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BRILL–NOETHER LOCI OF RANK 2 VECTOR BUNDLES ON A GENERAL ν-GONAL CURVE

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Abstract. In this paper we study the Brill Noether locus of rank 2, (semi)stable vector bundles with at least two sections and of suitable degrees on a general ν-gonal curve. We classify its reduced components whose dimensions are at least the corresponding Brill–Noether number. We moreover describe the general member $\mathcal{F}$ of such components only in terms of extensions of line bundles with suitable *minimality properties*, providing information on the birational geometry of such components as well as on the very ampleness of $\mathcal{F}$.

1. Introduction

Let C denote a smooth, irreducible, complex projective curve of genus $g \geq 2$. As in the statement of [10, Theorem] (cf. also Theorem 1.1 below), C is said to be *general* if C is a curve with general moduli (cf., e.g., [2], pp. 214–215). Let $U_C(n,d)$ be the moduli space of semistable, degree d, rank n vector bundles on C and let $U_C^s(n,d)$ be the open dense subset of stable bundles (when d is odd, more precisely one has $U_C(n,d) = U_C^s(n,d)$). Let $B^k_{n,d} \subseteq U_C(n,d)$ be the *Brill–Noether locus* which consists of vector bundles $\mathcal{F}$ having $h^0(\mathcal{F}) \geq k$, for a positive integer k.

Traditionally, we denote by W^k_d the Brill–Noether locus $B^{k+1}_{1,d}$ of line bundles $L \in \mathrm{Pic}^d(C)$ having $h^0(L) \geq k+1$, for a nonnegative integer k. With little abuse of notation, we will sometimes identify line bundles with corresponding divisor classes, interchangeably using multiplicative and additive notation.

For the case of rank 2 vector bundles, we simply put $B^k_d := B^k_{2,d}$, for which it is well known that the dimension of $B^k_d \cap U^s_C(2,d)$ is at least the Brill–Noether number $\rho^k_d := 4g-3-ik$, where $i := k+2g-2-d$ (cf. [9]). This is no longer true for possible components of B^k_d in $U_C(2,d) \setminus U^s_C(2,d)$, i.e., not containing stable points, which can occur only for d even (cf. [3, Remark 3.3] for more explanations and details).

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In the range $0 \leq d \leq 2g-2$, B^1_d has been deeply studied on any curve C by several authors (cf. [6,9]). Concerning B^2_d, using a degeneration argument, N. Sundaram [9] proved that B^2_d is nonempty for any C and for odd d such that $g \leq d \leq 2g-3$. M. Teixidor I Bigas generalizes Sundaram's result as follows.

Theorem 1.1 ([10]). *Given a nonsingular curve C and a d, $3 \leq d \leq 2g-1$, $B^2_d \cap U^s_C(2,d)$ has a component of dimension $\rho^2_d = 2d-3$ and a generic point on it corresponds to a vector bundle whose space of sections has dimension* 2 *and the generic section has no zeroes. If C is general, this is the only component of $B^2_d \cap U^s_C(2,d)$. Moreover, $B^2_d \cap U^s_C(2,d)$ has extra components if and only if W^1_n is nonempty and* $\dim W^1_n \geq d+2n-2g-1$ *for some n with $2n < d$.*

Inspired by Theorem 1.1, in this paper we focus on B^2_d for C a *general ν-gonal curve* of genus g, i.e., C corresponds to a general point of the ν-gonal stratum $\mathcal{M}^1_{g,\nu} \subset \mathcal{M}_g$. Precisely, we prove the following.

Theorem 1.2. *Let C be a general ν-gonal ($3 \leq \nu \leq \frac{g+8}{4}$) curve of genus g and let A be the unique line bundle of degree ν and $h^0(A)=2$. For any positive integer d with $2+2\nu \leq d \leq g-3$, the reduced components of B^2_d having dimension at least ρ^2_d are only two, which we denote by $B_{\rm reg}$ and $B_{\rm sup}$:*

(i) *$B_{\rm reg}$ is generically smooth, of dimension $\rho^2_d = 2d-3$ (*regular *for short). Moreover, $\mathcal{F}$ general in $B_{\rm reg}$ is stable, fitting in an exact sequence*
$$0 \to \mathcal{O}_C(p) \to \mathcal{F} \to L \to 0,$$
where $p \in C$ and $L \in W^0_{d-1}$ are general and where $h^0(\mathcal{F})=2$.

(ii) *$B_{\rm sup}$ is generically smooth, of dimension $d+2g-2\nu-2 > \rho^2_d$ (*superabundant *for short). Moreover, $\mathcal{F}$ general in $B_{\rm sup}$ is stable, fitting in an exact sequence*
$$0 \to A \to \mathcal{F} \to L \to 0,$$
where L is a general line bundle of degree $d-\nu$ and $h^0(\mathcal{F})=2$.

A more precise statement of this result is given in Theorem 3.1 for its *residual* version (i.e., concerning the isomorphic Brill–Noether locus B^{2g-d}_{4g-4-d}). Indeed, for any nonnegative integer i, if one sets $k_i := d-2g+2+i$ and
$$B^{k_i}_d := \{\mathcal{F} \in U_C(2,d) \mid h^0(\mathcal{F}) \geq k_i\} = \{\mathcal{F} \in U_C(2,d) \mid h^1(\mathcal{F}) \geq i\},$$
one has natural isomorphisms $B^{k_i}_d \simeq B^i_{4g-4-d}$, arising from the correspondence $\mathcal{F} \to \omega_C \otimes \mathcal{F}^*$, Serre duality, and semistability (cf. Section 2.2). The key ingredients of our approach are the geometric theory of extensions introduced by Atiyah, Newstead, Lange-Narasimhan et al. (cf., e.g., [5]), Theorem 2.3 below, and suitable parametric computations involving special and effective quotient line bundles and related families of sections of ruled surfaces, which make sense in the setup of Theorem 3.1. Finally, by Theorems 1.1 and 1.2, we can also see that a general vector bundle in $B_{\rm reg}$ admits a special section whose zero locus is of degree one while its general section has no zeros (cf. the proof of [10, Theorem] and Remark 3.14(ii) below).

For standard terminology, we refer the reader to [4].

2. Preliminaries

2.1. **Preliminary results on general ν-gonal curves.** In this section we will review some results concerning line bundles on general ν-gonal curves, which will be used in the paper.

