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ON THE RING OF DIFFERENTIAL OPERATORS OF CERTAIN REGULAR DOMAINS

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Abstract. Let $(A, \mathfrak{m})$ be a complete equicharacteristic Noetherian domain of dimension $d + 1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a regular local ring. Let $\operatorname{Sing}(A)$, the singular locus of A, be defined by an ideal J in A. Note that $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d. We show that R contains naturally a field $\mathbb{L} \cong k((X))$ such that $D(R)$, the ring of $\mathbb{L}$-linear differential operators on R, is a left and right Noetherian ring of global dimension d. This enables us to prove Lyubeznik's conjecture on R regarding finiteness of associate primes of local cohomology modules of R.

1. Introduction

Let K be a field of characteristic zero and let R be a commutative Noetherian domain containing K as a subring. Let $D(R) = D_K(R)$ be the ring of K-*linear differential operators* on R. In general $D(R)$ does not have good properties. However in the following cases it is known that $D(R)$ is both left and right Noetherian with finite global dimension:

(1) $R = K[X_1, \ldots, X_n]$. In this case $D(R) = A_n(K)$, the n^{th}-Weyl algebra over K. We have that the global dimension of $D(R)$ is equal to n; see [2, Chapter 2, Theorem 3.15].
(2) $R = K[[X_1, \ldots, X_n]]$. In this case the global dimension of $D(R)$ is equal to n; see [2, Chapter 3, Proposition 1.8].
(3) Let $K = \mathbb{C}$ and let V be a non-singular affine K-variety. Let R be the co-ordinate ring of V. In this case global dimension of $D(R)$ is equal to $\dim V$; see [2, Chapter 3, Theorem 2.5].
(4) Let $R = \mathbb{C}\{z_1, \ldots, z_n\}$ be the local ring of convergent power series in n-variables. In this case the global dimension of $D(R)$ is equal to n; see [2, p. 197].

More generally for rings of differentiable type, $D(R)$ behaves well; see [11] and [12]. In this paper we describe a *new* infinite class of Noetherian domains R with $D(R)$ both left and right Noetherian and with finite global dimension.

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1.1. **Setup.** Let $(A, \mathfrak{m})$ be a complete equicharacteristic Noetherian domain of dimension $d+1 \geq 2$. Assume $k = A/\mathfrak{m}$ has characteristic zero and that A is not a regular local ring. We note that A contains a field isomorphic to k. For convenience we also denote it with k. Let $\operatorname{Sing}(A)$, the singular locus of A, be defined by an ideal J in A. Note that $J \neq 0$. Let $f \in J$ with $f \neq 0$. Set $R = A_f$. Then R is a regular domain of dimension d. In subsection 2.3 we show that R contains naturally a field $\mathbb{L} \cong k((X))$. Let $\operatorname{Der}_{\mathbb{L}}(R)$ be the set of $\mathbb{L}$-linear derivations of R. Let $D(R) = D_{\mathbb{L}}(R)$ be the ring of $\mathbb{L}$-linear differential operators on R. The main result of this paper is:

Theorem 1.2. *Let R and $\mathbb{L}$ be as in Setup* 1.1*.*

(1) *R is a domain such that* $\operatorname{height} \mathfrak{n} = d$ *for each maximal ideal $\mathfrak{n}$ of R.*
(2) *For each maximal ideal $\mathfrak{n}$ of R the residual field $R/\mathfrak{n}$ is a finite extension of $\mathbb{L}$.*
(3) $\operatorname{Der}_{\mathbb{L}}(R)$ *is a finitely generated projective R-module of rank d such that* $\operatorname{Der}_{\mathbb{L}}(R_{\mathfrak{n}}) = R_{\mathfrak{n}} \otimes_R \operatorname{Der}_{\mathbb{L}}(R)$ *for each maximal ideal $\mathfrak{n}$ of R.*

As an immediate corollary we obtain (using Theorem 1.1 of [12])

Corollary 1.3. *Let R and $\mathbb{L}$ be as in Setup* 1.1*. The ring $D(R)$ is a left and right Noetherian ring with global dimension d. Moreover, the Bernstein class of $D(R)$ is closed under localization at one element of R.*

We note that the first half of the above result also holds by [2, Chapter 2, Theorem 1.2].

