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BEHAVIOR OF THE SQUEEZING FUNCTION NEAR H-EXTENDIBLE BOUNDARY POINTS

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ABSTRACT. It is shown that if the squeezing function tends to one at an h-extendible boundary point of a $\mathcal{C}^\infty$-smooth, bounded pseudoconvex domain, then the point is strictly pseudoconvex.

Denote by $\mathbb{B}_n$ the unit ball in $\mathbb{C}^n$. Let M be an n-dimensional complex manifold, and $z \in M$. For any holomorphic embedding $f : M \to \mathbb{B}_n$ with $f(z) = 0$, set

$$s_M(f, z) = \sup\{r > 0 : r\mathbb{B}_n \subset f(M)\}.$$

The squeezing function of M is defined by $s_M(z) = \sup\limits_f s_M(f, z)$ if such f's exist, and $s_M(z) = 0$ otherwise.

Many properties and applications of the squeezing function have been explored by various authors; see e.g. [3, 5, 6] and the references therein.

It was shown in [3] that if D is a $\mathcal{C}^2$-smooth strictly pseudoconvex domain in $\mathbb{C}^n$, then

$$\lim_{z \to \partial D} s_D(z) = 1. \tag{1}$$

A. Zimmer [10] proved the converse if D is a $\mathcal{C}^\infty$-smooth, bounded convex domain; namely, if (1) holds, then D is necessarily strictly pseudoconvex. Recently, he extended this result to the $\mathcal{C}^{2,\alpha}$-smooth case [11].

On the other hand, J.E. Fornæss and E.F. Wold [5] provided an example showing that $\mathcal{C}^2$-smoothness is not enough. They also asked if Zimmer's result holds for $\mathcal{C}^\infty$-smooth, bounded pseudoconvex domains.

S. Joo and K.-T. Kim [6] gave an affirmative answer for domains of finite type in $\mathbb{C}^2$.

This can be extended to a larger class of domains by using different arguments. Recall that a $\mathcal{C}^\infty$-smooth boundary point a of finite type of a domain D in $\mathbb{C}^n$ is said to be h-extendible [8, 9] (or semiregular [4]) if D is pseudoconvex near a and Catlin and D'Angelo's multitypes of a coincide.

For example, a is extendible if the Levi form at a has a corank at most one [9] or D is linearly convexifiable near a [1]. In particular, h-extendibility takes place in the strictly pseudoconvex, two-dimensional finite type and convex finite type cases.

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Theorem 1. *Let a be an h-extendible boundary point of a $\mathcal{C}^\infty$-smooth, bounded pseudoconvex domain D in $\mathbb{C}^n$. If $s_D(a_j) \to 1$ for a nontangential sequence $a_j \to a$, then a is a strictly pseudoconvex point.*

Nontangentiality means that $\liminf\limits_{j\to\infty} \dfrac{d_D(a_j)}{|a_j - a|} > 0$, where d_D is the distance to ∂D.

Before proving Theorem 1, we need some preparation.

Denote by $\mu = (m_1, m_2, \dots, m_n)$ Catlin's multitype of a ($m_1 = 1$ and $m_2 \le \dots \le m_n$ are even numbers). By [4, 8, 9], there exists a local change of variables $w = \Phi(z)$ near a such that $\Phi(a) = 0$, $J\Phi(a) = 1$,

$$r(\Phi^{-1}(w)) = \operatorname{Re}(w_1) + P(w') + o(\sigma(w)),$$

where r is the signed distance to ∂D, $\sigma(z) = \sum_{j=1}^n |w_j|^{m_j}$, and P is a $1/\mu$-homogeneous polynomial without pluriharmonic terms. Moreover, the so-called model domain

$$E = \{w \in \mathbb{C}^n : \operatorname{Re}(w_1) + P(w') < 0\}$$

(which depends on Φ) is of finite type.