Lemma 2.1 (cf. [7, Corollary 1])**.** *On a general ν-gonal curve of genus $g \geq 2\nu - 2$, with $\nu \geq 3$, there does not exist a $g^r_{\nu-2+2r}$ with $\nu - 2 + 2r \leq g - 1$, $r \geq 2$.*

The *Clifford index* of a line bundle L on a curve C is defined by

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(L) - 1).$$

Theorem 2.2 ([8], Theorem 2.1)**.** *Let C be a general ν-gonal curve of genus $g \geq 4$, let $\nu \geq 4$, and let g^1_ν be the unique pencil of degree ν on C. If C has a line bundle L with $\mathrm{Cliff}(L) \leq \frac{g-4}{2}$ and $\deg L \leq g - 1$, then $|L| = (\dim |L|) g^1_\nu + B$, for some effective divisor B.*

2.2. **Segre invariant and semistable vector bundles.** Given a rank 2 vector bundle $\mathcal{F}$ on C, the *Segre invariant* $s_1(\mathcal{F}) \in \mathbb{Z}$ of $\mathcal{F}$ is defined by

$$s_1(\mathcal{F}) = \min_{N \subset \mathcal{F}} \{\deg\mathcal{F} - 2\deg N\},$$

where N runs through all the subline bundles of $\mathcal{F}$. It easily follows from the definition that $s_1(\mathcal{F}) = s_1(\mathcal{F} \otimes L)$, for any line bundle L, and $s_1(\mathcal{F}) = s_1(\mathcal{F}^*)$, where $\mathcal{F}^*$ denotes the dual bundle of $\mathcal{F}$. A subline bundle $N \subset \mathcal{F}$ is called a *maximal subline bundle* of $\mathcal{F}$ if $\deg N$ is maximal among all subline bundles of $\mathcal{F}$. In such a case $\mathcal{F}/N$ is a *minimal quotient line bundle* of $\mathcal{F}$, i.e., is of minimal degree among quotient line bundles of $\mathcal{F}$. In particular, $\mathcal{F}$ is *semistable* (resp., *stable*) if and only if $s_1(\mathcal{F}) \geq 0$ (resp., $s_1(\mathcal{F}) > 0$).

2.3. **Extensions, secant varieties, and semistable vector bundles.** Let δ be a positive integer. Consider $L \in \mathrm{Pic}^\delta(C)$ and $N \in \mathrm{Pic}^{d-\delta}(C)$. The extension space $\mathrm{Ext}^1(L, N)$ parametrizes isomorphism classes of extensions, and any element $u \in \mathrm{Ext}^1(L, N)$ gives rise to a degree d, rank 2 vector bundle $\mathcal{F}_u$, fitting in an exact sequence

$$(u): \quad 0 \to N \to \mathcal{F}_u \to L \to 0. \tag{2.1}$$

We fix once and for all the following notation:

$$\begin{aligned} j &:= h^1(L), & l &:= h^0(L) = \delta - g + 1 + j, \\ r &:= h^1(N), & n &:= h^0(N) = d - \delta - g + 1 + r. \end{aligned} \tag{2.2}$$

In order to get $\mathcal{F}_u$ semistable, a necessary condition is

$$2\delta - d \geq s_1(\mathcal{F}_u) \geq 0. \tag{2.3}$$

In such a case, the Riemann–Roch theorem gives

$$\dim(\mathrm{Ext}^1(L, N)) = \begin{cases} 2\delta - d + g - 1 & \text{if } L \not\cong N, \\ g & \text{if } L \cong N. \end{cases} \tag{2.4}$$

Since we deal with *special* vector bundles, i.e., $h^1(\mathcal{F}_u) > 0$, they always admit a special quotient line bundle. Recall the following theorem.

Theorem 2.3 ([3], Lemma 4.1). *Let $\mathcal{F}$ be a semistable, special, rank 2 vector bundle on C of degree $d \geq 2g-2$. Then there exist a special, effective line bundle L on C of degree $\delta \leq d$, $N \in \mathrm{Pic}^{d-\delta}(C)$, and $u \in \mathrm{Ext}^1(L,N)$ such that $\mathcal{F} = \mathcal{F}_u$ as in Subsection* 2.1.

Tensor (2.1) by N^{-1} and consider $\mathcal{G}_e := \mathcal{F}_u \otimes N^{-1}$, which fits in

$$(e): \quad 0 \to \mathcal{O}_C \to \mathcal{G}_e \to L-N \to 0,$$

where $e \in \mathrm{Ext}^1(L-N, \mathcal{O}_C)$, so $\deg(\mathcal{G}_e) = 2\delta - d$. Then (u) and (e) define the same point in $\mathbb{P} := \mathbb{P}(H^0(K_C+L-N)^*)$. When the map $\varphi := \varphi_{|K_C+L-N|} : C \to \mathbb{P}$ is a morphism, set $X := \varphi(C) \subset \mathbb{P}$. For any positive integer h denote by $\mathrm{Sec}_h(X)$ the h^{st}-secant variety of X, defined as the closure of the union of all linear subspaces $\langle \varphi(D) \rangle \subset \mathbb{P}$, for general divisors D of degree h on C. One has

$$\dim(\mathrm{Sec}_h(X)) = \min\{\dim(\mathbb{P}), 2h-1\}.$$

Theorem 2.4 ([5, Proposition 1.1]). *Let $2\delta - d \geq 2$; then φ is a morphism and, for any integer $s \equiv 2\delta - d \pmod 2$ such that $4+d-2\delta \leq s \leq 2\delta - d$, one has*

$$s_1(\mathcal{E}_e) \geq s \Leftrightarrow e \notin \mathrm{Sec}_{\frac{1}{2}(2\delta-d+s-2)}(X).$$

3. The main result

In this section C will denote a general ν-gonal curve of genus $g \geq 4$ and A the unique line bundle of degree ν with $h^0(A) = 2$. As explained in the Introduction, from now on we will be concerned with the residual version of Theorem 1.2; therefore we set

$$3 \leq \nu \leq \frac{g+8}{4} \text{ and } 3g-1 \leq d \leq 4g-6-2\nu, \tag{3.1}$$

where d is an integer. For suitable line bundles L and N on C, we consider rank 2 vector bundles $\mathcal{F}$ arising as extensions. We will give conditions on L and N under which $\mathcal{F}$ is general in a certain component of the Brill–Noether locus $B_d^{k_2}$, where $k_2 = d-2g+4$ as in the Introduction. We moreover show that L is a quotient of $\mathcal{F}$ with suitable *minimality* properties. Finally, we prove the following theorem.

Theorem 3.1. *The reduced components of $B_d^{k_2}$ having dimension at least $\rho_d^{k_2}$ are only two, which we denote by B_{reg} and B_{sup}:*

(i) *The component B_{reg} is* regular, *i.e., generically smooth and of dimension $\rho_d^{k_2} = 8g-2d-11$. A general element $\mathcal{F}$ of B_{reg} is stable, fitting in an exact sequence*

$$0 \to K_C - D \to \mathcal{F} \to K_C - p \to 0, \tag{3.2}$$

where $p \in C$ and $D \in C^{(4g-5-d)}$ are general. Specifically, $s_1(\mathcal{F}) \geq 1$ (resp., 2) if d is odd (resp., even). Moreover, $K_C - p$ is minimal among special quotient line bundles of $\mathcal{F}$, and $\mathcal{F}$ is very ample for $\nu \geq 4$.

(ii) *The component B_{sup} is generically smooth, of dimension $6g-d-2\nu-6 > \rho_d^{k_2}$, i.e., B_{sup} is* superabundant. *A general element $\mathcal{F}$ of B_{sup} is stable, very ample, and fitting in an exact sequence*

$$0 \to N \to \mathcal{F} \to K_C - A \to 0, \tag{3.3}$$

for $N \in \mathrm{Pic}^{d-2g+2+\nu}(C)$ general. Moreover, $s_1(\mathcal{F}) = 4g-4-d-2\nu$ and $K_C - A$ is a minimal quotient of $\mathcal{F}$.

Proof. In Sections 3.1 and 3.2 we will construct the components B_{sup} and B_{reg}, respectively, and prove all the statements in Theorem 3.1 except for the minimality property of $K_C - p$ in (i) and the uniqueness of B_{sup} and B_{reg}, which will be proved in Section 3.3. □

Remark 3.2.

(i) As explained in the Introduction, Theorem 3.1 and the natural isomorphism $B_d^{k_2} \simeq B^2_{4g-4-d}$ also give a proof of Theorem 1.2.

