mathcal{Y} \subset \mathcal{O}$ , there is a sequence  $(A_n)_{n \in \omega}$  of open sets such that, for each  $(x, \mathcal{U}) \in X \times \mathcal{Y}$ , there is some  $n \in \omega$  such that  $x \in A_n \in \mathcal{U}$ .

After the comments above, it is easy to see the following.

**Proposition 2.3.** *Let  $(X, \tau)$  be a topological space. If  $X$  is productively Lindelöf, then  $X$  has the pL property.*

In the following, we will discuss when the pL property implies that a space is productively Lindelöf.

**2.2. From  $Y$  to  $\mathcal{Y}$ .** Let  $(X, \tau)$  and  $(Y, \rho)$  be topological spaces. Let  $\mathcal{W}$  be an open covering of  $X \times Y$  made by basic open sets. For each  $y \in Y$ , define  $\mathcal{U}_y = \{U \in \tau : \text{there is a } V \in \rho \text{ such that } U \times V \in \mathcal{W} \text{ and } y \in V\}$ . Note that each  $\mathcal{U}_y$  is an element of  $\mathcal{O}$ . We define  $\mathcal{Y} = \{\mathcal{U}_y : y \in Y\}$  with the topology defined as before. Note that  $\mathcal{Y}$  depends on  $\mathcal{W}$ , but we will not mark this dependency unless necessary.

**Proposition 2.4.** *If  $\mathcal{W}$  is an open covering of  $X \times Y$  and  $Y$  is Lindelöf, then  $\mathcal{Y}$  is Lindelöf.*

*Proof.* For each  $\mathcal{U}_y$ , let  $\{U_1^y, \dots, U_{n_y}^y\} \subset \mathcal{U}_y$  (thus  $\mathcal{U}_y \in \{U_1^y, \dots, U_{n_y}^y\}^*$ ). By the definition of  $\mathcal{U}_y$ , for each  $U_i^y$ , there is a  $V_i^y$  such that  $y \in V_i^y$  and  $U_i^y \times V_i^y \in \mathcal{W}$ . Define  $V_y = \bigcap_{i=1}^{n_y} V_i^y$ . Since  $Y$  is Lindelöf, there is a sequence  $(y_k)_{k \in \omega}$  such that  $Y \subset \bigcup_{k \in \omega} V_{y_k}$ . Let us prove that  $(\{U_1^{y_k}, \dots, U_{n_{y_k}}^{y_k}\}^*)_{k \in \omega}$  is a covering of  $\mathcal{Y}$ . Let  $\mathcal{U}_y \in \mathcal{Y}$ . Let  $k \in \omega$  such that  $y \in V_{y_k} = \bigcap_{i=1}^{n_{y_k}} V_i^{y_k}$ . So, by definition of  $\mathcal{U}_y$ , each  $U_i^{y_k} \in \mathcal{U}_y$ , which means that  $\mathcal{U}_y \in \{U_1^{y_k}, \dots, U_{n_{y_k}}^{y_k}\}^*$ , and thus this completes our proof.  $\square$

Now we will investigate when the pL property applied to  $\mathcal{Y}$  is enough to guarantee that  $\mathcal{W}$  has a countable subcovering. For this, the following definitions will be helpful.

**Definition 2.5.** Let  $\mathcal{W}$  be a covering of  $X \times Y$  made by basic open sets. Then we say that  $\mathcal{W}$  is an **injective covering** if whenever  $A \times B, A' \times B' \in \mathcal{W}$ , then  $A = A'$  implies  $B = B'$ . We say that  $\mathcal{W}$  is an  $\omega$ -**injective covering** if, for every  $A$ , the set  $\{B : A \times B \in \mathcal{W}\}$  is at most countable.

**Lemma 2.6.** *Let  $X$  and  $Y$  be Lindelöf spaces and let  $\mathcal{W}$  be an  $\omega$ -injective covering of  $X \times Y$ . If  $X$  has the pL property, then  $\mathcal{W}$  has a countable subcovering.*

*Proof.* Let  $\mathcal{Y}$  be such as before Proposition 2.4. The latter implies that  $\mathcal{Y}$  is a Lindelöf collection. Let  $(A_n)_{n \in \omega}$  be given by the pL property. For each  $A_n$ , let  $(B_n^m)_{m \in \omega}$  be an enumeration of all open sets  $B \subset Y$  such that  $A_n \times B \in \mathcal{W}$ . We will show that  $X \times Y \subset \bigcup_{n, m \in \omega} A_n \times B_n^m$ . Let  $(x, y) \in X \times Y$ . Let  $n \in \omega$  be such that  $x \in A_n \in \mathcal{U}_y$ . Note that, since  $A_n \in \mathcal{U}_y$ , there is a  $B$  such that  $y \in B$  and  $A_n \times B \in \mathcal{W}$ . Thus there is an  $m$  such that  $B_n^m = B$ , and therefore  $(x, y) \in A_n \times B_n^m$ .  $\square$

Note that if  $X \times Y$  is Lindelöf, then every open covering of  $X \times Y$  has an  $\omega$ -injective refinement (just go for a countable refinement made by basic open sets). Therefore, the following result is an easy consequence of the previous ones.

**Theorem 2.7.** *Let  $X$  be a Lindelöf space. Then  $X$  is productively Lindelöf if and only if  $X$  has the pL property and, for every Lindelöf  $Y$  and every  $\mathcal{W}$  covering for  $X \times Y$ , there is an  $\omega$ -injective refinement for  $\mathcal{W}$ .*

Thus, for every class of spaces such that it is always possible to find  $\omega$ -injective refinements as above, the productively Lindelöf property is equivalent to the pL property. In the following we will discuss such classes of spaces.

**2.3. Getting  $\omega$ -injective refinements.** The objective of this section is to show that it is possible to find  $\omega$ -injective refinements for every regular space. But some of the results of this section will also be used in the next section, where we consider nonregular spaces.

**Lemma 2.8.** *Let  $X$  and  $Y$  be any spaces where  $Y$  is Lindelöf and  $X$  is  $T_1$ . Let  $\mathcal{W}$  be an open covering of  $X \times Y$  and let  $\mathcal{B}$  be a base for  $X$ . Then there is an open refinement  $\mathcal{R}$  of  $\mathcal{W}$  such that every element of  $\mathcal{R}$  is of the form  $B \times C$  where  $B \in \mathcal{B}$  and, for every  $B \in \mathcal{B}$ ,  $|\{C : B \times C \in \mathcal{R}\}| \leq |B| + \aleph_0$ .*

*Proof.* For every isolated point  $x \in X$  let us fix a countable open covering  $\mathcal{C}_x$  of  $Y$  such that for each element of  $\{\{x\} \times C : C \in \mathcal{C}_x\}$  there is an element of  $\mathcal{W}$  containing it. Set  $\mathcal{R}_0 = \{\{x\} \times C : x \in X \text{ is isolated and } C \in \mathcal{C}_x\}$ .