1.4. **Application.** In [4], Huneke asked whether for any Noetherian ring T and ideal I, the set $\operatorname{Ass} H^i_I(T)$ is finite for all $i \geq 0$. In addition, he also stated that it is more reasonable to expect this for regular rings. In general, Huneke's question has a negative answer. Singh gave the first example of a singular ring R having an ideal I such that $\operatorname{Ass}_R H^i_I(R)$ is infinite; see [13]. In this example the ring R did not contain a field. Later Katzman (see [6]) gave an example of an affine algebra R over a field (and also a local ring containing a field) having an ideal I such that $\operatorname{Ass}_R H^i_I(R)$ is infinite. Later Singh and Swanson gave similar examples of a ring having only rational singularities; see [14]. Lyubeznik conjectured (see [8]), that if S is a regular ring and I is an ideal in S, then for any $i \geq 0$ the set $\operatorname{Ass}_S H^i_I(S)$ is finite. Previously this conjecture was known to be true in the following cases:

(1) S contains a field K with $\operatorname{char} K = p > 0$; see [5].
(2) S is local and contains a field K with $\operatorname{char} K = 0$; see [7].
(3) S is a regular affine K-algebra (here $\operatorname{char} K = 0$; see [7].
(4) S is an unramifed regular local ring; see [9, 17].
(5) S is a smooth $\mathbb{Z}$-algebra; see [1].

As an application of our results, we verify Lyubeznik's conjecture for regular rings satisfying Setup 1.1. We stress that there exist regular rings that satisfy Setup 1.1 and that are not included in the results previously known.

Corollary 1.5. *Let R and $\mathbb{L}$ be as in Setup* 1.1*. Let I be an ideal in R. Then for any $i \geq 0$ the set* $\operatorname{Ass}_R H^i_I(R)$ *is finite.*

Here is an overview of the contents of the paper. In section 2 we discuss some preliminaries that we need. In section 3 we discuss our result on ranks of certain modules of derivations. We prove Theorem 1.3, Corollary 1.3, and Corollary 1.5 in section 4. Finally in section 5 we give an infinite number of examples of non-isomorphic regular rings satisfying our hypothesis 1.1.

2. Preliminaries

2.1. Our running hypotheses will be as in Setup 1.1. In this section we prove some preliminary facts about R and A.

We first prove

Proposition 2.2. *Let R and $\mathbb{L}$ be as in Setup 1.1. Let $\mathfrak{n}$ be a maximal ideal in R. Then $\mathfrak{n} = \mathfrak{q}R$ where $\mathfrak{q}$ is a prime ideal of height d in A. In particular, $\dim R = d$.*

Proof. Suppose if possible there exists a prime ideal $\mathfrak{q}$ in A of height $\leq d-1$ such that $\mathfrak{n} = \mathfrak{q}R$ is a maximal ideal of R. Note $f \notin \mathfrak{q}$. As A is complete it is catenary. So height$(\mathfrak{m}/\mathfrak{q}) \geq 2$. In particular $\dim A/\mathfrak{q} \geq 2$. The image of f is non-zero in $A/\mathfrak{q}$. By [10, Theorem 31.2] we get that $A/\mathfrak{q}$ has infinitely many prime ideals of height 1. We can choose one, say $\overline{P} = P/\mathfrak{q}$, not containing $\overline{f}$. Thus P is a prime ideal in A not containing f, and P strictly contains $\mathfrak{q}$. It follows that $\mathfrak{n} = \mathfrak{q}R$ is not a maximal ideal of R, a contradiction. $\square$

2.3. Consider the map $\phi\colon k[[X]] \to A$ which maps k identically to k and X to f. As A is a domain it follows that ϕ is an injective map. Inverting X we get a map $\psi\colon k((X)) \to A_X$. Notice that $A_X = A_f = R$. Thus R naturally contains a field $\mathbb{L} \cong k((X))$. We also note that image $\phi = k[[f]]$ and $\mathbb{L} = k((f))$.

The following is a crucial ingredient to prove Theorem 1.3.

Lemma 2.4. *Let R and $\mathbb{L}$ be as in Setup 1.1. Let $\mathfrak{n}$ be a maximal ideal of R. Then $R/\mathfrak{n}$ is a finite extension of $\mathbb{L}$.*

Proof. By Proposition 2.2 we get that $\mathfrak{n} = \mathfrak{q}R$, where $\mathfrak{q}$ is a prime ideal of height d in A not containing f. The map $\phi\colon k[[X]] \to A$ as in subsection 2.3 descends to a map $\overline{\phi}\colon k[[X]] \to A/\mathfrak{q}$. Set $T = A/\mathfrak{q}$ and $S = k[[X]]$. As $(\mathfrak{q}, f)$ is $\mathfrak{m}$-primary in A we get that $T/XT = A/(\mathfrak{q}, f)$ is a finite dimensional k-vector space. We also get that

$$\bigcap_{n\geq 1} X^n T \subseteq \bigcap_{n\geq 1} f^n T \subseteq \bigcap_{n\geq 1} \mathfrak{m}^n T = 0.$$

Thus T is separated with respect to (X)-topology of S. It follows that T is a finite S-module; see [10, Theorem 8.4]. Therefore the quotient field of $A/\mathfrak{q}$ is a finite extension of the quotient field of S. The result follows. $\square$

We will use the next result in the next section.