(ii) It is well known how the study of rank 2 vector bundles on curves is related to that of (surface) scrolls in projective space. Therefore, the very ampleness condition in Theorem 3.1 is a key for the study of components of Hilbert schemes of smooth scrolls, in a suitable projective space, dominating $\mathcal{M}^1_{g,\nu}$. This will be the subject of a forthcoming paper.

3.1. **The superabundant component** B_{sup}**.** In this section we first construct the component B_{sup} as in Theorem 3.1. We consider the line bundle $L := K_C - A \in W^{g-\nu}_{2g-2-\nu}$ and a general $N \in \operatorname{Pic}^{d-2g+2+\nu}(C)$; since $d - 2g + 2 + \nu \geq g + 1 + \nu$ from (3.1), in particular $h^1(N) = 0$. We first need the following preliminary result.

Lemma 3.3. *Let $N \in \operatorname{Pic}^{d-2g+2+\nu}(C)$ be general. Then, for a general $u \in \operatorname{Ext}^1(K_C - A, N)$, the corresponding rank 2 vector bundle $\mathcal{F}_u$ is stable with:*

(a) $h^1(\mathcal{F}_u) = h^1(K_C - A) = 2$*;*
(b) $s_1(\mathcal{F}_u) = 4g - 4 - 2\nu - d$*; more precisely, $K_C - A$ is a minimal quotient line bundle of $\mathcal{F}_u$;*
(c) *$\mathcal{F}_u$ is very ample.*

Proof. To ease notation, set $L = K_C - A$ and $\delta := \deg L$. To show that $\mathcal{F}_u$ is stable, note that the upper bound on d in (3.1) implies $2\delta - d = 2(2g - 2 - \nu) - d \geq 2$; so we are in a position to apply Theorem 2.4. We consider the natural morphism

$$\varphi := \varphi_{|K_C+L-N|} : C \longrightarrow \mathbb{P} := \mathbb{P}(\operatorname{Ext}^1(L, N)).$$

Set $X := \varphi(C)$. Let s be an integer such that $s \equiv 2\delta - d \pmod 2$ and $0 < s \leq 2\delta - d$. Since $s \leq 2\delta - d = 4g - 4 - 2\nu - d < g - 3$, we have

$$\dim\left(\operatorname{Sec}_{\frac{1}{2}(2\delta-d+s-2)}(X)\right) = 2\delta - d + s - 3 < 2\delta - d + g - 2 = \dim(\mathbb{P}),$$

where the last equality follows from (2.4) and $L \ncong N$. One can therefore take $s = 2\delta - d$ so that the general $\mathcal{F}_u$ arising from (3.3) is of degree d, with $h^1(\mathcal{F}_u) = h^1(L) = 2$, and it is stable, since $s_1(\mathcal{F}_u) = 2\delta - d = 4g - 4 - 2\nu - d \geq 2$; the equality $s_1(\mathcal{F}_u) = 2\delta - d$ follows from Theorem 2.4 and from (3.3). This proves the stability of $\mathcal{F}_u$ together with (a) and (b).

Finally, to prove (c), observe first that $K_C - A$ is very ample: indeed, if $K_C - A$ is not very ample, by the Riemann–Roch theorem there exists a $g^2_{\nu+2}$ on C. This is contrary to Lemma 2.1, since the hypothesis $3 \leq \nu \leq \frac{g+8}{4}$ implies $g \geq 2\nu - 2 + (2\nu - 6) \geq 2\nu - 2$. At the same time, since $\deg(N) = d - 2g + 2 + \nu \geq g + 4$ by (3.1), a general N is also very ample. Thus any $\mathcal{F}_u$ as in (3.3) is very ample, too. □

We now want to show that vector bundles constructed in Lemma 3.3 fill up the component B_{sup}, as N varies in $\operatorname{Pic}^{d-2g+2+\nu}(C)$. To do this, we need to consider a parameter space of rank 2 vector bundles on C, arising as extensions of $K_C - A$ by N, as N varies. If $\mathcal{N} \to \mathrm{Pic}^{d-2g+2+\nu}(C) \times C$ is a Poincaré line bundle, we have the following diagram:

$$\begin{tikzcd}
 & \mathcal{N} \arrow[d] & & \\
 & \mathrm{Pic}^{d-2g+2+\nu}(C) \times C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & K_C - A \arrow[dl] \\
\mathrm{Pic}^{d-2g+2+\nu}(C) & & C &
\end{tikzcd}$$

Set $\mathcal{E}_{d,\nu} := R^1 p_{1*}(\mathcal{N} \otimes p_2^*(A - K_C))$. By [2, pp. 166–167], $\mathcal{E}_{d,\nu}$ is a vector bundle on a suitable open, dense subset $S \subseteq \mathrm{Pic}^{d-2g+2+\nu}(C)$ of rank $\dim \mathrm{Ext}^1(K_C - A, N) = 5g - 5 - 2\nu - d$ as in (2.4), since $K_C - A \not\cong N$. Consider the projective bundle $\mathbb{P}(\mathcal{E}_{d,\nu}) \to S$, which is the family of $\mathbb{P}\left(\mathrm{Ext}^1(K_C - A, N)\right)$'s as N varies in S. One has

$$\dim \mathbb{P}(\mathcal{E}_{d,\nu}) = g + (5g - 5 - 2\nu - d) - 1 = 6g - 6 - 2\nu - d.$$

Consider the natural (rational) map

$$\begin{array}{rcl} \mathbb{P}(\mathcal{E}_{d,\nu}) & \stackrel{\pi_{d,\nu}}{\dashrightarrow} & U_C(2,d), \\ (N,u) \to & & \mathcal{F}_u; \end{array}$$

from Lemma 3.3 we know that $\mathrm{im}(\pi_{d,\nu}) \subseteq B_d^{k_2} \cap U_C^s(2,d)$.

Proposition 3.4. *The closure B_{sup} of $\mathrm{im}(\pi_{d,\nu})$ in $U_C(2,d)$ is a generically smooth component of $B_d^{k_2}$, having dimension $6g - 6 - 2\nu - d$. In particular, B_{sup} is* super-abundant.

Proof. The result will follow once we prove that

$$\dim T_{\mathcal{F}}(B_d^{k_2}) = \dim B_{\mathrm{sup}},$$

for a general $\mathcal{F}$ in $\mathrm{im}(\pi_{d,\nu})$. First we claim that $\dim B_{\mathrm{sup}} = 6g - 6 - 2\nu - d$. Indeed, let $\Gamma \subset F = \mathbb{P}(\mathcal{F}_u)$ be the section corresponding to the quotient $\mathcal{F}_u \twoheadrightarrow K_C - A$. Its normal bundle is $N_{\Gamma/F} \simeq K_C - A - N$ (cf. [4, Sect. V, Prop. 2.9]). Since N is general of degree at least $g + 4$ by (3.1), we have $h^0(K_C - A - N) = 0$; in other words Γ is an algebraically isolated section of F. This guarantees that $\pi_{d,\nu}$ is generically finite (for more details see the proof of [3, Lemma 6.2] and apply the same arguments). Hence we get $\dim \mathrm{im}(\pi_{d,\nu}) = 6g - 6 - 2\nu - d$.