Now let  $\kappa$  be an infinite cardinal. We define  $X_\kappa = \{x \in X : \kappa \text{ is the least } \rho \text{ such that there is a } B \in \mathcal{B} \text{ such that } x \in B \text{ and } |B| = \rho\}$ . Let  $\mathcal{B}_\kappa = \{B \in \mathcal{B} : |B| = \kappa\}$ .

Let  $\{x_\xi : \xi < \lambda\}$  be an enumeration of  $X_\kappa$ . Let  $M_0$  be an elementary submodel such that  $x_0, X, Y, \mathcal{W}, \mathcal{B} \in M_0$ , and  $|M_0| = \kappa$ . Then  $A \subset M_0$  for every  $A \in M_0$  with  $|A| \leq \kappa$ . Let

$$\mathcal{R}_0^\kappa = \{B \times C \in M_0 : B \times C \text{ refines } \mathcal{W} \text{ and } B \in \mathcal{B}_\kappa\}.$$

Note that  $|\mathcal{R}_0^\kappa| \leq \kappa$ .

*Claim 2.9.* If  $x \in X_\kappa$  and there is a  $B \in \mathcal{B}_\kappa$  such that  $B \times C \in \mathcal{R}_0^\kappa$  for some  $C$ , then  $\{x\} \times Y \subset \bigcup \mathcal{R}_0^\kappa$ .

*Proof.* Let  $B \in \mathcal{B}_\kappa$  be such that  $x \in B$  and  $B \times C \in \mathcal{R}_0^\kappa$ . Since  $|B| = \kappa$ ,  $B \subset M_0$ . Thus, since  $Y$  is Lindelöf, there is a sequence  $(B_n \times C_n)_{n \in \omega}$  covering  $\{x\} \times Y$ , refining  $\mathcal{W}$ , and such that  $B_n \in \mathcal{B}_\kappa$  for all  $n \in \omega$ . By elementarity, we may suppose that this sequence is in  $M_0$  and, therefore, each  $B_n \times C_n \in \mathcal{R}_0^\kappa$ , which proves the claim. □

Now let  $x_\xi$  be the first one such that  $x_\xi$  is not covered by any  $B$  such that  $B \times C \in \mathcal{R}_0^\kappa$  for some  $C$ . Let  $M_\xi$  be an elementary submodel as before, but this time containing  $x_\xi$ . Define

$$\mathcal{R}_\xi^\kappa = \{B \times C \in M_\xi : B \in \mathcal{B}_\kappa, B \setminus A \neq \emptyset \text{ and } B \times C \text{ refines } \mathcal{W}\},$$

where  $A$  is the collection of all  $x \in X$  that are covered by some  $B$  such that  $B \times C \in \mathcal{R}_0^\kappa$  for some  $C$ .

Note that, again,  $|\mathcal{R}_\xi^\kappa| \leq \kappa$ . Also, if  $B \times C \in \mathcal{R}_0^\kappa$  and  $B' \times C' \in \mathcal{R}_\xi^\kappa$ , then  $B \neq B'$ . Finally, note that the analogy of the claim for  $\mathcal{R}_\xi^\kappa$  also holds. Thus, we can proceed like this until there is no  $x_\eta \in X_\kappa$  not covered. Then, define  $\mathcal{R}_\kappa = \bigcup_{\xi < \lambda} \mathcal{R}_\xi^\kappa$  (if  $\mathcal{R}_\xi^\kappa$  was not defined, just let it be empty). Finally, note that  $\mathcal{R} = \bigcup_{\kappa < |X|} \mathcal{R}_\kappa$  is the refinement we were looking for. □

**Lemma 2.10.** *Let  $X$  be a  $T_1$  space without isolated points and let  $\mathcal{B}$  be a base of  $X$  such that there are no different  $A, B \in \mathcal{B}$  with  $|A \Delta B|$  finite. Then for every Lindelöf space  $Y$  and every  $\mathcal{W}$  covering for  $X \times Y$  there is an injective refinement of  $\mathcal{W}$ .*

*Proof.* First, applying Lemma 2.8, we may suppose that every element of  $\mathcal{W}$  is of the form  $B \times C$  with  $B \in \mathcal{B}$  and such that

$$|\{C : B \times C \in \mathcal{W}\}| \leq |B|$$

for every  $B \in \mathcal{B}$ . Let  $B \in \mathcal{B}$ . For each  $C$  such that  $B \times C \in \mathcal{W}$ , select different  $b_C, d_C \in B$  in such a way that  $\{b_C, d_C\} \cap \{b_{C'}, d_{C'}\} = \emptyset$  if  $C \neq C'$ . Note that  $\{(B \setminus \{b_C\}) \times C, (B \setminus \{d_C\}) \times C\}$  refines  $B \times C$ . Also, note that if we repeat this process with all the infinite elements of  $\mathcal{B}$ , we get an injective refinement. □

**Lemma 2.11.** *Any regular space with no isolated points has a base  $\mathcal{B}$  such that there are no different  $A, B \in \mathcal{B}$  such that  $|A\Delta B| < \aleph_0$ .*

*Proof.* Note that the regular open sets form a base. Also, note that if  $A$  and  $B$  are different regular open sets, then  $A\Delta B \supset (A \setminus \overline{B}) \cup (B \setminus \overline{A})$ , and the latter is a nonempty open set. Since  $X$  has no isolated points, this set is infinite.  $\square$

**Lemma 2.12.** *Let  $X$  be a scattered  $T_1$  space and let  $Y$  be a Lindelöf space. Let  $\mathcal{W}$  be an open covering for  $X \times Y$ . Then there is an  $\omega$ -injective open refinement of  $\mathcal{W}$ .*

*Proof.* Let  $X_0$  be the set of all isolated points of  $X$ . Then, let  $X_\xi$  be the set of all isolated points of  $X \setminus \bigcup_{\eta < \xi} X_\eta$ .

