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NEUMANN ISOPERIMETRIC CONSTANT ESTIMATE FOR CONVEX DOMAINS

XIANZHE DAI, GUOFANG WEI, AND ZHENLEI ZHANG

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ABSTRACT. We present a geometric and elementary proof of the local Neumann isoperimetric inequality on convex domains of a Riemannian manifold with Ricci curvature bounded below.

1. Introduction

Isoperimetric and Sobolev inequalities are equivalent inequalities (see e.g. Theorem 1.3 below) which play an important role in geometric analysis on manifolds. Indeed, doing analysis on manifolds usually depends on the estimate of the Sobolev constant which could then be obtained via the isoperimetric constant. There is extensive work on isoperimetric constant estimates. An important method pioneered by Gromov relies on the geometric measure theory and its regularity theory, which works for closed manifolds or convex domains with smooth boundary; see e.g. the survey article [10] and the recent paper [15]. One may also obtain an estimate through the Li-Yau gradient estimate for the heat kernel [13] and the equivalence of heat kernel bounds, Sobolev inequality, isoperimetric inequality; see [18, page 448], which again requires smooth and convex boundary. Another method using needle decomposition from convex geometry has also been very successful and, very recently, has been combined with optimal transport and extended to the non-smooth case; see [4] and the references therein. For star-shaped domains in a manifold with Ricci curvature bounded from below, Buser [3] gave an elementary proof for a Neumann isoperimetric constant (the Cheeger constant) estimate using comparison geometry, but the estimate depends on the in and out radius of the domain, which does not give a uniform estimate for convex domain as the in-radius might be small. In the presence of positive Ricci lower bound, see [2, 16, 17]; in particular [16] also treats the non-smooth boundary in that setting. For general convex domains with non-smooth boundaries, the estimate for the isoperimetric constant is only obtained in the very recent paper [4] mentioned above. In this short note we give a very geometric and elementary proof of a Neumann isoperimetric inequality, albeit with non-optimal constant, for convex domains whose boundaries need not be smooth.

First we recall some definitions.

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Definition 1.1. When M is compact (with or without boundary), the Neumann α-isoperimetric constant of M is defined by

$$\mathrm{IN}_\alpha(M) = \sup_H \frac{\min\{\mathrm{vol}(M_1), \mathrm{vol}(M_2)\}^{1-\frac{1}{\alpha}}}{\mathrm{vol}(H)},$$

where H varies over the compact $(n-1)$-dim submanifold of M which divides M into two disjoint open submanifolds M_1, M_2 (with or without boundary).

Definition 1.2. The Neumann α-Sobolev constant of M is defined by

$$\mathrm{SN}_\alpha(M) = \sup_{f \in C^\infty(M)} \frac{\inf_{a\in\mathbb{R}} \|f-a\|_{\frac{\alpha}{\alpha-1}}}{\|\nabla f\|_1}.$$

The isoperimetric constant and Sobolev constant are equivalent.

Theorem 1.3 ([5]; see also [12])**.** *For all $n \leq \alpha \leq \infty$,*

$$\mathrm{IN}_\alpha(M) \geq \mathrm{SN}_\alpha(M) \geq \frac{1}{2}\,\mathrm{IN}_\alpha(M).$$

For convenience we consider the normalized Neumann α-isoperimetric and α-Sobolev constant:

$$\mathrm{IN}^*_\alpha(M) = \mathrm{IN}_\alpha(M)\ \mathrm{vol}(M)^{1/\alpha}, \qquad \mathrm{SN}^*_\alpha(M) = \mathrm{SN}_\alpha(M)\ \mathrm{vol}(M)^{1/\alpha}.$$

Using comparison geometry and Vitali covering we give an estimate on the normalized Neumann isoperimetric constant for convex domain in terms of the Ricci curvature lower bound and the diameter of the domain.