Now we prove that $\dim T_{\mathcal{F}}(B_d^{k_2}) = 6g - 6 - 2\nu - d$. To show this, consider the Petri map of a general $\mathcal{F} \in \mathrm{im}(\pi_{d,\nu})$:

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*).$$

By (3.3) and $h^1(N) = 0$, we have

$$H^0(\mathcal{F}) \simeq H^0(N) \oplus H^0(K_C - A) \quad \text{and} \quad H^0(\omega_C \otimes \mathcal{F}^*) \simeq H^0(A).$$

Thus $\mu_{\mathcal{F}}$ reads as

$$\left(H^0(N) \oplus H^0(K_C - A)\right) \otimes \ H^0(A) \ \xrightarrow{\mu_{\mathcal{F}}} \ H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*).$$

Consider the following natural multiplication maps:

$$\mu_{A,N}: \quad H^0(N) \otimes H^0(A) \to H^0(N+A), \tag{3.4}$$

$$\mu_{0,A}: \quad H^0(K_C - A) \otimes H^0(A) \to H^0(K_C). \tag{3.5}$$

Claim 3.5. $\ker(\mu_{\mathcal{F}}) \simeq \ker(\mu_{0,A}) \oplus \ker(\mu_{A,N})$.

Proof of Claim 3.5. Consider the exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N+A-K_C & \to & \mathcal{F}\otimes(A-K_C) & \to & \mathcal{O}_C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N\otimes\mathcal{F}^* & \to & \mathcal{F}\otimes\mathcal{F}^* & \to & (K_C-A)\otimes F^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_C & \to & \mathcal{F}\otimes N^{-1} & \to & (K_C-A)\otimes N^{-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

which arises from (3.3) and its dual sequence $0 \to A - K_C \to \mathcal{F}^* \simeq \mathcal{F}(A - K_C - N) \to N^{-1} \to 0$. If we tensor the column in the middle by ω_C, we get $H^0(\mathcal{F}\otimes A) \hookrightarrow H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*)$.

Observe, moreover, that $H^0(N+A) \oplus H^0(K_C) \simeq H^0(\mathcal{F}\otimes A)$, which follows from (3.3) tensored by A and the fact that $h^1(N+A) = 0$. Therefore, there is no intersection between $\mathrm{im}(\mu_{0,A})$ and $\mathrm{im}(\mu_{A,N})$, and the statement is proved. □

By Claim 3.5,

$$\begin{aligned} \dim T_{\mathcal{F}}(B_d^{k_2}) &= 4g-3-h^0(\mathcal{F})h^1(\mathcal{F}) + \dim(\ker \mu_{\mathcal{F}}) \\ &= 4g-3-2(d-2g+4) + \dim(\ker(\mu_0(A))) + \dim(\ker(\mu_{A,N})). \end{aligned}$$

From (3.4) and (3.5), we have

$$\ker(\mu_{0,A}) \simeq H^0(K_C - 2A) \cong H^1(2A)^* \quad \text{and} \quad \ker(\mu_{A,N}) \simeq H^0(N-A),$$

as it follows from the basepoint-free pencil trick. Under the numerical assumption $\nu \leq \frac{g+8}{4}$, from Theorem 2.2 we have $h^0(2A) = 3$, which implies $h^1(2A) = g+2-2\nu$. The inequality $\deg N \geq g+1+\nu$ given by (3.1) and the generality of N show that $h^1(N-A) = 0$, which yields $h^0(N-A) = d-3g+3$. So we have

$$\begin{aligned} \dim T_{\mathcal{F}}(B_d^{k_2}) &= 4g-3-2(d-2g+4)+(g+2-2\nu)+(d-3g+3) \\ &= 6g-6-2\nu-d = \dim B_{\mathrm{sup}}. \end{aligned}$$

To complete the proof, it suffices to observe that $\rho_d^{k_2} = 8g-11-2d \leq 5g-10-d < 6g-6-2\nu-d$, as it follows by (3.1). □

3.2. **The regular component B_{reg}.** In this subsection we construct the regular component B_{reg} as in Theorem 3.1. In what follows, we use notation as in (2.2), i.e., $l = h^0(L)$, $j = h^1(L)$, $r = h^1(N)$, which will be considered all positive (cf. Theorem 2.3 for L). For any exact sequence (u) as in (2.1), let $\partial_u : H^0(L) \to H^1(N)$ be the corresponding coboundary map. For any integer $t > 0$, consider

$$\mathcal{W}_t := \{u \in \mathrm{Ext}^1(L,N) \mid \mathrm{corank}(\partial_u) \geq t\} \subseteq \mathrm{Ext}^1(L,N), \tag{3.6}$$

which has a natural structure of determinantal scheme; its expected codimension is $t(l-r+t)$ (cf. [3, Sect. 5.2]). In this setup, one has the following theorem.

Theorem 3.6 ([3, Theorem 5.8 and Corollary 5.9])**.** *Let C be a smooth curve of genus $g \geq 3$. Let*

$$r = h^1(N) \geq 1,\ l = h^0(L) \geq \max\{1, r-1\},\ m := \dim(\mathrm{Ext}^1(L,N)) \geq l+1.$$

Then, we have:

(i) $l - r + 1 \geq 0$.
(ii) $\mathcal{W}_1$ *is irreducible of (expected) dimension* $m - (l - r + 1)$.
(iii) *if* $l \geq r$, *then* $\mathcal{W}_1 \subset \mathrm{Ext}^1(L,N)$. *Moreover for general* $u \in \mathrm{Ext}^1(L,N)$, ∂_u *is surjective, whereas for general* $w \in \mathcal{W}_1$, $\mathrm{corank}(\partial_w) = 1$.

To construct B_{reg}, observe first that by (3.1) W^0_{4g-5-d} is not empty or irreducible and that $h^0(D) = 1$, for general $D \in W^0_{4g-5-d}$. We will prove the following preliminary result.

Lemma 3.7. *Let both $D \in W^0_{4g-5-d}$ and $p \in C$ be general and let $\mathcal{W}_1 \subseteq \mathrm{Ext}^1(K_C - p, K_C - D)$ be as in (3.6). Then, for $u \in \mathcal{W}_1$ general, the corresponding rank 2 vector bundle $\mathcal{F}_u$ is stable, with:*

(a) $h^1(\mathcal{F}_u) = 2$;
(b) $s_1(\mathcal{F}) \geq 1$ *(resp., 2) if d is odd (resp., even);*
(c) $\mathcal{F}_u$ *is very ample when* $\nu \geq 4$.

Proof. From the assumptions we have

$$\begin{array}{llcccc} & (u)\colon 0 \to & K_C - D & \to \mathcal{F} \to & K_C - p & \to 0 \\ & \deg & d-2g+3 & d & 2g-3 & \\ (3.7) & h^0 & d-3g+5 & & g-1 & \\ & h^1 & 1 & & 1 & \end{array}$$

By (3.1) $\deg D = 4g - d - 5 \geq 2\nu + 1$; therefore $K_C - D \ncong K_C - p$. Thus, using (2.4) and notation as in Theorem 3.6, one has

$$l = g-1,\ r = 1 \ \text{ and } \ m = \dim \mathrm{Ext}^1(K_C - p, K_C - D) = 5g - 7 - d.$$

By (3.1) one has $d \leq 4g - 7$, so $m \geq l + 1 = g$. Hence we can apply Theorem 3.6 to

$$\mathcal{W}_1 = \{u \in \mathrm{Ext}^1(K_C - p, K_C - D) \mid \mathrm{corank}(\partial_u) \geq 1\},$$

which therefore is irreducible, of (expected) dimension $\dim \mathcal{W}_1 = m - 1(l - r + 1) = 4g - 6 - d$. Moreover, by Theorem 3.6(iii) and formula (3.7), for general $u \in \mathcal{W}_1$ one has $h^1(\mathcal{F}_u) = 2$, which proves (a).