Lemma 2.5. *Let $A, R,$ and $\mathbb{L}$ be as in Setup 1.1. Let $\mathfrak{q}$ be a prime of height d in A such that $f \notin \mathfrak{q}$. Let $\kappa(\mathfrak{q})$ be the residue field of $A_\mathfrak{q}$. Then there exists $y_1, \ldots, y_d \in \mathfrak{q}$ such that:*

(1) height$(f, y_1, \ldots, y_j) = j + 1$ *for $j = 0, \ldots, d$.*
(2) *The images of $y_1, \ldots, y_j$ in the $\kappa(\mathfrak{q})$-vector space $\mathfrak{q}A_\mathfrak{q}/\mathfrak{q}^2A_\mathfrak{q}$ is linearly independent for $j = 1, \ldots, d$.*

(3) $f, y_1, \dots, y_d$ *is a system of parameters of* A.
(4) $(y_1, \dots, y_d)A_\mathfrak{q} = \mathfrak{q}A_\mathfrak{q}$.

Proof. (1) and (2): As A is a domain we get that $\operatorname{height}(f) = 1$. Suppose $y_1, \dots, y_j$ is already chosen where $0 \le j < d$. We choose y_{j+1} as follows:

(a) Set

$$J = \big((y_1, \dots, y_j)A_\mathfrak{q} + \mathfrak{q}^2 A_\mathfrak{q}\big) \cap A.$$

Then $J \subseteq \mathfrak{q}$. We claim that $\mathfrak{q} \nsubseteq J$. If this is so we get $\mathfrak{q} = J$ and therefore

$$\mathfrak{q}A_\mathfrak{q} = (y_1, \dots, y_j)A_\mathfrak{q} + \mathfrak{q}^2 A_\mathfrak{q}, \text{ and so by Nakayama's Lemma } \mathfrak{q}A_\mathfrak{q} = (y_1, \dots, y_j)A_\mathfrak{q}.$$

This implies that $\dim A_\mathfrak{q} \le j < d$, a contradiction.

(b) Let $P_1, \dots, P_s$ be all the minimal primes of $(f, y_1, \dots, y_j)$ of height $j+1$. We claim that $\mathfrak{q} \nsubseteq P_i$ for all $i = 1, \dots, s$. We have to consider two cases:

Case (i). $j \le d-2$. Then as $\operatorname{height} P_i < d$ for all i, we get the result.

Case (ii). $j = d-1$. If $\mathfrak{q} \subseteq P_i$ for some i, then as both these prime ideals have height d we get $\mathfrak{q} = P_i$. We then get $f \in \mathfrak{q}$, a contradiction.

Using (a), (b), and prime avoidance, there exists

$$y_{j+1} \in \mathfrak{q} \setminus J \cup \left(\bigcup_{i=1}^{s} P_i\right).$$

Then note that $y_1, \dots, y_{j+1}$ satisfies the conditions of (1) and (2).

(3) This follows since by (1) we have height $(f, y_1, \dots, y_d) = d + 1 = \dim A$.

(4) As $A_\mathfrak{q}$ is a regular local ring of dimension d we get that $\mathfrak{q}A_\mathfrak{q}/\mathfrak{q}^2A_\mathfrak{q}$ is a d-dimensional $\kappa(\mathfrak{q})$-vector space. The result follows from (2). □

3. Ranks of modules of derivations

Let T, S be commutative Noetherian rings. Assume S is a T-algebra. Let $\operatorname{Der}_T(S)$ denote the set of T-linear derivations on S. The main goal of this section is to prove Theorem 3.4. As an easy consequence we get proofs of our results Theorem 1.2, Corollary 1.3, and Corollary 1.5. Our running hypotheses will be as in Setup 1.1.

We first prove:

Proposition 3.1. *Let A, R, and $\mathbb{L}$ be as in Setup* 1.1. *The A-module* $\operatorname{Der}_k(A)$ *is finitely generated with rank* $= d + 1$.

Proof. By [10, Theorem 30.7], $\operatorname{Der}_k(A)$ is a finitely generated A-module of rank $\le d + 1$. Let $A = Q/\mathfrak{q}$, where $Q = k[[x_1, \dots, x_n]]$ and $\mathfrak{q} \subseteq (x_1, \dots, x_n)^2$ is a prime ideal in Q. Let $r = \operatorname{height} \mathfrak{q}$. Then $n = d + 1 + r$.