Theorem 1.4. *Let (M,g) be a complete Riemannian manifold of dimension n, with $\mathrm{Ric} \geq -(n-1)K$ for some $K \geq 0$. Let Ω be a bounded convex domain. Then*

$$\mathrm{IN}^*_n(\Omega) \leq 40^n e^{11(n-1)\sqrt{K}d} \cdot d, \tag{1.1}$$

where d is the diameter of the domain Ω. In particular, if M is closed with diameter d, then

$$\mathrm{IN}^*_n(M) \leq 40^n e^{11(n-1)\sqrt{K}d} \cdot d. \tag{1.2}$$

Corollary 1.5. *Let (M,g) be a complete Riemannian manifold of dimension n, with non-negative Ricci curvature. Let Ω be a bounded convex domain. Then*

$$\mathrm{IN}^*_n(\Omega) \leq 40^n \cdot d, \tag{1.3}$$

where d is the diameter of the domain Ω. In particular, if M is closed with diameter d, then

$$\mathrm{IN}^*_n(M) \leq 40^n \cdot d. \tag{1.4}$$

Remark 1.6. The case when Ω equals the whole manifold is well-known. The reference we mentioned earlier for convex domain in the literature deals with domains with (smooth) convex boundary which is a stronger condition.

Remark 1.7. For balls we can obtain both Dirichlet and Neumann isoperimetric constant estimates even under the much weaker integral Ricci lower bound assumption [8,21]. On the other hand it is not clear if that will remain true for convex domains.

Remark 1.8. Using the mean curvature estimate from [19] one gets a similar estimate when the Bakry-Emery Ricci curvature is bounded from below and oscillation of the potential function is bounded.

2. Proof of Theorem 1.4

The proof goes by a covering argument of Anderson [1], combined with an observation of Gromov [11]. See [1] or [8] for a similar argument of estimating the local Dirichlet isoperimetric constant. First of all we recall a lemma whose proof is a slight modification of Gromov's observation [11, 5.(C)].

Lemma 2.1. *Let M^n be a complete Riemannian manifold. Let Ω be a convex domain of M and let H be any hypersurface dividing Ω into two parts Ω_1, Ω_2. For any Borel subsets $W_i \subset \Omega_i$, there exists x_1 in one of W_i, say W_1, and a subset W in another one, W_2, such that*

$$\mathrm{vol}(W) \geq \frac{1}{2}\,\mathrm{vol}(W_2) \tag{2.1}$$

and any $x_2 \in W$ has a unique minimal geodesic connecting to x_1 which intersects H at some z such that

$$\mathrm{dist}(x_1, z) \geq \mathrm{dist}(x_2, z). \tag{2.2}$$

The convexity assumption of Ω is essential. It implies that any minimal geodesic with endpoints in different parts must intersect H. The Bishop-Gromov relative volume comparison theorem gives the following:

Lemma 2.2. *Let H, W, and x_1 be as in the lemma above. Then*

$$\mathrm{vol}(W) \leq 2^{n-1} D e^{(n-1)\sqrt{K}D}\,\mathrm{vol}(H'), \tag{2.3}$$

where $D = \sup_{x \in W} \mathrm{dist}(x_1, x)$ and H' is the set of intersection points with H of geodesics $\gamma_{x_1,x}$ for all $x \in W$.

Proof. Let $\Gamma \subset S_{x_1}$ be the set of unit vectors such that $\gamma_v = \gamma_{x_1,x_2}$ for some $x_2 \in W$. We compute the volume in the polar coordinate at x_1. Write $dv = \mathcal{A}(\theta, t)d\theta \wedge dt$ in the polar coordinate $(\theta, t) \in S_{x_1} \times \mathbb{R}^+$. For any $\theta \in \Gamma$, let $r(\theta)$ be the radius such that $\exp_{x_1}(r\theta) \in H$. Then $W \subset \{\exp_{x_1}(r\theta) | \theta \in \Gamma,\, r(\theta) \leq r \leq 2r(\theta)\}$. So, by relative volume comparison,