We now want to show that $\mathcal{F}_u$ also satisfies (b), for $u \in \mathcal{W}_1$ general; in particular, it is stable. To do this, set $\mathbb{P} := \mathbb{P}\left(\mathrm{Ext}^1(K_C - p, K_C - D)\right)$ and consider the projective scheme $\widehat{\mathcal{W}_1} := \mathbb{P}(\mathcal{W}_1) \subset \mathbb{P}$, which therefore has dimension $4g - 7 - d$. Posing $\delta := 2g - 3$ and considering (3.1), one has $2\delta - d \geq 2\nu \geq 6$. We are therefore in a position to apply Theorem 2.4. We consider the natural morphism $C \xrightarrow{\varphi} \mathbb{P}$, given by the complete linear system $|K_C + D - p|$. Set $X = \varphi(C)$, as in the proof of Lemma 3.3. Let s be an integer such that $s \equiv 2\delta - d \pmod 2$ and $0 \leq s \leq 2\delta - d$. Then we have

$$\dim \mathrm{Sec}_{\frac{1}{2}(2\delta - d + s - 2)}(X) = 2\delta - d + s - 3 = 4g - 9 - d + s \leq 4g - 7 - d = \dim \widehat{\mathcal{W}_1}$$

if and only if $s \leq 2$, where the equality holds if and only if $s = 2$.

Therefore, for d odd, by Theorem 2.4 one has $s_1(\mathcal{F}_u) \geq 1$ for $u \in \mathcal{W}_1$ general; in particular, $\mathcal{F}_u$ is stable and (b) is proved in this case.

For d even, if one dualizes the exact sequence (3.2) and tensors via ω_C, one gets

$$(e): \quad 0 \to p \to \mathcal{E}_e := \mathcal{F}_u^* \otimes \omega_C \to D \to 0,$$

where (e) defines the same point as (u) in the projective space $\mathbb{P}$; in particular, $s_1(\mathcal{F}_u) = s_1(\mathcal{E}_e)$ (cf. Section 2.2) and $h^0(\mathcal{E}_e) = 2$, by Serre duality and the fact that $(u) \in \widehat{\mathcal{W}}_1$. Following the same strategy as in the first part of the proof of [10, Theorem], one deduces that (e) belongs to the linear span $\langle \varphi(D) \rangle \subset \mathbb{P}$. On the other hand, any point $x \in \langle \varphi(D) \rangle$ gives rise to an extension,

$$(x): \quad 0 \to p \to \mathcal{E}_x \to D \to 0,$$

which belongs to $\widehat{\mathcal{W}}_1$, since $h^0(\mathcal{E}_x) = 2$ (cf. diagram (2) and the subsequent details in the proof of [10, Theorem]). Thus $\langle \varphi(D) \rangle \subseteq \widehat{\mathcal{W}}_1$. By the Riemann–Roch theorem,

$$\dim \langle \varphi(D) \rangle = h^0(K_C + D - p) - h^0(K_C - p) - 1 = 4g - 7 - d = \dim \widehat{\mathcal{W}}_1.$$

Since they are both closed and irreducible, one gets $\widehat{\mathcal{W}}_1 = \langle \varphi(D) \rangle$. On the other hand,

$$\mathrm{Sec}_{\frac{1}{2}(2\delta - d + 2 - 2)}(X) = \mathrm{Sec}_{\frac{1}{2}(4g-6-d)}(X),$$

which is of dimension $4g - 7 - d$, too, is nondegenerate in $\mathbb{P}$ as $X \subset \mathbb{P}$ is not. Thus, we conclude that $\widehat{\mathcal{W}}_1 \neq \mathrm{Sec}_{\frac{1}{2}(4g-6-d)}(X)$. In particular, from Theorem 2.4, for a general $u \in \widehat{\mathcal{W}}_1$ one has $s_1(\mathcal{F}_u) \geq 2$, so $\mathcal{F}_u$ is stable and (b) is also proved in this case.

To prove (c) observe first that, since $\nu \geq 4$ by assumption, $K_C - p$ is very ample, as it follows by the Riemann–Roch theorem. Now we have the following claim.

Claim 3.8. For general $D \in W^0_{4g-5-d}$, $K_C - D$ is very ample if $\nu \geq 4$.

Proof of Claim 3.8. Assume by contradiction that $K_C - D$ is not very ample for general $D \in W^0_{4g-5-d}$. For a nonnegative integer τ, define the following:

$$\Xi_\tau := \{(D, p + q) \in W^0_{4g-5-d} \times W^0_2 \mid h^0(D + p + q) = \tau + 1\}.$$

If $\Xi_\tau \neq \emptyset$, then we have the diagram

$$\begin{tikzcd} & \Xi_\tau \arrow[dl, "\pi_\tau"'] \arrow[dr, "\wp_\tau"] & \\ W^0_{4g-5-d} & & W^\tau_{4g-3-d} \end{tikzcd}$$

which is given by $\pi_\tau(D, p+q) := D$ and $\wp_\tau(D, p+q) := D+p+q$. The assumption implies that, for some $\tau \in \{1, 2\}$, the image of π_τ is dense in W^0_{4g-5-d}. Considering the map $\wp_\tau$, we get $\dim \Xi_\tau \leq \dim W^\tau_{4g-3-d} + \tau$. By Martens's and Mumford's theorems (cf. [2, Thm. (5.1), (5.2)]), we have $\dim W^\tau_{4g-3-d} \leq 4g - 5 - d - 2\tau$, since C is a general ν-gonal curve with $\nu \geq 4$ and $4g - 3 - d \leq g - 2$ by (3.1). In summation, it turns out that

$$\dim W^0_{4g-5-d} \leq \dim \Xi_r \leq 4g - 5 - d - \tau,$$

which cannot occur. This completes the proof of the claim. $\square$

The above arguments prove (c) and complete the proof of the lemma. $\square$

To construct the component $B_{\rm reg}$ notice that, as in Section 3.1, one has a projective bundle $\mathbb{P}(\mathcal{E}_d) \to S$, where $S \subseteq W^0_{4g-5-d} \times C$ is a suitable open dense subset: $\mathbb{P}(\mathcal{E}_d)$ is the family of $\mathbb{P}(\mathrm{Ext}^1(K_C - p, K_C - D))$'s as $(D,p) \in S$ varies. Since, for any such $(D,p) \in S$, $\widehat{\mathcal{W}_1}$ is irreducible of constant dimension $4g-7-d$, one has an irreducible subscheme $\widehat{\mathcal{W}}_1^{Tot} \subset \mathbb{P}(\mathcal{E}_d)$ which therefore has dimension

$$\dim \widehat{\mathcal{W}}_1^{Tot} = \dim S + 4g - 7 - d = 4g - d - 4 + 4g - 7 - d = 8g - 2d - 11 = \rho_d^{k_2}.$$

From Lemma 3.7, one has the natural (rational) map

$$\begin{array}{ccc} \widehat{\mathcal{W}}_1^{Tot} & \stackrel{\pi}{\dashrightarrow} & U_C(d), \\ (D,p,u) & \longrightarrow & \mathcal{F}_u, \end{array}$$

and $\mathrm{im}(\pi) \subset B_d^{k_2} \cap U_C^s(2,d)$.