Let T be a finitely generated A-module. By equation (6) in the proof of [10, Theorem 25.2] we get an exact sequence of A-modules:

$$0 \to \operatorname{Der}_k(A, T) \to \operatorname{Der}_k(Q, T) \to \operatorname{Hom}_A(\mathfrak{q}/\mathfrak{q}^2, T).$$

We note that $\mathrm{Der}_k(Q,T) \cong T^n$. Set $T = A$ in the above equation:

$$0 \to \mathrm{Der}_k(A) \to A^n \to \mathrm{Hom}_A(\mathfrak{q}/\mathfrak{q}^2, A).$$

We localize the above equation at (0). We note that $(\mathfrak{q}/\mathfrak{q}^2)_{(0)} \cong \mathfrak{q}Q_{\mathfrak{q}}/\mathfrak{q}^2 Q_{\mathfrak{q}} \cong \kappa(\mathfrak{q})^r$; here $\kappa(\mathfrak{q})$ is the residue field of $Q_{\mathfrak{q}}$ (this is so as $Q_{\mathfrak{q}}$ is a regular local ring of dimension r). Note that $\kappa(\mathfrak{q})$ is also the quotient field of A. So we have an exact sequence

$$0 \to \mathrm{Der}_k(A)_{(0)} \to \kappa(\mathfrak{q})^n \to \kappa(\mathfrak{q})^r.$$

Therefore $\operatorname{rank} \mathrm{Der}_k(A) \geq n - r = d + 1$. The result follows. □

To prove Theorem 3.4 we will need the following two easily proved facts:

Fact 3.2. Let K be a field of characteristic zero and let $S = K[[X_1, \ldots, X_n]]$. Let T be an S-module, not necessarily finitely generated, such that T is complete with respect to $(X_1, \ldots, X_n)$-adic topology. Then $\mathrm{Der}_K(S,T) \cong T^n$.

Fact 3.3. Let $R \subseteq S$ be an inclusion of Noetherian domains. Let I be an ideal in R such that R is complete with respect to I-adic topology. Let J be an ideal in S such that S is complete with respect to J-adic topology. Assume $IS \subseteq J$. Let $\{r_n\}$ be a convergent sequence in R (in the I-adic topology) with $r_n \to r$. Then $\{r_n\}$ considered as a sequence in S is convergent in the J-adic topology, and $\{r_n\}$ converges to r in S.

The following is the main result of this section:

Theorem 3.4. *Let $A, R,$ and $\mathbb{L}$ be as in Setup* 1.1. *Let T be the subring $k[[f]]$ of A. Consider* $\mathrm{Der}_T(A)$. *Then*

(1) $\mathrm{Der}_T(A)$ *is a finitely generated A-module and* $\operatorname{rank} \mathrm{Der}_T(A) \geq d$.
(2) $\mathrm{Der}_T(A)_f = \mathrm{Der}_{\mathbb{L}}(R)$. *In particular* $\mathrm{Der}_{\mathbb{L}}(R)$ *is finitely generated as an R-module.*
(3) *Let $\mathfrak{n}$ be a maximal ideal of R. Then*
  (a) $\mathrm{Der}_{\mathbb{L}}(R_{\mathfrak{n}}) = (\mathrm{Der}_{\mathbb{L}}(R))_{\mathfrak{n}}$.
  (b) $\mathrm{Der}_{\mathbb{L}}(R_{\mathfrak{n}})$ *is a free $R_{\mathfrak{n}}$-module of rank d.*
(4) $\mathrm{Der}_{\mathbb{L}}(R)$ *is a projective R-module of rank d.*

Proof. (1) Consider the inclusion of rings $k \subseteq T \subseteq A$. By equation (3) of the proof of [10, Theorem 25.1], for any A-module W we have the following exact sequence of A-modules:

$$0 \to \mathrm{Der}_T(A,W) \to \mathrm{Der}_k(A,W) \to \mathrm{Der}_k(T,W). \tag{3.4.1}$$

We now put $W = A$ in (3.4.1). Notice that $T \cong k[[X]]$. As A is complete with respect to $\mathfrak{m}$-adic topology it is also complete with respect to (f)-adic topology. So $\mathrm{Der}_k(T,A) \cong A$; see Fact 3.2. By Proposition 3.1 we get that $\mathrm{Der}_k(A)$ is finitely generated as an A-module and $\operatorname{rank} \mathrm{Der}_k(A) = d + 1$. The result follows from (3.4.1).

We need some work to prove the remaining assertions:

Claim 1. $\mathrm{Der}_T(A)_f \subseteq \mathrm{Der}_{\mathbb{L}}(R)$. In particular $\operatorname{rank} \mathrm{Der}_{\mathbb{L}}(R) \geq d$. *Note that we are not yet asserting that* $\mathrm{Der}_{\mathbb{L}}(R)$ *is finitely generated as an R-module.*

Remark. Let K be the quotient field of R. By the rank of a not-necessarily finitely generated R-module M, we mean the cardinality of a basis of the K-vector space $M \otimes_R K$.