For every  $x \in X_\xi$ , since  $Y$  is Lindelöf, we can find  $(A_n^x \times B_n^x)_{n \in \omega}$  that is a refinement of  $\mathcal{W}$  that covers  $\{x\} \times Y$ . We may also assume that  $A_n^x \setminus \{x\} \subset \bigcup_{\eta < \xi} X_\eta$  for all  $x \in X_\xi$  and  $n \in \omega$ , i.e.,  $X_\xi \cap A_n^x = \{x\}$ . Note that the union of all  $(A_n^x \times B_n^x)$ 's covers the whole  $X \times Y$  and that this is  $\omega$ -injective since  $A_n^x \neq A_m^y$  if  $x \neq y$ .  $\square$

Combining Lemmata 2.10, 2.11, 2.12, we obtain our main result as follows.

**Theorem 2.13.** *For every regular Lindelöf space,  $X$  is productively Lindelöf if and only if  $X$  has the  $pL$  property.*

Finally, it is worth mentioning that the regularity hypothesis in the characterization above was used only to guarantee refinements of open coverings with as many different open sets as necessary for the proof. In general, we don't know whether the regularity can be dropped. The next subsection is devoted to the particular situation when we do not need it.

**2.4. Nonregular spaces.** Even for nonregular spaces we can sometimes obtain the same characterization. For a space  $X$  we denote by  $kc(X)$  the minimal cardinality of a covering of  $X$  by its compact subspaces. Note that if  $Y$  is Lindelöf, then any open covering of  $X \times Y$  has a subcovering of size  $\leq kc(X)$ .

**Lemma 2.14.** *Let  $X$  and  $Y$  be Lindelöf spaces, with  $X$  being  $T_1$  and such that every open subset of  $X$  has cardinality at least  $kc(X)$ . Then every open covering  $\mathcal{W}$  of  $X \times Y$  has an injective refinement  $\mathcal{W}'$  consisting of standard basic open subsets of  $X \times Y$ . Moreover, if  $\mathcal{B}$  is a base for  $X$ , then we may additionally assume that any  $W \in \mathcal{W}'$  has the form  $(B \setminus \{x\}) \times V$  for some  $B \in \mathcal{B}$ ,  $x \in X$ , and open  $V \subset Y$ .*

*Proof.* We may assume that  $\mathcal{W}$  consists of standard basic open subsets of  $X \times Y$  and  $|\mathcal{W}| \leq kc(X) = \kappa$ . Let us write  $\mathcal{W}$  in the form  $(A_\xi \times B_\xi)_{\xi < \kappa}$ . We will construct two sequences  $(x_\xi^1)_{\xi < \kappa}$ ,  $(x_\xi^2)_{\xi < \kappa}$  of elements of  $X$  in such a way that for every  $\xi < \kappa$  we have the following properties:

- (1)  $x_\xi^i \in A_\xi$  for  $i = 1, 2$ ;
- (2)  $x_\xi^1 \neq x_\xi^2$ ;
- (3)  $A_\xi \setminus \{x_\xi^i\} \neq A_\eta \setminus \{x_\eta^j\}$  for every  $i, j \in \{1, 2\}$  and every  $\eta < \xi$ .

This is easily done by induction of our assumption on the cardinality of  $A_\xi$ . Now note that  $\{(A_\xi \setminus \{x_\xi^i\}) \times B_\xi : \xi < \kappa \text{ and } i = 1, 2\}$  is the refinement we were looking for.  $\square$

**Corollary 2.15.** *Let  $X$  be a Lindelöf  $T_1$  space such that every open subset has cardinality at least  $kc(X)$ . Then  $X$  is productively Lindelöf if and only if  $X$  has the  $pL$  property.*

*Proof.* It follows directly from Theorem 2.7 and Lemma 2.14.  $\square$

**Proposition 2.16.** *Let  $X$  be a Lindelöf  $T_1$  space of size  $\aleph_1$  and let  $Y$  be a Lindelöf space. Then, for every  $\mathcal{W}$  covering for  $X \times Y$ , there is an  $\omega$ -injective refinement.*

*Proof.* Let  $X'$  be the collection of all points of  $X$  that have a countable neighborhood. Let  $\mathcal{W}'$  be an  $\omega$ -injective refinement for  $X' \times Y$  given by the Lemma 2.8. Note that  $X'' = X \setminus X'$  is closed and, therefore, Lindelöf. Thus we can use the same argument as in the proof of Lemma 2.14 and get  $\mathcal{W}''$  as an injective refinement for  $X'' \times Y$  (note that for each open set, in the argument of the lemma we can remove points from  $X'$ ). Thus,  $\mathcal{W}' \cup \mathcal{W}''$  is the refinement we were looking for.  $\square$

**Corollary 2.17.** *Let  $X$  be a Lindelöf  $T_1$  space of size  $\aleph_1$ . Then  $X$  is productively Lindelöf if and only if  $X$  has the  $pL$  property.*

### 3. ABOUT ALSTER COVERINGS

For this section and the next one, we need some more definitions.

**Definition 3.1.** Let  $X$  be a topological space. We say that  $\mathcal{G}$  is an **Alster covering** for  $X$  if each  $G \in \mathcal{G}$  is a  $G_\delta$  set and, for every compact  $K \subset X$ , there is a  $G \in \mathcal{G}$  such that  $K \subset G$ . We say that  $X$  is an **Alster space** if every Alster covering has a countable subcovering.

In this section, we present a generalization of the following.

**Theorem 3.2** (Alster [2]). *Suppose CH. Let  $X$  be a Tychonoff space with weight  $\leq \aleph_1$ . Then  $X$  is productively Lindelöf if and only if for every Alster covering, there is a countable subcovering, i.e.,  $X$  is an Alster space.*

In the following, we use the standard notation  $I = [0, 1]$ . Our main result in this section is the following.

**Theorem 3.3.** *Let  $X \subset I^{\omega_1}$  be a productively Lindelöf Menger space and let  $\mathcal{G}$  be an Alster covering of  $X$  of size  $|\mathcal{G}| = \omega_1$  consisting of Lindelöf  $G_\delta$ -subsets of  $I^{\omega_1}$ . Then there exists  $\mathcal{G}' \in [\mathcal{G}]^\omega$  such that  $X \subset \bigcup \mathcal{G}'$ .*

First, note that if a space is an Alster space, then it is productively Lindelöf. This is the “easy” part of [2], Theorem 3.2. Also, note that any Tychonoff space  $X$  with weight  $\leq \aleph_1$  can be viewed as a subspace of  $I^{\omega_1}$  and, under CH, given any Alster covering, there is a refinement of size  $\aleph_1$  for this covering made by compact  $G_\delta$ -subsets of  $I^{\omega_1}$ . Finally, under CH, every productively Lindelöf space is Menger ([7, Proposition 3.1]), since there is a Michael space. With this, we obtain that Theorem 3.2 follows from Theorem 3.3.