$$\begin{aligned}
\mathrm{vol}(W) &\leq \int_\Gamma \int_{r(\theta)}^{2r(\theta)} \mathcal{A}(\theta, t)\,dt\,d\theta \\
&\leq \frac{\sinh^{n-1}(2\sqrt{K}D)}{\sinh^{n-1}(\sqrt{K}D)} \int_\Gamma r(\theta)\mathcal{A}(\theta, r(\theta))\,d\theta \\
&\leq D\frac{\sinh^{n-1}(2\sqrt{K}D)}{\sinh^{n-1}(\sqrt{K}D)}\,\mathrm{vol}(H').
\end{aligned}$$

The required estimate follows from $\frac{\sinh(2t)}{\sinh t} = 2\cosh t \leq 2e^t$ whenever $t \geq 0$. □

Corollary 2.3. *Let H be any hypersurface dividing a convex domain Ω into two parts Ω_1, Ω_2. For any ball $B = B_r(x)$ we have*

$$\min\big(\mathrm{vol}(B \cap \Omega_1), \mathrm{vol}(B \cap \Omega_2)\big) \leq 2^{n+1} r e^{(n-1)\sqrt{K}d}\,\mathrm{vol}(H \cap B_{2r}(x)) \tag{2.4}$$

where $d = \mathrm{diam}(\Omega)$. In particular, if $B \cap \Omega$ is divided equally by H, we have

$$\mathrm{vol}(B_r(x) \cap \Omega) \leq 2^{n+2} r e^{(n-1)\sqrt{K}d}\,\mathrm{vol}(H \cap B_{2r}(x)). \tag{2.5}$$

Proof. Put $W_i = B \cap \Omega_i$ in the above lemma and notice that $D \leq 2r$ and $H' \subset H \cap B_{2r}(x)$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1.4. We may assume that $\mathrm{vol}(\Omega_1) \leq \mathrm{vol}(\Omega_2)$. For any $x \in \Omega_1$, let r_x be the smallest radius such that

$$\mathrm{vol}(B_{r_x}(x) \cap \Omega_1) = \mathrm{vol}(B_{r_x}(x) \cap \Omega_2) = \frac{1}{2}\,\mathrm{vol}(B_{r_x}(x) \cap \Omega).$$

Let $d = \mathrm{diam}(\Omega)$. By the above corollary,

$$\mathrm{vol}(B_{r_x}(x) \cap \Omega) \leq 2^{n+2} r_x e^{(n-1)\sqrt{K}d}\,\mathrm{vol}(H \cap B_{2r}(x)). \tag{2.6}$$

The domain Ω_1 has a covering

$$\Omega_1 \subset \bigcup_{x \in \Omega_1} B_{2r_x}(x).$$

By the Vitali Covering Lemma (cf. [14, Section 1.3]), we can choose a countable family of disjoint balls $B_i = B_{2r_{x_i}}(x_i)$ such that $\bigcup_i B_{10r_{x_i}}(x_i) \supset \Omega_1$. Applying the relative volume comparison theorem and the convexity of Ω we have

$$\begin{aligned}
\mathrm{vol}(\Omega_1) &\leq \sum_i \frac{\int_0^{10r_{x_i}} \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^{r_{x_i}} \sinh^{n-1}(\sqrt{K}t)dt}\,\mathrm{vol}\left(B_{r_{x_i}}(x_i) \cap \Omega_1\right) \\
&\leq 10 \sum_i \frac{\sinh^{n-1}(10\sqrt{K}r_{x_i})}{\sinh^{n-1}(\sqrt{K}r_{x_i})}\,\mathrm{vol}\left(B_{r_{x_i}}(x_i) \cap \Omega_1\right) \\
&\leq 10 \frac{\sinh^{n-1}(10\sqrt{K}d)}{\sinh^{n-1}(\sqrt{K}d)} \sum_i \mathrm{vol}\left(B_{r_{x_i}}(x_i) \cap \Omega_1\right) \\
&\leq 10^n e^{9(n-1)\sqrt{K}d} \sum_i \mathrm{vol}\left(B_{r_{x_i}}(x_i) \cap \Omega_1\right) \\
&= 2^{-1} \cdot 10^n \cdot e^{9(n-1)\sqrt{K}d} \sum_i \mathrm{vol}\left(B_{r_{x_i}}(x_i) \cap \Omega\right).
\end{aligned}$$