Proposition 3.9. *The closure $B_{\rm reg}$ of* $\mathrm{im}(\pi)$ *in $U_C(2,d)$ is a generically smooth component of $B_d^{k_2}$ with dimension $\rho_d^{k_2} = 8g - 11 - 2d$, i.e., $B_{\rm reg}$ is* regular.

Proof. From the fact that $\mathrm{im}(\pi)$ contains stable bundles, any component of $B_d^{k_2}$ containing it has dimension at least $\rho_d^{k_2}$. We concentrate in computing $\dim T_{\mathcal{F}}(B_d^{k_2})$, for general $\mathcal{F} \in \mathrm{im}(\pi)$. Consider the Petri map

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*)$$

for a general $\mathcal{F} \in \mathrm{im}(\pi)$. From diagram (3.7) and the fact that $\mathcal{F} = \mathcal{F}_u$, for some u in some fiber $\widehat{\mathcal{W}_1}$ of $\widehat{\mathcal{W}}_1^{Tot}$, one has that the corresponding coboundary map ∂_u is the zero-map; in other words,

$$H^0(\mathcal{F}) \cong H^0(K_C - D) \oplus H^0(K_C - p) \quad \text{and} \quad H^1(\mathcal{F}) \cong H^1(K_C - D) \oplus H^1(K_C - p).$$

This means that, for any such bundle, the domain of the Petri map $\mu_{\mathcal{F}}$ coincides with that of $\mu_{\mathcal{F}_0}$, where $\mathcal{F}_0 := (K_C - D) \oplus (K_C - p)$ corresponds to the zero vector in $\mathcal{W}_1 \subset \mathrm{Ext}^1(K_C - p, K_C - D)$. We will concentrate on $\mu_{\mathcal{F}_0}$; observe that

$$\begin{aligned} H^0(\mathcal{F}_0) \otimes H^0(\omega_C \otimes \mathcal{F}_0^*) \cong & \left(H^0(K_C - D) \otimes H^0(D)\right) \oplus \left(H^0(K_C - D) \otimes H^0(p)\right) \\ & \oplus \left(H^0(K_C - p) \otimes H^0(D)\right) \oplus \left(H^0(K_C - p) \otimes H^0(p)\right). \end{aligned}$$

Moreover,

$$\omega_C \otimes \mathcal{F}_0 \otimes \mathcal{F}_0^* \cong K_C \oplus (K_C + p - D) \oplus (K_C + D - p) \oplus K_C.$$

Therefore, for Chern classes reason,

$$\mu_{\mathcal{F}_0} = \mu_{0,D} \oplus \mu_{K_C - D, p} \oplus \mu_{K_C - p, D} \oplus \mu_{0,p},$$

where the maps

$$\begin{array}{rc} \mu_{0,D}: & H^0(D) \otimes H^0(K_C - D) \to H^0(K_C), \\ \mu_{K_C - D,p}: & H^0(K_C - D) \otimes H^0(p) \to H^0(K_C - D + p), \\ \mu_{K_C - p,D}: & H^0(K_C - p) \otimes H^0(D) \to H^0(K_C + D - p), \\ \mu_{0,p}: & H^0(p) \otimes H^0(K_C - p) \to H^0(K_C) \end{array}$$

are natural multiplication maps. Since $h^0(D) = h^0(p) = 1$, the maps $\mu_{0,D}$, $\mu_{K_C - D,p}$, $\mu_{K_C - p,D}$ $\mu_{0,p}$ are all injective and so is $\mu_{\mathcal{F}_0}$. By semicontinuity on $\mathcal{W}_1$, one has that $\mu_{\mathcal{F}}$ is injective, for $\mathcal{F}$ general in $\widehat{\mathcal{W}_1}$.

The previous argument shows that a general $\mathcal{F} \in \mathrm{im}(\pi)$ is contained in only one irreducible component, say B_{reg}, of $B_d^{k_2}$ for which

$$\begin{aligned} \dim B_{\mathrm{reg}} = \dim T_{\mathcal{F}}(B_{\mathrm{reg}}) &= 4g-3-h^0(\mathcal{F})h^1(\mathcal{F}) \\ &= 4g-3-2(d-2g+4) = 8g-11-2d, \end{aligned}$$

i.e., B_{reg} is generically smooth and of dimension $\rho_d^{k_2}$.

To conclude that B_{reg} is the closure of $\mathrm{im}(\pi)$, it suffices to show that the rational map π is generically finite onto its image. To do this, let $F = \mathbb{P}(\mathcal{F}_u)$ be the ruled surface, for general $\mathcal{F}_u \in \widehat{\mathcal{W}}_1^{Tot}$, and let Γ be the section corresponding to the quotient $\mathcal{F}_u \twoheadrightarrow K_C - p$. Then its normal bundle is $N_{\Gamma/F} \simeq D - p$, which has no sections. Thus, one deduces the generic finiteness of π by reasoning as in the proof of Proposition 3.4. $\square$

3.3. **No other reduced components of dimension at least $\rho_d^{k_2}$.**

In this section, we will show that no other reduced components of $B_d^{k_2}$, having dimension at least $\rho_d^{k_2} = 8g-11-2d$, exist except for B_{reg} and B_{sup} constructed in the previous sections.

Let $B \subset B_d^{k_2}$ be any reduced component with $\dim B \geq \rho_d^{k_2} = 8g-11-2d$. From Theorem 2.3, $\mathcal{F} \in B$ general fits in an exact sequence of the form

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{3.8}$$

where L is a special, effective line bundle of degree $\delta \leq d$, i.e., $l,\ j > 0$ and $h^1(\mathcal{F}) \geq 2$.

We first focus on the case of $h^1(\mathcal{F}) = 2$. We start with the following proposition.

Proposition 3.10. *Let B be any reduced component of $B_d^{k_2}$, with $\dim B \geq \rho_d^{k_2}$. For $\mathcal{F}$ general in B, assume that it fits in an exact sequence like* (3.8)*, with $h^1(\mathcal{F}) = h^1(L) = 2$. Then, B coincides with the component B_{sup} as in Section* 3.1.

Proof. Since $\mathcal{F}$ is semistable, from (2.3) and (3.1) one has $\deg L \geq \frac{3g-1}{2}$. Moreover, since C is a general ν-gonal curve and $h^1(L) = 2$, from [1, Theorem 2.6] we have $|\omega_C \otimes L^{-1}| = g^1_\nu + B_b$, where B_b is a base locus of degree b. Hence $L \simeq K_C - A - B_b$, where $b \leq \frac{g-3}{2} - \nu$. For simplicity, put $\delta := \deg L = 2g-2-\nu-b$ so $\deg N = d-\delta$.

Since B is reduced, one must have

$$\dim B = \dim T_{\mathcal{F}} B$$

for general $\mathcal{F} \in B$. We will prove the proposition by showing that $\dim B = \dim T_{\mathcal{F}} B$ can occur only if $L = K_C - A$ and N is nonspecial, general of its degree.