Before proving Theorem 3.3 let us draw some corollaries from it.

**Corollary 3.4.** *Suppose that there exists a Michael space. Let  $X \subset I^{\omega_1}$  be a productively Lindelöf space and let  $\mathcal{G}$  be an Alster covering of  $X$  of size  $|\mathcal{G}| = \omega_1$  consisting of Lindelöf  $G_\delta$ -subsets of  $I^{\omega_1}$ . Then there exists  $\mathcal{G}' \in [\mathcal{G}]^\omega$  such that  $X \subset \bigcup \mathcal{G}'$ .*

*Proof.* By [7, Proposition 3.1] there exists a Michael space if and only if all productively Lindelöf spaces are Menger.  $\square$

**Corollary 3.5.** *Suppose that  $\mathfrak{b} = \omega_1$  or  $\text{cov}(\mathcal{M}) = \mathfrak{d}$ . Let  $X \subset I^{\omega_1}$  be a productively Lindelöf space and let  $\mathcal{G}$  be an Alster covering of  $X$  of size  $|\mathcal{G}| = \omega_1$  consisting of Lindelöf  $G_\delta$ -subsets of  $I^{\omega_1}$ . Then there exists  $\mathcal{G}' \in [\mathcal{G}]^\omega$  such that  $X \subset \bigcup \mathcal{G}'$ .*

*Proof.* If  $\mathfrak{b} = \omega_1$  or  $\text{cov}(\mathcal{M}) = \mathfrak{d}$ , then there exists a Michael space. More precisely, its existence under  $\mathfrak{b} = \omega_1$  follows by an almost literal repetition of Michael's proof [4] that  $\omega^\omega$  is not productively Lindelöf under CH, and the case  $\text{cov}(\mathcal{M}) = \mathfrak{d}$  has been treated in [5, Theorem 2.2].  $\square$

We shall divide the proof of Theorem 3.3 into a sequence of lemmas. For every  $\alpha \in \omega_1$  we shall denote by  $p_\alpha$  the projection map from  $I^{\omega_1}$  to  $I^\alpha$ , i.e.,  $p_\alpha : (x_\xi)_{\xi < \omega_1} \mapsto (x_\xi)_{\xi < \alpha}$ . Let us denote by  $\mathcal{B}_\alpha$  the family of open subsets of  $I^{\omega_1}$  of the form  $\prod_{\xi \in \omega_1} U_\xi$ , where  $U_\xi = I$  for all  $\xi \notin F$  for some finite  $F \subset \alpha$ , and  $U_\xi$  is an interval with rational end-points for all  $\xi \in F$ . Thus  $\mathcal{B} := \bigcup_{\alpha \in \omega_1} \mathcal{B}_\alpha$  is the standard base for the topology on  $I^{\omega_1}$ .

The following fact may be thought of as folklore.

**Lemma 3.6.** *A  $G_\delta$  subset  $G$  of  $I^{\omega_1}$  is Lindelöf if and only if there exists  $\alpha < \omega_1$  such that  $G = p_\beta^{-1}[p_\beta[G]]$  for all  $\beta \geq \alpha$ .*

*Proof.* For the “if” part note that  $G = p_\alpha^{-1}[p_\alpha[G]]$  implies  $G = p_\alpha[G] \times I^{\omega_1 \setminus \alpha}$ , i.e.,  $G$  is a product of a metrizable separable space and a compact space. Such products are obviously Lindelöf.

Let us prove now the “only if” part. Write  $G$  in the form  $\bigcap_{n \in \omega} U_n$ , where  $U_n$  is open. Given any  $y \in G$  and  $n \in \omega$ , fix  $B(y, n) \in \mathcal{B}$  such that  $y \in B(y, n) \subset U_n$ . Let  $\alpha(y, n)$  be such that  $B(y, n) \in \mathcal{B}_{\alpha(y, n)}$ . Since  $G$  is Lindelöf there exists a countable  $Y \subset G$  such that  $G \subset \bigcup_{y \in Y} B(y, n)$  for all  $n \in \omega$ . Then  $G = \bigcap_{n \in \omega} \bigcup_{y \in Y} B(y, n)$ . It is easy to see that  $\alpha = \sup\{\alpha(y, n) : y \in Y, n \in \omega\} + 1$  is as required.  $\square$

We shall need the following result which is a direct consequence of [1, Corollary 2.5].

**Proposition 3.7.** *Let  $Z$  be a metrizable space and let  $\{A_\xi : \xi < \omega_1\}$  be an increasing covering of  $Z$  such that for every compact  $K \subset Z$  there exists  $\alpha \in \omega_1$  with the property  $K \subset A_\alpha$ . If  $A_\xi \neq Z$  for all  $\xi$ , then  $Z$  is not productively Lindelöf.*

We shall call  $A \subset I^{\omega_1}$  *big* if  $A \cap X$  is not covered by any countable subfamily of  $\mathcal{G}$ . If  $X$  is not big, then there is nothing to prove. So we shall assume that  $X$  is big in what follows.

**Lemma 3.8.** *For every big closed subspace  $Z$  of  $X$  and  $\alpha \in \omega_1$  there exists a compact  $K \subset p_\alpha[Z]$  such that  $Z \cap p_\alpha^{-1}[K]$  is big.*

*Proof.* Let  $\mathcal{G} = \{G_\xi : \xi < \omega_1\}$  and let  $A_\beta = p_\alpha[Z] \setminus p_\alpha[Z \setminus \bigcup_{\xi < \beta} G_\xi]$ . If there is no compact  $K \subset p_\alpha[Z]$  such that  $Z \cap p_\alpha^{-1}[K]$  is big, then  $\{A_\xi : \xi < \omega_1\}$  is easily seen to be an increasing covering of  $p_\alpha[Z]$  such that for every compact  $K \subset p_\alpha[Z]$  there exists  $\xi \in \omega_1$  with the property  $K \subset A_\xi$ , and  $A_\xi \neq p_\alpha[Z]$  for all  $\xi$  because  $Z$  is big. Now Proposition 3.7 implies that  $p_\alpha[Z]$  is not productively Lindelöf, a contradiction.  $\square$

For a topological space  $Z$  we shall denote by  $\text{O}(Z)$  the family of all open coverings of  $Z$ . For a subspace  $T$  of  $Z$  we shall denote by  $\text{cl}_Z(T)$  the closure of  $T$  in  $Z$ .