It is well-known that $\operatorname{Der}_T(A)_f \subseteq \operatorname{Der}_T(R)$. Now $T = k[[f]]$ and f is invertible in R. Let $\delta \in \operatorname{Der}_T(R)$. We assert that it is $\mathbb{L} = k((f))$-linear. To see this let

$$v = \frac{1}{f^i} r \quad \text{for some } i \geq 1 \text{ and } r \in R.$$

Then $r = f^i v$. As δ is T-linear we get $\delta(r) = f^i \delta(v)$. It follows that

$$\delta(v) = \frac{1}{f^i}\delta(r).$$

Any $\xi \in \mathbb{L} \setminus T$ is of the form t/f^i where $t \in T$ and $i \geq 1$. By the previous argument we get that $\delta(\xi r) = \xi\delta(r)$ for any $r \in R$. Thus δ is $\mathbb{L}$-linear.

Now let $\mathfrak{n}$ be a maximal ideal of R. By Proposition 2.2 we get $\mathfrak{n} = \mathfrak{q}R$, where $\mathfrak{q}$ is a prime ideal in A of height d and $f \notin \mathfrak{q}$. By Lemma 2.5 there exists $y_1, \ldots, y_d \in \mathfrak{q}$ such that $f, y_1, \ldots, y_d$ is a system of parameters of A and $(y_1, \ldots, y_d)A_\mathfrak{q} = \mathfrak{q}A_\mathfrak{q}$. Set

$$V = k[[f, y_1, \ldots, y_d]] = T[[y_1, \ldots, y_d]]. \quad \text{Note that } V \cong k[[Y_0, Y_1, \ldots, Y_d]].$$

Also note that A is finitely generated as a V-module.

Claim 2. $\operatorname{Der}_\mathbb{L}(R_\mathfrak{n})$ is a free $R_\mathfrak{n}$-module of rank d. There also exists $\delta_i \in \operatorname{Der}_\mathbb{L}(R_\mathfrak{n})$ such that

$$\delta_i(y_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

(In particular $\delta_1, \ldots, \delta_d$ generate $\operatorname{Der}_\mathbb{L}(R_\mathfrak{n})$ as an $R_\mathfrak{n}$-module.)

We note that $(\operatorname{Der}_\mathbb{L}(R))_\mathfrak{n} \subseteq \operatorname{Der}_\mathbb{L}(R_\mathfrak{n})$. By Claim 1 we get $\operatorname{rank} \operatorname{Der}_\mathbb{L}(R_\mathfrak{n}) \geq d$ as an $R_\mathfrak{n}$-module. Using [10, Theorem 30.7] and Lemma 2.4 we get that $\operatorname{rank} \operatorname{Der}_\mathbb{L}(R_\mathfrak{n}) \leq d$ as an $R_\mathfrak{n}$-module. So $\operatorname{rank} \operatorname{Der}_\mathbb{L}(R_\mathfrak{n}) = d$. Set $z_i =$ image of y_i in $A_\mathfrak{q}$. As $R_\mathfrak{n} = A_\mathfrak{q}$ is regular local and $z_1, \ldots, z_d$ is a regular system of parameters of $A_\mathfrak{q}$, by [10, Theorem 30.6] we get that there also exists $\delta_i \in \operatorname{Der}_\mathbb{L}(R_\mathfrak{n})$ such that

$$\delta_i(z_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i, j \leq d. \tag{3.4.2}$$

The result follows since as A is a domain we get $A \subseteq A_\mathfrak{q}$.

By an argument similar to that in Claim 1 we get $\operatorname{Der}_T(A)_\mathfrak{q} \subseteq \operatorname{Der}_\mathbb{L}(A)_\mathfrak{q}$. More is true. In fact we show

Claim 3. For $i = 1, \ldots, d$ there exists $D_i \in \operatorname{Der}_T(A)$ such that $\delta_i = D_i/s_i$ for some $s_i \notin \mathfrak{q}$. In particular $\operatorname{Der}_T(A)_\mathfrak{q} = \operatorname{Der}_\mathbb{L}(A_\mathfrak{q})$.

We prove it for $i = 1$. The argument for $i > 1$ is similar. Set $W = T[y_1, \ldots, y_d]$. As δ_1 is T-linear and $\delta_1(y_j) = 1$ if $j = 1$ and 0 if $j > 1$ we get that restricted map $(\delta_1)_W \in \operatorname{Der}_T(W)$. We note that $W \cong T[Y_1, \ldots, Y_d]$ and $(\delta_1)_W$ is the usual differentiation with respect to Y_1.

We now show

Claim 4. $\delta_1(V) \subseteq V$.