Claim 3.11. $\dim B \leq \begin{cases} 6g-d-2\nu-6-b & \text{if } h^1(N) = 0, \\ 9g-2d-3\nu-2r-2b-7 & \text{if } h^1(N) \geq 1. \end{cases}$

Proof of Claim 3.11. We will use notation as in (2.2). Since B is irreducible, all integers in (2.2) are constant for a general $\mathcal{F} \in B$. From (3.8) combined with $L = K_C - A - B_b$, it follows there exists an open dense subset S of a closed subvariety of $\mathrm{Pic}^{d-\delta} \times C^{(b)}$ and a projective bundle $\mathcal{P} \to S$, whose general fiber identifies with $\mathbb{P} = \mathbb{P}(H^0(K_C + L - N)^*) = \mathbb{P}(\mathrm{Ext}^1(L, N)) \cong \mathbb{P}^{m-1}$, where $m := \dim(\mathrm{Ext}^1(L, N))$. Since $h^1(\mathcal{F}) = h^1(L)$, as in [3, Sect. 6], the component B has to be the image of $\mathcal{P}$

via a dominant rational map

$$\begin{array}{lcl} \mathcal{P} & \overset{\pi}{\dashrightarrow} & B \subset B_d^{k_2} \\ \downarrow & & \\ S & & \end{array}$$

(cf. [3, Sect. 6] for details). Therefore we obtain $\dim B \leq \dim \mathcal{P} = \dim S + m - 1$ since $\mathcal{P}$ is a projective bundle over S whose general fiber is $(m-1)$-dimensional. Specifically, if $r \geq 1$, then S is a subset of $W_{d-\delta}^{d-\delta-g+r} \times C^{(b)}$, the latter being equivalent to $W_{2g-2+\delta-d}^{r-1} \times C^{(b)}$ by Serre duality, and $\dim W_{2g-2+\delta-d}^{r-1} \leq 2g-2+\delta-d-2(r-1)$ by using Martens's theorem (cf. [2, Theorem 5.1]) for $r \geq 2$. Therefore, we get

$$\dim S \leq \begin{cases} g+b & \text{if } r=0, \\ 2g-2+\delta-d-2r+2+b & \text{if } r \geq 1. \end{cases}$$

This inequality, combined with (2.4), gives

$$\dim B \leq \begin{cases} (g+b)+2\delta-d+g-2 & \text{if } r=0, \\ (2g-2+\delta-d-2r+2+b)+2\delta-d+g-1 & \text{if } r \geq 1, \end{cases}$$

since a nonspecial line bundle cannot be isomorphic to a special one. By substituting $\delta = 2g-2-\nu-b$, we get the conclusion of Claim 3.11. □

Claim 3.12. $\dim T_{\mathcal{F}}(B) \geq 6g-d-2\nu-2r-6.$

Proof of Claim 3.12. The tangent space $T_{\mathcal{F}}(B)$ is the orthogonal space to the image of the Petri map:

$$\mu_{\mathcal{F}} : H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \to H^0(\omega_C \otimes \mathcal{F}^* \otimes \mathcal{F}),$$

so $\dim T_{\mathcal{F}}(B) = \dim(\mathrm{im}(\mu_{\mathcal{F}})^{\perp}) = h^0(K_C \otimes F^* \otimes F) - h^0(\mathcal{F})h^1(\mathcal{F}) + \dim\ker\mu_{\mathcal{F}}$.

From the exact sequence (3.8), we get $H^0(\mathcal{F}) \simeq H^0(N) \oplus W$, where $W := \mathrm{im}(H^0(\mathcal{F}) \to H^0(L))$. Since $H^1(\mathcal{F}) \simeq H^1(L)$, the connecting homomorphism in (3.8) is surjective, hence $\dim W = l - r = h^0(L) - h^1(N)$. Let $\mu_{N,\omega_C \otimes L^{-1}}$ and $\mu_{0,W}$ be the maps defined as follows:

$$\begin{aligned} \mu_{N,\omega_C\otimes L^{-1}} &: \quad H^0(N) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(N \otimes \omega_C \otimes L^{-1}), \\ \mu_{0,W} &: \quad W \otimes H^0(\omega_C \otimes L^{-1}) \hookrightarrow H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C). \end{aligned}$$

Then we have

$$\dim\ker\mu_{\mathcal{F}} \geq \dim\ker\mu_{N,\omega_C\otimes L^{-1}} + \dim\ker\mu_{0,W} \tag{3.9}$$

by the following commutative diagram:

$$\begin{tikzcd} H^0(\mathcal{F}) \otimes H^0(\omega_C \otimes \mathcal{F}^*) \arrow[rr, "\mu_{\mathcal{F}}"] \arrow[dd, equal, "\wr"'] & & H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*) & \\ & & H^0(\omega_C \otimes L^{-1} \otimes N) \arrow[u, "\alpha"] & H^0(\omega_C) \arrow[ul, "\beta"'] \\ (H^0(N) \oplus W) \otimes H^0(\omega_C \otimes L^{-1}) \arrow[r, "\cong"] & & (H^0(N) \otimes H^0(\omega_C \otimes L^{-1})) \arrow[u, "\mu_{N,\omega_C\otimes L^{-1}}"] & \oplus\, (W \otimes H^0(\omega_C \otimes L^{-1})) \arrow[u, "\mu_{0,W}"] \end{tikzcd}$$

where the map β comes from the trivial section of $H^0(\mathcal{F}\otimes\mathcal{F}^*)$ after tensoring via ω_C. To explain the map α, if one takes the diagram determined by the exact sequence (3.8) and its dual sequence and tensors it by ω_C, one gets

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_C\otimes N\otimes L^{-1} & \to & \omega_C\otimes\mathcal{F}\otimes L^{-1} & \to & \omega_C & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_C\otimes N\otimes\mathcal{F}^* & \to & \omega_C\otimes\mathcal{F}\otimes\mathcal{F}^* & \to & \omega_C\otimes L\otimes\mathcal{F}^* & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_C & \to & \omega_C\otimes\mathcal{F}\otimes N^{-1} & \to & \omega_C\otimes L\otimes N^{-1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The map α is the composition of the two injections

$$H^0(\omega_C\otimes N\otimes L^{-1})\hookrightarrow H^0(\omega_C\otimes\mathcal{F}\otimes L^{-1})\hookrightarrow H^0(\omega_C\otimes\mathcal{F}\otimes\mathcal{F}^*).$$

Since $K_C-L=A+B_b$, by the basepoint-free pencil trick, we have

$$\begin{aligned}
\dim\ker\mu_{N,\omega_C\otimes L^{-1}}=h^0(N-A) &= \deg N-\deg A-g+h^0(K_C-N+A)+1\\
&\geq d-\delta-\nu-g+1=d-3g+3+b.
\end{aligned}$$

From $\dim W=h^0(L)-r$, it follows that $\dim\ker\mu_{0,W}\geq\dim\ker\mu_0(L)-2r$, where

$$\mu_0(L):H^0(L)\otimes H^0(K_C-L)\to H^0(K_C).$$

To compute $\dim\ker\mu_0(L)$, we apply once again the basepoint-free pencil trick which gives

$$\begin{aligned}
\dim\ker\mu_0(L) &= h^0(L-A)=h^0(K_C-2A-B_b)\\
&= 2g-2-2\nu-b-g+h^0(2A+B_b)+1\\
&\geq g-2\nu-b+2,
\end{aligned}$$

the latter inequality following from the fact that $h^0(2A+B_b)\geq 3$. Hence, from (3.9), one has

$$\begin{aligned}
\dim\ker\mu_{\mathcal{F}} &\geq d-3g+3+b+g-2\nu-b+2-2r\\
&= d-2g-2\nu-2r+5.
\end{aligned}$$

The previous inequality gives $\dim T_{\mathcal{F}}(B)\geq 6g-d-2\nu-2r-6$, proving Claim 3.12. □

Assume that $h^1(N)\geq 1$. Then, Claims 3.11 and 3.12 and (3.1) imply that

$$\dim T_{\mathcal{F}}B-\dim B\geq d-3g+\nu+2b+1\geq\nu+2b.$$

Thus the equality $\dim B=\dim T_{\mathcal{F}}B$ cannot occur for $h^1(N)\geq 1$; therefore, N must be nonspecial. In this case, $\dim B=\dim T_{\mathcal{F}}B$ holds if and only if $b=0$ and N is general of its degree. Consequently, the proposition is proved. □

Thus, the only remaining case is the following proposition.