**Lemma 3.9.** *Let  $A \subset X$  be a big Lindelöf subspace. Then there exists  $\alpha \in \omega_1$  such that for every  $\beta > \alpha$  there exists an open covering  $\mathcal{W}_\beta$  of  $p_\beta[A]$  such that  $A \setminus p_\beta^{-1}[\cup \mathcal{V}]$  is big for all  $\mathcal{V} \in [\mathcal{W}_\beta]^{<\omega}$ .*

*Proof.* Suppose that contrary to our claim the set  $\Lambda$  of all  $\alpha \in \omega_1$  such that for every open covering  $\mathcal{W}$  of  $p_\alpha[A]$  there exists  $\mathcal{V}_\mathcal{W} \in [\mathcal{W}]^{<\omega}$  for which  $A \setminus p_\alpha^{-1}[\cup \mathcal{V}_\mathcal{W}]$  is not big, is cofinal in  $\omega_1$ . For every  $\alpha \in \Lambda$  the set  $K_\alpha = \bigcap_{\mathcal{W} \in \mathcal{O}(p_\alpha[A])} \text{cl}_{p_\alpha[A]}(\cup \mathcal{V}_\mathcal{W})$  is a compact subspace of  $p_\alpha[A]$ . Since  $p_\alpha[A]$  is metrizable separable, there exists a countable  $W_\alpha \subset \mathcal{O}(p_\alpha[A])$  such that  $K_\alpha = \bigcap_{\mathcal{W} \in W_\alpha} \text{cl}_{p_\alpha[A]}(\cup \mathcal{V}_\mathcal{W})$ . Since for every  $\mathcal{W} \in W_\alpha$  the set  $A \setminus p_\alpha^{-1}[\cup \mathcal{V}_\mathcal{W}]$  is not big, so is the set

$$A \setminus p_\alpha^{-1}[K_\alpha] = A \setminus \bigcap_{\mathcal{W} \in W_\alpha} p_\alpha^{-1}[\cup \mathcal{V}_\mathcal{W}] = \bigcup_{\mathcal{W} \in W_\alpha} (A \setminus p_\alpha^{-1}[\cup \mathcal{V}_\mathcal{W}])$$

because nonbig sets are obviously closed under countable unions. Therefore

$$\bigcup_{\alpha \in \Lambda \cap \gamma} (A \setminus p_\alpha^{-1}[K_\alpha]) = A \setminus \bigcap_{\alpha \in \Lambda \cap \gamma} p_\alpha^{-1}[K_\alpha]$$

is not big for every  $\gamma \in \omega_1$ , and hence  $A \cap \bigcap_{\alpha \in \Lambda \cap \gamma} p_\alpha^{-1}[K_\alpha]$  is big for every  $\gamma \in \omega_1$ . Note that  $K := \bigcap_{\alpha \in \Lambda} p_\alpha^{-1}[K_\alpha]$  is a compact subspace of  $I^{\omega_1}$ . Since  $K_\alpha \subset p_\alpha[A]$  for all  $\alpha \in \Lambda$  and  $A$  is Lindelöf, we have that  $K \subset A$ . (Suppose there is a  $k \in K \setminus A$ . Note that  $(U_\alpha)_{\alpha < \omega_1}$  is an open covering for  $A$ , where each  $U_\alpha = \{a \in A : a(\alpha) \neq k(\alpha)\}$ . But such a covering cannot have a countable subcovering since  $p_\alpha(k) \in p_\alpha[A]$  for every  $\alpha \in \omega_1$ .) Thus there exists  $G \in \mathcal{G}$  such that  $K \subset G$ , and hence  $\{A \setminus p_\alpha^{-1}[K_\alpha] : \alpha \in \Lambda\}$  is an open covering of  $A \setminus G$ . Since  $A$  is Lindelöf, so is  $A \setminus G$  being an  $F_\sigma$ -subset of a Lindelöf space, and hence  $A \setminus G \subset \bigcup_{\alpha \in \Lambda \cap \gamma} (A \setminus p_\alpha^{-1}[K_\alpha])$  for some  $\gamma < \omega_1$ . Therefore  $G \supset A \cap \bigcap_{\alpha \in \Lambda \cap \gamma} p_\alpha^{-1}[K_\alpha]$ , which means that the latter set cannot be big, a contradiction.  $\square$

Let us fix a map  $\psi : \mathcal{G} \rightarrow \omega_1$  such that  $\psi^{-1}(\alpha)$  is countable for all  $\alpha$ , and  $\psi(G)$  satisfies Lemma 3.6 for  $G$ , i.e.,  $G = p_\beta^{-1}[p_\beta[G]]$  for all  $\beta \geq \psi(G)$ . For every  $\alpha$  let us consider the set  $C_\alpha = \{z \in I^{\omega_1} : p_\beta(z) \in p_\beta[X] \text{ for all } \beta < \alpha, p_\alpha(z) \notin p_\alpha[X], \text{ and } z \notin G \text{ for all } G \in \mathcal{G} \text{ such that } \psi(G) < \alpha\}$ . Note that  $C_\alpha$  depends on  $\psi$ .

We shall need the following game of length  $\omega$ : In the  $n$ th round Player *I* chooses an open covering  $\mathcal{U}_n$  of  $X$ , and Player *II* responds by choosing a finite  $\mathcal{V}_n \subset \mathcal{U}_n$ . Player *II* wins the game if  $\bigcup_{n \in \omega} \bigcup \mathcal{V}_n = X$ . Otherwise, Player *I* wins. We shall call this game *the Menger game* on  $X$ . Since  $X$  is Menger, Player *I* has no winning strategy in this game on  $X$ ; see [3] or [8, Theorem 13].

**Lemma 3.10.** *The set  $\Lambda = \{\alpha : C_\alpha \neq \emptyset\}$  is unbounded in  $\omega_1$ .*

*Proof.* Given  $\alpha_0 \in \omega_1$ , we shall find  $\alpha > \alpha_0$  such that  $C_\alpha \neq \emptyset$ . Now we shall describe a strategy of Player *I* in the Menger game on  $X$ .