In [7, 9], the nontangential boundary behavior of the Kobayashi-Royden and Carathéodory-Reiffen metrics of D near a are expressed in terms of r, Φ, and the respective metrics of E_Φ at its interior point $e = (-1, 0')$. Obvious modifications in the proofs of these results allows us to obtain similar results for the Kobayashi-Eisenman and Carathéodory-Eisenman volumes of D :

$$\mathcal{K}_D(u) = \inf\{|Jf(0)|^{-1} : f \in \mathcal{O}(\mathbb{B}^n, D), f(0) = u\},$$
$$\mathcal{C}_D(u) = \sup\{|Jf(u)| : f \in \mathcal{O}(D, \mathbb{B}^n), f(u) = 0\}.$$

Proposition 2. *Let a be an h-extendible boundary point of a domain D in $\mathbb{C}^n$. Let μ be Catlin's multitype of a and let $m = \sum_{j=1}^n \frac{1}{m_j}$. Then*

$$\mathcal{K}_D(a_j)(d_D(a_j))^m \to \mathcal{K}_E(e) \tag{2}$$

for any nontangential sequence $a_j \to a$.

If, in addition, D is $\mathcal{C}^\infty$-smooth, bounded pseudoconvex, then

$$\mathcal{C}_D(a_j)(d_D(a_j))^m \to \mathcal{C}_E(e). \tag{3}$$

Since E is hyperbolic with respect to the Carathéodory-Reiffen metric [7], it is easy to see that $\mathcal{C}_E > 0$. So, the limits above are positive.

Sketch of the proof of Proposition 2. Let $\varepsilon > 0$ and let

$$E_{\pm\varepsilon} = \{w \in \mathbb{C}^n : \operatorname{Re}(w_1) + P(w') \pm \varepsilon\sigma(w) < 0\}.$$

There exists a neighborhood U_ε of a such that

$$E_{+\varepsilon} \cap V_\varepsilon \subset \Phi(D \cap U_\varepsilon) \subset E_{-\varepsilon} \cap V_\varepsilon,$$

where $V_\varepsilon = \Phi(U_\varepsilon)$. Since $a \in \partial D$ is a local holomorphic peak point [4, 8], the localization $\dfrac{\mathcal{K}_{D\cap U_\varepsilon}(a_j)}{K_D(a_j)} \to 1$ holds. On the other hand, $E_{\pm\varepsilon}$ are taut domains if $\varepsilon \le \varepsilon_0$ [9]; in particular, $\mathcal{K}_{E_{\pm\varepsilon}}$ are continuous functions. Let $-c_j + id_j$ be the first coordinate of b_j ($c_j, d_j \in \mathbb{R}$). Since d_j/c_j is a bounded sequence, it suffices to show (2) when $d_j/c_j \to s$. Set $b_j = \Phi(a_j)$ and $\pi_j(w) = (w_1 c_j^{-1/m_1}, \dots, w_n c_j^{-1/m_n})$. Note that $\pi_j(b_j) \to e_s := (-1+is, 0')$. Now, applying the scaling of coordinates π_j and using normal family arguments, we obtain that $\mathcal{K}_{E_{\pm\varepsilon}\cap V_\varepsilon}(b_j)c_j^m \to \mathcal{K}_{E_{\pm\varepsilon}}(e_s)$. Finally, following [9, Theorem 2.1], one can prove that $\mathcal{K}_{E_{\pm\varepsilon}}(e_s) \to \mathcal{K}_E(e)$ as $\varepsilon \to 0$. These facts, together with $\dfrac{c_j}{d_D(a_j)} \to 1$, imply (2).

The proof of (3) follows by similar but more delicate arguments as in [7]. □

Proof of Theorem 1. By [3], one has that

$$(s_D(a_j))^n \mathcal{K}_D(a_j) \le \mathcal{C}_D(a_j) \le \mathcal{K}_D(a_j).$$

It follows by Proposition 2 that

$$\mathcal{C}_E(e) = \mathcal{K}_E(e).$$

Since $\mathbb{B}_n$ and E are taut domains [8, 9], there exist extremal functions for $\mathcal{C}_E(e)$ and $\mathcal{K}_E(e)$. Then the Carathéodory-Cartan-Kaup-Wu theorem implies that E and $\mathbb{B}_n$ are biholomorphic. Since E is a model domain of finite type, the main result in [2] shows that $m_2 = \dots = m_n = 2$; that is, a is a strictly pseudoconvex point. □

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