Assume the claim for the moment. Now A is finitely generated as a V-module. Say $A = Va_1 + \cdots + Va_c$. Say $\delta_1(a_j) = u_j/t_j$, where $a_j, t_j \in A$, and $t_j \notin \mathfrak{q}$. Set $D_1 = s_1\delta_1$, where $s_1 = t_1 \cdots t_c$. Notice that $D_1(V) \subseteq A$. Also $D_1 \in \mathrm{Der}_{\mathbb{L}}(A_{\mathfrak{q}})$. Notice that $D_1(A) \subseteq A$ and D_1 is T-linear. Thus $(D_1)_A \in \mathrm{Der}_T(A)$, and so $D_1 = s_1\delta_1$ in $\mathrm{Der}_{\mathbb{L}}(A_{\mathfrak{q}})$. This proves Claim 3.

We now give a proof of Claim 4:

Set $K = R/\mathfrak{n} = \kappa(\mathfrak{q})$ as the residue field of $A_{\mathfrak{q}}$. Note that the $\mathfrak{q}A_{\mathfrak{q}}$ completion of $A_{\mathfrak{q}}$ is $\widehat{A_{\mathfrak{q}}} = K[[z_1, \ldots, z_d]]$. (Recall that z_i is the image of y_i in $A_{\mathfrak{q}}$.) Furthermore δ_1 extends to a K-linear derivation on $\widehat{A_{\mathfrak{q}}}$, and it is in fact differentiation with respect to z_1.

We now note that we have an inclusion of rings $V = T[[y_1, \ldots, y_d]] \subseteq \widehat{A_{\mathfrak{q}}}$. Furthermore V is complete with respect to $I = (y_1, \ldots, y_d)$ and $I\widehat{A_{\mathfrak{q}}} = \mathfrak{q}\widehat{A_{\mathfrak{q}}}$. Let $\xi \in V$. Write

$$\xi = \sum_{j \geq 0} t_j y_1^j \quad \text{where } t_j \in T[[y_2, \ldots, y_d]].$$

Set

$$\xi_n = \sum_{j=0}^{n} t_j y_1^j.$$

Notice that $\xi_n \in W$ and $\xi_n \to \xi$ in V (with respect to the I-adic topology on V). Set

$$\eta = \sum_{j \geq 1} j t_j y_1^{j-1} \quad \text{and} \quad \eta_n = \sum_{j=1}^{n} j t_j y_1^{j-1}.$$

We note that $\eta_n \to \eta$ in V.

By Fact 3.3 we get that $\xi_n \to \xi$ in $\widehat{A_{\mathfrak{q}}}$. As δ_1 is continuous with respect to $\mathfrak{q}\widehat{A_{\mathfrak{q}}}$-adic topology in $\widehat{A_{\mathfrak{q}}}$ we get that $\delta_1(\xi_n) \to \delta_1(\xi)$. Notice that $\delta_1(\xi_n) = \eta_n$. It follows (using Fact 3.3) that $\delta_1(\xi) = \eta$. Thus $\delta_1(V) \subseteq V$, and we have proved Claim 4.

(2) We have an inclusion of R-modules $\mathrm{Der}_T(A)_f \subseteq \mathrm{Der}_{\mathbb{L}}(R)$. If $\mathfrak{n} = \mathfrak{q}R$ is a maximal ideal in R (where $\mathfrak{q}$ is a height d prime ideal in A and $f \notin \mathfrak{q}$), then we have $\mathrm{Der}_{\mathbb{L}}(R)_{\mathfrak{n}} \subseteq \mathrm{Der}_{\mathbb{L}}(R_{\mathfrak{n}})$. Note that $R_{\mathfrak{n}} = A_{\mathfrak{q}}$, and by Claim 3 we have that $\mathrm{Der}_T(A)_{\mathfrak{q}} = \mathrm{Der}_{\mathbb{L}}(A_{\mathfrak{q}})$. In particular we have $(\mathrm{Der}_T(A)_f)_{\mathfrak{n}} = (\mathrm{Der}_{\mathbb{L}}(R))_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of R. Therefore $\mathrm{Der}_T(A)_f = \mathrm{Der}_{\mathbb{L}}(R)$.

(3)(a) This follows from (2) and Claim 3.

(3)(b) This follows from Claim 2 and 3(a).

(4) This follows from (3). □

Finally we give a proof of our results. We first give

Proof of Theorem 1.2. (1) As A is a domain we get that $R = A_f$ is a domain. Furthermore by Proposition 2.2 we get that height $\mathfrak{n} = d$ for each maximal ideal $\mathfrak{n}$ of R.

(2) This follows from Lemma 2.4.

(3) This follows from Theorem 3.4. □

We recall the following result from [12].