*Round 0.* Set  $A_0 = X$ . Since  $A_0$  is big and productively Lindelöf, by Lemma 3.8 there exists a compact  $K_0 \subset p_{\alpha_0}[A_0]$  such that  $A'_0 := A_0 \cap p_{\alpha_0}^{-1}[K_0]$  is big. Let  $G_0 \in \mathcal{G}$  be such that  $\psi(G_0) < \alpha_0$ . Since  $A'_0 \setminus G_0$  is big and nonbig sets are closed under countable unions, there exists an open set  $S_0 \supset G_0$  such that  $p_\beta^{-1}[p_\beta[S_0]] = S_0$  for all  $\beta \geq \psi(G_0)$ , and  $A''_0 := A'_0 \setminus S_0$  is big. By Lemma 3.9 applied to  $A''_0$  there exists  $\alpha_1 > \alpha_0$  and  $\mathcal{W}_0 \in \mathcal{O}(p_{\alpha_1}[A''_0])$  such that  $A''_0 \setminus p_{\alpha_1}^{-1}[\cup \mathcal{V}]$  is big for all  $\mathcal{V} \in [\mathcal{W}_0]^{<\omega}$ . Then Player *I* starts by choosing the open covering  $\mathcal{U}_0 = \{X \setminus A''_0\} \cup \{p_{\alpha_1}^{-1}[W] : W \in \mathcal{W}_0\}$  of  $X$ . Suppose that Player *II* replies by choosing  $\{X \setminus A''_0\} \cup \{p_{\alpha_1}^{-1}[W] : W \in \mathcal{V}_0\}$



for some finite  $\mathcal{V}_0 \subset \mathcal{W}_0$ . Then we set  $A_1 = A_0'' \setminus p_{\alpha_1}^{-1}[\cup \mathcal{V}_0]$ . It follows from above that  $A_1$  is a big closed subspace of  $X$ .

*Round  $n$ .* Suppose that after  $n - 1$  rounds the set  $A_n \subset X$  of those  $x \in X$  which have not yet been covered by the choices of Player *II* is closed<sup>1</sup> and big. Suppose also that in the course of the previous rounds player *I* has constructed ordinals  $\alpha_0 < \alpha_1 < \dots < \alpha_n$ . Then player *I* acts in basically the same way as in round 0. For the sake of completeness we repeat the argument. Since  $A_n$  is big and productively Lindelöf, by Lemma 3.8 there exists a compact  $K_n \subset p_{\alpha_n}[A_n]$  such that  $A'_n := A_n \cap p_{\alpha_n}^{-1}[K_n]$  is big. Let  $G_n \in \mathcal{G}$  be such that  $\psi(G_n) < \alpha_n$ . Since  $A'_n \setminus G_n$  is big and nonbig sets are closed under countable unions, there exists an open set  $S_n \supset G_n$  such that  $p_{\beta}^{-1}[p_{\beta}[S_n]] = S_n$  for all  $\beta \geq \psi(G_n)$ , and  $A''_n := A'_n \setminus S_n$  is big. By Lemma 3.9 applied to  $A''_n$  there exists  $\alpha_{n+1} > \alpha_n$  and  $\mathcal{W}_n \in \mathcal{O}(p_{\alpha_{n+1}}[A''_n])$  such that  $A''_n \setminus p_{\alpha_{n+1}}^{-1}[\cup \mathcal{V}]$  is big for all  $\mathcal{V} \in [\mathcal{W}_n]^{<\omega}$ . Then Player *I* plays by choosing the open covering  $\mathcal{U}_n = \{X \setminus A''_n\} \cup \{p_{\alpha_{n+1}}^{-1}[W] : W \in \mathcal{W}_n\}$  of  $X$ . Suppose that Player *II* replies by choosing  $\{X \setminus A''_n\} \cup \{p_{\alpha_{n+1}}^{-1}[W] : W \in \mathcal{V}_n\}$  for some finite  $\mathcal{V}_n \subset \mathcal{W}_n$ . Then we set  $A_{n+1} = A''_n \setminus p_{\alpha_{n+1}}^{-1}[\cup \mathcal{V}_n]$ . It follows from the above that  $A_{n+1}$  is a big closed subspace of  $X$ .

In addition, by choosing  $G_n$  in the  $n$ th round, Player *I* makes sure that each  $G \in \mathcal{G}$  with  $\psi(G) < \sup_{k \in \omega} \alpha_k$  has been chosen, using some straightforward book-keeping. This completes our definition of a strategy of player *I*.

Since  $X$  is Menger, this strategy cannot be winning, and hence there is a play in which Player *I* uses the strategy described above and loses. Let

$$\langle \alpha_n, A_n, A'_n, A''_n, \mathcal{W}_n, \mathcal{V}_n, K_n, G_n, S_n : n \in \omega \rangle$$

be the corresponding objects constructed in the course of this play. Since this play has been lost by Player *I* we have

$$X = \bigcup_{n \in \omega} (X \setminus A''_n) \cup \bigcup_{n \in \omega} p_{\alpha_{n+1}}^{-1}[\cup \mathcal{V}_n].$$

Letting  $\alpha = \sup_{n \in \omega} \alpha_n$ , we claim that  $C_\alpha \supset K := \bigcap_{n \in \omega} p_{\alpha_n}^{-1}[K_n]$ . Note that this would prove our lemma, as  $K$  is nonempty because  $\langle p_{\alpha_n}^{-1}[K_n] : n \in \omega \rangle$  is a decreasing sequence of compact subspaces of  $I^\omega$ . Let us fix  $z \in K$ .

Given any  $\beta < \alpha$ , find  $n$  with  $\beta < \alpha_n$ . Then  $p_{\alpha_n}(z) \in K_n \subset p_{\alpha_n}[A_n] \subset p_{\alpha_n}[X]$ , and hence  $p_{\beta}(z) \in p_{\beta}[X]$ .

Let us fix  $G \in \mathcal{G}$  with  $\psi(G) < \alpha$ . By the requirement on the strategy of Player *I* that we have made at the end of its definition, we have that  $G = G_n$  for some  $n \in \omega$ . Then  $p_{\alpha_{n+1}}(z) \in K_{n+1} \subset p_{\alpha_{n+1}}[A_{n+1}]$  and  $A_{n+1} \cap S_n = A_{n+1} \cap p_{\alpha_{n+1}}^{-1}[p_{\alpha_{n+1}}[S_n]] = \emptyset$  by the construction. Therefore  $p_{\alpha_{n+1}}(z) \notin p_{\alpha_{n+1}}[S_n]$ , and hence  $z \notin S_n \supset G_n$ .

It suffices to show that  $p_\alpha(z) \notin p_\alpha[X]$ . Suppose to the contrary that  $p_\alpha(z) = p_\alpha(x)$  for some  $x \in X$ . Two cases are possible.