Moreover, since the balls B_i are disjoint, (2.6) gives

$$\mathrm{vol}(H) \geq \sum_i \mathrm{vol}(B_i \cap H) \geq 2^{-n-2} e^{-(n-1)\sqrt{K}d} \sum_i r_{x_i}^{-1}\,\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega).$$

These two estimates lead to

$$\begin{aligned}
\frac{\mathrm{vol}(\Omega_1)^{\frac{n-1}{n}}}{\mathrm{vol}(H)} &\leq 2 \cdot 20^n e^{10(n-1)\sqrt{K}d} \frac{\left(\sum_i \mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)\right)^{\frac{n-1}{n}}}{\sum_i r_{x_i}^{-1}\,\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \\
&\leq 40^n e^{10(n-1)\sqrt{K}d} \frac{\sum_i \mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{\frac{n-1}{n}}}{\sum_i r_{x_i}^{-1}\,\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \\
&\leq 40^n e^{10(n-1)\sqrt{K}d} \sup_i \frac{\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{\frac{n-1}{n}}}{r_{x_i}^{-1}\,\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)} \\
&= 40^n e^{10(n-1)\sqrt{K}d} \sup_i \left(\frac{r_{x_i}^n}{\mathrm{vol}(B_{r_{x_i}}(x_i) \cap \Omega)}\right)^{\frac{1}{n}}.
\end{aligned}$$

On the other hand, since $\text{vol}(\Omega_1) \le \text{vol}(\Omega_2)$, we have $r_x \le d$ for any $x \in \Omega_1$. Thus, by the relative volume comparison and convexity of Ω again, we have

$$\text{vol}(\Omega) \le \frac{\int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^{r_x} \sinh^{n-1}(\sqrt{K}t)dt} \text{vol}(B_{r_x}(x) \cap \Omega).$$

Therefore,

$$\text{vol}(\Omega)^{\frac{1}{n}} \cdot \frac{\text{vol}(\Omega_1)^{\frac{n-1}{n}}}{\text{vol}(H)} \le 40^n e^{10(n-1)\sqrt{K}d} \sup_{0<r\le d} \left(\frac{r^n \int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^r \sinh^{n-1}(\sqrt{K}t)dt} \right)^{\frac{1}{n}}.$$

The last term on the right hand side has the estimate

$$\begin{aligned} \frac{r^n \int_0^d \sinh^{n-1}(\sqrt{K}t)dt}{\int_0^r \sinh^{n-1}(\sqrt{K}t)dt} &\le r^n \cdot \frac{d}{r} \cdot \frac{\sinh^{n-1}(\sqrt{K}d)}{\sinh^{n-1}(\sqrt{K}r)} \le d^n \cdot \frac{\sinh^{n-1}(\sqrt{K}d)}{(\sqrt{K}d)^{n-1}} \\ &\le d^n e^{(n-1)\sqrt{K}d}. \end{aligned}$$

The required normalized Neumann isoperimetric constant estimate now follows. □

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Department of Mathematics, East China Normal University, Shanghai, 200062 People's Republic of China – and – University of California Santa Barbara, Santa Barbara, California 93106

Email address: `dai@math.ucsb.edu`

Department of Mathematics, University of California Santa Barbara, Santa Barbara, California 93106

Email address: `wei@math.ucsb.edu`

Department of Mathematics, Capital Normal University, Beijing, 10048 People's Republic of China

Email address: `zhleigo@aliyun.com`