Proposition 3.13. *Let B be any reduced component of $B^{k_2}_d$, with $\dim B \geq \rho^{k_2}_d$. Assume that a general element $\mathcal{F}$ of B fits in the following exact sequence:*

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{3.10}$$

where $h^1(\mathcal{F}) = 2$ and $h^1(L) = 1$. Then, B coincides with the component B_{reg} as in Section 3.2.

Proof. We will use notation as in (2.2). Since B is irreducible, all integers in (2.2) are constant for a general $\mathcal{F} \in B$. Then $\frac{3g-1}{2} \leq \delta \leq 2g-2$, since L is special and $\mathcal{F}$ is semistable. Hence

$$g-1 \leq \deg N = d - \delta \leq d/2 \leq 2g - 3\nu. \tag{3.11}$$

By (3.10), the line bundle N is special and the corresponding coboundary map ∂ is of corank one. As in the proof of Proposition 3.10, for a suitable open dense subset S of $W^{r-1}_{2g-2+\delta-d} \times C^{(2g-2-\delta)}$, one has a projective bundle $\mathbb{P}(\mathcal{E}) \to S$ whose general fiber is $\widehat{\mathcal{W}_1} := \mathbb{P}(\mathcal{W}_1)$, where $\mathcal{W}_1 := \{u \in \mathrm{Ext}^1(L, N) \mid \mathrm{corank}(\partial_u) \geq 1\}$. Then the component B is the image of $\mathcal{P}$ via a dominant rational map $\mathcal{P} \overset{\pi}{\dashrightarrow} B \subset B^{k_2}_d$ (cf. [3, Sect. 6] for details). Hence

$$\dim B \leq \dim W^{r-1}_{2g-2-d+\delta} + 2g - 2 - \delta + \dim \widehat{\mathcal{W}_1}.$$

Since from (3.11) $\deg(K_C - N) \leq g-1$, by Martens's theorem [2, Thm. (5.1)] we obtain

$$\dim W^{r-1}_{2g-2+\delta-d} \leq \begin{cases} 2g-2-d+\delta = \deg(K_C - N) & \text{if } r = 1, \\ 2g-2-d+\delta-2r+1 & \text{if } r \geq 2. \end{cases}$$

Note that $m \geq g + 2\delta - d - 1$ by (2.4), where $m := \dim(\mathrm{Ext}^1(L, N))$. Thus it follows that $l \geq r$ and $m \geq l+1$ since $l = h^0(L) = \delta - g + 2 \geq \frac{g+3}{2}$ and $r - 1 \leq \frac{\deg(K_C-N)}{2}$. Applying Theorem 3.6, we get $\dim \widehat{\mathcal{W}_1} = m - l + r - 2 = m - \delta + g + r - 4$, whence

$$\begin{aligned} \dim B &\leq \dim W^{r-1}_{2g-2-d+\delta} + (2g-2-\delta) + m - \delta + g + r - 4 \\ &\leq \begin{cases} 5g-d-\delta-7+m & \text{if } r = 1, \\ 5g-d-\delta-r-7+m & \text{if } r \geq 2. \end{cases} \end{aligned}$$

Assume that $r \geq 2$; this implies that N cannot be isomorphic to L. Therefore (2.4) gives $m = 2\delta - d + g - 1$. Thus we have

$$\rho^{k_2}_d \leq \dim B \leq 6g - 2d + \delta - r - 8,$$

which cannot occur since $\rho^{k_2}_d = 8g - 2d - 11$ and $\delta \leq 2g-2$. Therefore, we must have $r = 1$. Then by (2.4) we get

$$\dim B \leq \begin{cases} (5g-d-\delta-7) + 2\delta - d + g - 1 & \text{if } L \ncong N, \\ (5g-d-\delta-7) + g & \text{if } L \cong N. \end{cases} \tag{3.12}$$

If $L \cong N$, then we have $8g - 2d - 11 \leq \dim B \leq 6g - d - \delta - 7$, which yields $\deg N = d - \delta \geq 2g - 4$. This is a contradiction to (3.11). Accordingly, we have $L \ncong N$, and hence by (3.12),

$$8g - 2d - 11 \leq \dim B \leq 6g - 2d + \delta - 8,$$

which implies $\delta \geq 2g-3$. Since L is a special line bundle, it turns out that either $L \simeq K_C$ or $L \simeq K_C(-p)$ for some $p \in C$.

If $L \simeq K_C$, let Γ be the section of the ruled surface $F = \mathbb{P}(\mathcal{F})$ corresponding to the quotient $\mathcal{F} \twoheadrightarrow K_C$; then $\dim |\mathcal{O}_F(\Gamma)| = 1$ by [3, (2.6)] and the fact that $h^1(\mathcal{F}) = 2$. By [3, Prop. 2.12] any such $\mathcal{F}$ admits; therefore $K_C - p$ as a quotient line bundle, for some $p \in C$. This completes the proof since N is special. $\square$

Remark 3.14.

(i) From the proof of Proposition 3.13, it also follows that $K_C - p$ is minimal among special quotient line bundles for $\mathcal{F}$ general in the component B_{reg}, completely proving Theorem 3.1(i).

(ii) Notice moreover that, from the same proof, $\mathcal{F}$ general in B_{reg} also admits a *presentation* via a canonical quotient, i.e., $0 \to K_C - D - p \to \mathcal{F} \to K_C \to 0$, which on the other hand is not via a quotient line bundle of $\mathcal{F}$ of minimal degree among special quotients and whose residual presentation coincides with that in the proof of [10, Theorem], i.e., $0 \to \mathcal{O}_C \to \mathcal{E} \to L \to 0$, where $\mathcal{E} = \omega_C \otimes \mathcal{F}^*$ and $L = \mathcal{O}_C(D + p)$. In other words, the component B_{reg} coincides with that in [10, Theorem]; the minimality of $K_C - p$ for $\mathcal{F}$ reflects in our Theorem 1.2(i) via a special section of $\mathcal{E}$ whose zero locus is of degree one.

We now consider the case $h^1(\mathcal{F}) = i \geq 3$.

Proposition 3.15. *There is no reduced component of $B_d^{k_2}$ whose general member $\mathcal{F}$ is of speciality $i \geq 3$.*

Proof. If $\mathcal{F} \in B_d^{k_2}$ is such that $h^1(\mathcal{F}) = i \geq 3$, then by the Riemann–Roch theorem $h^0(\mathcal{F}) = d - 2g + 2 + i = k_2 + (i-2) = k_i > k_2$. Thus $\mathcal{F} \in \mathrm{Sing}(B_d^{k_2})$ (cf. [2, p. 189]). Therefore the statement follows. $\square$

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