Theorem 3.5. *Let R be a regular commutative Noetherian ring with unity that contains a field F of characteristic 0 satisfying the following conditions:*

1. *R is equidimensional of dimension n,*
2. *every residual field with respect to a maximal ideal is an algebraic extension of F,*
3. *$\mathrm{Der}_F(R)$ is a finitely generated projective R-module of rank n such that $R_{\mathfrak{m}} \otimes_R \mathrm{Der}_F(R) = \mathrm{Der}_F(R_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ of R.*

Then, the ring of F-linear differential operators $D_F(R)$ is a ring of differentiable type of weak global dimension equal to $\dim(R)$. Moreover, the Bernstein class of $D_F(R)$ is closed under localization at one element of R.

Remark 3.6.

1. As an immediate corollary of Theorems 3.5 and 1.2 we get Corollary 1.3.
2. An immediate corollary of Corollary 1.3 is Corollary 1.5. For instance see section 4 of [12].

4. Examples

In this section we show that for each $d \geq 1$ there exist infinitely many examples of regular rings which satisfy our Setup 1.1. For simplicity we will assume that k is an algebraically closed field of characteristic zero. The author expects that there should be many more examples than those described in this section.

Example 4.1. Let $Q = k[x_1, \cdots, x_d, x_{d+1}]$ where $d \geq 2$. Set

$$S = \widehat{Q} = k[[x_1, \cdots, x_d, x_{d+1}]].$$

Let $n \geq 2$ be a positive integer, let ζ be a primitive n^{th}-root of unity, and let $G = \langle \zeta^i \colon 0 \leq i \leq n-1 \rangle$. Then G acts on both Q and S with the action $x_i \mapsto \zeta x_i$. Let $B = Q^G$ and let $A = S^G$. Note that $B \cong Q^{\langle n \rangle}$ the n^{th} Veronese subring of Q and that $A = \widehat{B}$ the completion of B at its irrelevant maximal ideal. As $\mathrm{Proj}(B)$ is smooth we get that A is an isolated singularity. It is well-known that $Cl(B)$, the class group of B, is $\mathbb{Z}/n\mathbb{Z}$ (for instance this follows from [15, Theorem 1.6]). As $\mathrm{Proj}(B)$ is smooth and $\dim B = d + 1 \geq 3$ we get that B satisfies the R_2 property of Serre. So by a result of Flenner (see [3]) $Cl(A) \cong Cl(B)$.

Let $f = x_1^n + \cdots + x_{d+1}^n$. As $d \geq 2$, it is well-known that f is irreducible in Q. Note that $f \in A$. Let $\mathfrak{m}$ be the maximal ideal of S. If T is a quotient ring of S, then set $G(T) = \bigoplus_{n \geq 0} \mathfrak{m}^n T/\mathfrak{m}^{n+1} T$ as the associated graded ring of T with respect to its maximal ideal $\mathfrak{m}T$. Note that $G(S/fS) \cong G(S)/fG(S) = Q/fQ$, which is a domain. So S/fS is a domain. In particular fS is a prime ideal in S. As $fA = fS \cap A$ we get that fA is a prime ideal in A.

Set $R_{n,d} = A_f$. By the localization sequence of class groups we have $Cl(R_{n,d}) = \mathbb{Z}/n\mathbb{Z}$. Also note that $\dim R_{n,d} = d \geq 2$.

In Example 4.1 we had the restriction that $d \geq 2$ and that R is not a UFD. Next we give infinitely many one-dimensional examples satisfying Setup 1.1. We also give infinitely many examples satisfying Setup 1.1 of dimension $d \geq 3$ which are also UFD's. We need to recall the notion of simple singularities.

4.2. **Simple singularities.** Let $S = k[[x, y, z_2, \dots, z_d]]$ with $d \geq 1$. Simple singularities are defined by the following equations:

$$(A_n) \quad x^2 + y^{n+1} + \sum_{j=2}^{d} z_j^2 \qquad (n \geq 1),$$

$$(D_n) \quad x^2 y + y^{n-1} + \sum_{j=2}^{d} z_j^2 \qquad (n \geq 4),$$

$$(E_6) \quad x^3 + y^4 + \sum_{j=2}^{d} z_j^2,$$

$$(E_7) \quad x^3 + xy^3 + \sum_{j=2}^{d} z_j^2,$$

$$(E_8) \quad x^3 + y^5 + \sum_{j=2}^{d} z_j^2.$$

4.3. Let $A = S/(f)$ be a simple singularity. Then A is an isolated singularity. In particular by a result due to Grothendieck, A is a UFD if $\dim A \geq 4$. We also note that if $d \geq 2$, then $A/(z_d)$ is a simple singularity of the same type.