1.  $x \in X \setminus A''_n$  for some  $n \in \omega$ . Then  $x \in X \setminus A_{n+1}$  since  $A_{n+1} \subset A''_n$ . By the construction of  $A_k$  we get that  $p_\beta[A_k \setminus A_{k+1}] = p_\beta[A_k] \setminus p_\beta[A_{k+1}]$  for all  $k \in \omega$  and  $\beta \geq \alpha_{k+1}$ . Indeed to get  $A_{k+1}$  we make several steps, and at each of them we remove from  $A_k$  a set  $T$  such that  $T = p_{\beta}^{-1}[p_{\beta}[T]]$  for all  $\beta \geq \alpha_{k+1}$ . Since  $X = A_0$ ,

<sup>1</sup>It is always closed by the definition of the game.

we have that  $X \setminus A_{n+1} = \bigcup_{k \leq n} (A_k \setminus A_{k+1})$ , and hence

$$\begin{aligned} p_{\alpha_{n+1}}(x) &\in p_{\alpha_{n+1}}[X \setminus A_{n+1}] = p_{\alpha_{n+1}}\left[\bigcup_{k \leq n} (A_k \setminus A_{k+1})\right] \\ &= \bigcup_{k \leq n} p_{\alpha_{n+1}}[A_k] \setminus p_{\alpha_{n+1}}[A_{k+1}] = p_{\alpha_{n+1}}[X] \setminus p_{\alpha_{n+1}}[A_{n+1}]. \end{aligned}$$

However,  $p_{\alpha_{n+1}}(x) = p_{\alpha_{n+1}}(z) \in K_{n+1} \subset p_{\alpha_{n+1}}[A_{n+1}]$ , a contradiction.

2.  $x \in p_{\alpha_{n+1}}^{-1}[\bigcup \mathcal{V}_n]$  for some  $n \in \omega$ . Then  $x \in X \setminus A_{n+1}$  because  $A_{n+1} = A_n'' \setminus p_{\alpha_{n+1}}^{-1}[\bigcup \mathcal{V}_n]$  by the construction, and we have already seen in case 1 that  $x \in X \setminus A_{n+1}$  leads to a contradiction. This completes our proof.  $\square$

Let us denote by  $Y$  the union  $\bigcup_{\alpha \in \omega_1} C_\alpha$ .

**Lemma 3.11.**  *$Y$  is Lindelöf.*

*Proof.* Let  $\mathcal{U}$  be an open covering  $Y$ . Without loss of generality we may assume that  $\mathcal{U}$  is closed under finite unions and taking open subsets of its elements. Let  $\mathcal{U}_\alpha = \mathcal{U} \cap \mathcal{B}_\alpha$ . It suffices to show that  $Y \subset \bigcup \mathcal{U}_\alpha$  for some  $\alpha$ . Suppose that this is not the case; set  $\mathcal{G}' = \mathcal{G} \cup \{\bigcup \mathcal{U}_\alpha : \alpha \in \omega_1\}$ , and extend  $\psi$  to  $\mathcal{G}'$  by letting  $\psi(\bigcup \mathcal{U}_\alpha) = \alpha$ . We claim that  $X$  is not covered by any countable subfamily of  $\mathcal{G}'$ . Indeed, fix such a subfamily  $\mathcal{G}''$  and find  $\alpha$  with  $\psi[\mathcal{G}''] \subset \alpha$ , i.e.,  $\mathcal{G}'' \subset \{\bigcup \mathcal{U}_\beta : \beta < \alpha\} \cup \{G \in \mathcal{G} : \psi(G) < \alpha\}$ . Pick  $z \in Y \setminus \bigcup_{\beta < \alpha} \bigcup \mathcal{U}_\beta = Y \setminus \bigcup \mathcal{U}_\alpha$ . Then  $z \in C_\gamma$  for some  $\gamma > \alpha$  by the following.

*Claim 3.12.*  $C_\xi \subset \bigcup_{\eta < \xi} \bigcup \mathcal{U}_\eta = \mathcal{U}_\xi$  for all  $\xi < \omega_1$ .

*Proof.* Given  $u \in C_\xi$ , set  $K_u = \{p_\xi(u)\} \times I^{\omega_1 \setminus \xi}$  and note that  $K_u$  is a compact subspace of  $Y$ . Since  $K_u \subset \bigcup \mathcal{U}$ , there exists a finite  $\mathcal{V} \subset \mathcal{U}$  and an open set  $U = \prod_{\eta \in \omega_1} U_\eta$ , where  $U_\eta = I$  for all  $\eta \notin F$  for some finite  $F \subset \omega_1$  and  $U_\eta$  is an interval with rational end-points for all  $\eta \in F$ , with the following properties:

$$u \in K_u \subset U \subset \bigcup \mathcal{V}.$$

$U \in \mathcal{U}$  by our convention. Moreover,  $U_\eta$  must obviously be equal to  $I$  for all  $\eta \geq \xi$ , and hence  $U \in \mathcal{U}_\xi$ .  $\square$

We proceed with the proof of Lemma 3.11. Since  $z \in C_\gamma$  for some  $\gamma > \alpha$ , we have  $z \notin G$  for all  $G \in \mathcal{G}$  with  $\psi(G) < \alpha$ , and hence  $z \notin G$  for any  $G \in \mathcal{G}''$ . Pick  $x \in X$  such that  $p_\alpha(x) = p_\alpha(z)$  and note that  $x \notin G$  for any  $G \in \mathcal{G}''$  because  $G = p_\alpha^{-1}[p_\alpha[G]]$  for all such  $G$ .

Now repeating the proof of Lemma 3.10 for  $\mathcal{G}'$  and  $\psi$  extended to  $\mathcal{G}'$ , we get that there exists  $\alpha \in \omega_1$  such that the set  $C'_\alpha = \{z \in I^{\omega_1} : p_\beta(z) \in p_\beta[X] \text{ for all } \beta < \alpha, p_\alpha(z) \notin p_\alpha[X], \text{ and } z \notin G \text{ for all } G \in \mathcal{G}' \text{ such that } \psi(G) < \alpha\}$  is nonempty. However,  $C'_\alpha = C_\alpha \setminus \bigcup_{\beta < \alpha} \bigcup \mathcal{U}_\beta = C_\alpha \setminus \bigcup \mathcal{U}_\alpha$  by the definition of  $\mathcal{G}'$  and  $\psi \upharpoonright \mathcal{G}' \setminus \mathcal{G}$ , and  $C_\alpha \setminus \bigcup \mathcal{U}_\alpha = \emptyset$  by Claim 3.12, a contradiction.  $\square$

**Lemma 3.13.** *Let  $U$  be an open neighborhood of  $Y$ . Then  $X \setminus U \subset \bigcup \mathcal{H}$  for some countable  $\mathcal{H} \subset \mathcal{G}$ .*