4.4. **Grothendieck groups.** Let T be a commutative Noetherian ring and let $\operatorname{mod}(T)$ denote the category of all finitely generated T-modules. Let $\mathfrak{U}$ be an additive subcategory of $\operatorname{mod}(T)$ closed under extensions and let $\operatorname{Gr}(\mathfrak{U})$ denote the *Grothendieck group* of $\mathfrak{U}$. We recall the following three facts of Grothendieck groups that we need.

(1) Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local domain. Let $\mathfrak{C}$ be the additive subcategory of $\operatorname{mod}(A)$ consisting of all maximal Cohen-Macaulay A-modules. Then
   (a) The inclusion $i : \mathfrak{C} \to \operatorname{mod}(A)$ induces an isomorphism of Grothendieck groups $\operatorname{Gr}(\mathfrak{C})$ and $\operatorname{Gr}(\operatorname{mod}(A))$; cf. [16, 13.2].
   (b) The map $\operatorname{rk}\colon \operatorname{Gr}(\mathfrak{C}) \to \mathbb{Z}$ defined by $[M] \mapsto \operatorname{rank}(M)$ is a well-defined surjective group homomorphism. We have an isomorphism $\mathbb{Z} \oplus \ker \operatorname{rk} \to \operatorname{Gr}(\mathfrak{C})$ where $(1, 0) \mapsto [A]$.

(2) Let T be a regular ring of finite Krull dimension and let $K(T)$ be its K-group. Then the natural map $K(T) \to \operatorname{Gr}(\operatorname{mod}(T))$ is an isomorphism.

(3) Let $f \in T$. The sequence

$$\operatorname{Gr}(T/(f)) \xrightarrow{d_1} \operatorname{Gr}(T) \xrightarrow{d_0} \operatorname{Gr}(T_f) \to 0$$

is exact. Here

$$d_1([M]) = [M] \quad \text{and} \quad d_0([N]) = [N_f].$$

Remark 4.5. If f is T-regular, then note that the class of $[T/(f)]$ is zero in $\operatorname{Gr}(T)$. The reason is that we have an exact sequence $0 \to T \xrightarrow{f} T \to T/(f) \to 0$.

Remark 4.6. The Grothendieck groups of all simple singularities is known; see [16, 13.10]. We will only need the following fact: Let A be an A_n singularity of dimension l. Then

(1) If n is even, then $\mathrm{Gr}(A) = \mathbb{Z}$ if l is odd and is equal to $\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$ if l is even.
(2) If n is odd, then $\mathrm{Gr}(A) = \mathbb{Z}^2$ if l is odd and is equal to $\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$ if l is even.

Example 4.7. Let $S = k[[x, y, z_2, \ldots, z_d]]$ with $d \geq 2$ and let $A = S/(f)$ be an A_n-singularity with n *even*. Note that $\dim A = d + 1$. Set $R_{n,d} = A_{z_d}$. We note that if $\dim A \geq 4$, then A is a UFD and so $R_{n,d}$ is also a UFD.

Case 1. $\dim A = d + 1$ is *even*.

Consider the exact sequence

$$\mathrm{Gr}(A/(z_d)) \xrightarrow{d_1} \mathrm{Gr}(A) \xrightarrow{d_0} \mathrm{Gr}(R_{n,d}) \to 0.$$

Note that $A/(z_d)$ is an A_n singularity of dimension d. Also for all $d \geq 1$ the ring $A/(z_d)$ is a domain. By subsection 4.4 and Remark 4.6 we have that $\mathrm{Gr}(A/(z_d)) = \mathbb{Z}$ and is generated by the class of $A/(z_d)$. By subsection 4.4(3) it follows that $d_1 = 0$. It follows that

$$\mathbb{Z} \oplus \mathbb{Z}/(n+1) = \mathrm{Gr}(A) \cong \mathrm{Gr}(R_{n,d}) \cong K(R_{n,d}).$$

Case 2. $\dim A = d + 1$ is *odd*.

We again consider the exact sequence

$$\mathrm{Gr}(A/(z_d)) \xrightarrow{d_1} \mathrm{Gr}(A) \xrightarrow{d_0} \mathrm{Gr}(R_{n,d}) \to 0.$$

We again assert that $d_1 = 0$. Notice that $\mathrm{Gr}(A/(z_d)) = \mathbb{Z} \oplus \mathbb{Z}/(n+1)$ and $\mathrm{Gr}(A) = \mathbb{Z}$. Notice that $d_1(\mathbb{Z}/(n+1)) = 0$. By subsection 4.4(1)(b) the element $(1, 0)$ of $\mathrm{Gr}(A/(z_d))$ is generated by the class of $A/(z_d)$. By subsection 4.4(3) we get $d_1([A/(z_d)]) = 0$. Thus again $d_1 = 0$. So we have

$$\mathbb{Z} = \mathrm{Gr}(A) \cong \mathrm{Gr}(R_{n,d}) \cong K(R_{n,d}).$$

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