*Proof.* Let  $\mathcal{U}$  be the family of all open subsets of  $U$  and let  $\mathcal{U}_\alpha$  be the intersection  $\mathcal{U} \cap \mathcal{B}_\alpha$ . In the proof of Lemma 3.11 we have established that there exists  $\alpha \in \omega_1$  and  $\mathcal{H} \in [\mathcal{G}]^\omega$  such that  $X \subset \bigcup \mathcal{U}_\alpha \cup \bigcup \mathcal{H}$ . (More precisely, by the last paragraph of the proof of Lemma 3.11 we have a contradiction if  $X \not\subset \bigcup \mathcal{U}_\alpha \cup \bigcup \mathcal{H}$  for all  $\alpha$  and  $\mathcal{H} \in [\mathcal{G}]^\omega$ ). Therefore  $X \setminus U \subset X \setminus \bigcup \mathcal{U}_\alpha \subset \bigcup \mathcal{H}$ .  $\square$

We can now finish the proof of Theorem 3.3 by deriving a contradiction from  $X$  being big as follows. Let  $\mathcal{G} = \{G_\alpha : \alpha < \omega_1\}$  and pick  $x_\alpha \in X \setminus \bigcup_{\beta < \alpha} G_\beta$ . Consider the space  $Z = \{x_\alpha : \alpha < \omega_1\} \cup Y$  with the following topology  $\tau$ : all  $x_\alpha$ 's are isolated, and the basic open neighborhoods of  $y \in Y$  have the form  $U \cap Z$ , where  $U \ni y$  is open in  $I^{\omega_1}$ .  $(Z, \tau)$  is Lindelöf:  $(Y, \tau \upharpoonright Y)$  is Lindelöf by Lemma 3.11, and by Lemma 3.13 and the definition of  $\tau$  we have that  $Z \setminus U$  is countable for any open  $U \in \tau$  containing  $Y$ . Now  $X \times Z$  is not Lindelöf when  $Z$  is considered with the topology  $\tau$  as  $\{(x_\alpha, x_\alpha) : \alpha < \omega_1\}$  is a closed discrete uncountable subspace of this product, a contradiction to  $X$  being productively Lindelöf.

#### 4. WEAKLY ALSTER SPACES

The inspiration for the results in this section is the characterization for Hurewicz spaces presented in [9], Theorem 6.

**Theorem 4.1** (Tall). *A regular Lindelöf space  $X$  is Hurewicz if and only if for every Čech-complete space  $G \supset X$  there is a  $\sigma$ -compact space  $K$  such that  $X \subset K \subset G$ .*

The standard definition for Hurewicz spaces is in terms of a selection principle involving  $\gamma$ -coverings. It was proved in [9, Theorem 7] that every regular Alster space is Hurewicz. Also, as in the previous section, the Alster property implies productively Lindelöf. In this section, we present a new definition, called weakly Alster, that is in between the Alster and productively Lindelöf properties. It is not known by us if this new property is actually equivalent to any of the other two in general. But one advantage is that the relation between weakly Alster and Hurewicz properties is immediate because of Theorem 4.1.

**Definition 4.2.** Let  $X$  be a regular space. We say that it is *weakly Alster* if for every covering  $\mathcal{G}$  made by  $G_\delta$  sets of  $\beta X$  such that  $\{G \cap X : G \in \mathcal{G}\}$  is an Alster covering for  $X$ , there is a  $\sigma$ -compact space  $Y \subset \beta X$  such that  $X \subset Y \subset \bigcup \mathcal{G}$ .

**Proposition 4.3.** *Every Alster space is weakly Alster.*

*Proof.* Let  $X$  be an Alster space. Let  $\mathcal{G}$  be a covering for  $X$  as in the definition of weakly Alster. We may suppose, taking a refinement if necessary, that for each  $G \in \mathcal{G}$  there is a sequence  $(A_n^G)_{n \in \omega}$  of open sets in  $\beta X$  such that  $G = \bigcap_{n \in \omega} A_n^G$  and  $\text{cl}_{\beta X}(A_{n+1}^G) \subset A_n^G$  for all  $n$ . In other words, we may assume that each  $G \in \mathcal{G}$  is compact. Since  $X$  is Alster and  $\{X \cap G : G \in \mathcal{G}\}$  is an Alster cover of  $X$ , there is a sequence  $(G_n)_{n \in \omega}$  of elements of  $\mathcal{G}$  such that  $X \subset \bigcup_{n \in \omega} G_n$ . This completes our proof.  $\square$

**Proposition 4.4.** *Every weakly Alster space is productively Lindelöf.*

*Proof.* Let  $X$  be a weakly Alster space and let  $Z$  be a Lindelöf space. Let  $\mathcal{C}$  be a basic open covering for  $X \times Z$ . We may suppose that for each compact  $K \subset X$  and each  $z \in Z$ , there are open sets  $A_{K,z}, B_{K,z}$  such that  $K \times \{z\} \subset A_{K,z} \times B_{K,z} \in \mathcal{C}$ . Also, we may suppose that each  $A_{K,z}$  is an open set in  $\beta X$ . Since  $Z$  is Lindelöf for each compact  $K \subset X$ , there is a sequence  $(z_n)_{n \in \omega}$  (depending on  $K$ ) such that  $K \times Z \subset \bigcup_{n \in \omega} A_{K,z_n} \times B_{K,z_n}$ . Let  $G_K = \bigcap_{n \in \omega} A_{K,y_n}$ . Let  $Y \subset \beta X$  be a  $\sigma$ -compact such that  $X \subset Y \subset \bigcup_{K \in \mathcal{K}(X)} G_K$ .

Fix  $(y, z) \in Y \times Z$ . Let  $K$  be such that  $y \in G_K$ . Let  $z_n$  be such that  $z \in B_{K,z_n}$ . Note that  $(y, z_n) \in A_{K,z_n} \times B_{K,z_n}$ . Since  $Y$  is  $\sigma$ -compact, there is a countable subcovering for  $Y \times Z$ . Note that this is also a countable subcovering for  $X \times Z$ .  $\square$

Since under CH, productively Lindelöf and Alster is the same for Tychonoff spaces with weight  $\leq \aleph_1$ , the following questions are natural.

**Question 4.5.** Is every productively Lindelöf Tychonoff space a weakly Alster space? Under CH?

**Question 4.6.** Is every weakly Alster space an Alster space under CH?

**Question 4.7.** Is every Tychonoff weakly Alster space with weight  $\leq \mathfrak{c}$  an Alster space?

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