PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 12, December 2018, Pages 5269–5278
https://doi.org/10.1090/proc/14195
Article electronically published on September 17, 2018

MONOMIAL BASIS IN KORENBLUM TYPE SPACES OF ANALYTIC FUNCTIONS

JOSÉ BONET, WOLFGANG LUSKY, AND JARI TASKINEN

(Communicated by Thomas Schlumprecht)

Abstract. It is shown that the monomials $\Lambda = (z^n)_{n=0}^{\infty}$ are a Schauder basis of the Fréchet spaces $A_+^{-\gamma}$, $\gamma \geq 0$, that consists of all the analytic functions f on the unit disc such that $(1-|z|)^{\mu}|f(z)|$ is bounded for all $\mu > \gamma$. Lusky proved that Λ is not a Schauder basis for the closure of the polynomials in weighted Banach spaces of analytic functions of type H^{∞}. A sequence space representation of the Fréchet space $A_+^{-\gamma}$ is presented. The case of (LB)-spaces $A_-^{-\gamma}$, $\gamma > 0$, that are defined as unions of weighted Banach spaces is also studied.

1. Introduction and preliminaries

We consider analytic functions $f \in H(\mathbb{D})$ on the unit complex disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f : \mathbb{D} \to \mathbb{C}$ and $0 \leq r < 1$ we put $M_{\infty}(f,r) = \sup_{|z|=r} |f(z)|$. If f is analytic, then $M_{\infty}(f,r)$ is increasing with respect to r. For $\mu > 0$ let

$$||f||_{\mu} = \sup_{0 \leq r < 1} M_{\infty}(f,r)(1-r)^{\mu}$$

and $A^{-\mu} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ analytic }, ||f||_{\mu} < \infty\}$. Moreover let

$$A_0^{-\mu} = \{f \in A^{-\mu} : \lim_{r \to 1} M_{\infty}(f,r)(1-r)^{\mu} = 0\},$$

and for $\gamma \in [0,\infty[$ let

$$A_+^{-\gamma} = \bigcap_{\mu > \gamma} A^{-\mu} = \bigcap_{\mu > \gamma} A_0^{-\mu}.$$

We consider the norms $||\cdot||_{\mu}$, $\mu > \gamma$, with which $A_+^{-\gamma}$ becomes a Frechet space. By definition we have

$$||\cdot||_{\mu_1} \leq ||\cdot||_{\mu_2} \quad \text{and} \quad A^{-\mu_2} \subset A^{-\mu_1} \quad \text{whenever } \mu_1 > \mu_2.$$

Similarly, for $\gamma \in]0,\infty]$, let

$$A_-^{-\gamma} := \bigcup_{\mu < \gamma} A^{-\mu} = \bigcup_{\mu < \gamma} A_0^{-\mu}$$

be endowed with the finest locally convex topology such that all inclusions $A^{-\mu} \subset A_-^{-\gamma}$ are continuous. With this topology $A_-^{-\gamma}$ is an (LB)-space, i.e., a Hausdorff countable inductive limit of Banach spaces.

Received by the editors January 17, 2018, and, in revised form, April 16, 2018.
2010 *Mathematics Subject Classification.* Primary 46E10; Secondary 46A35, 46A45, 46E15.
Research of the first author was partially supported by the project MTM2016-76647-P. .
Research of the third author was partially supported by the Väisälä Foundation of the Finnish Academy of Sciences and Letters.

The *Korenblum space* $A_-^{-\infty}$, denoted simply by $A^{-\infty}$ [6], is defined via

$$A^{-\infty} := \bigcup_{0<\gamma<\infty} A^{-\gamma} = \bigcup_{n\in\mathbb{N}} A^{-n}.$$

Spaces of this type play a relevant role in interpolation and sampling of analytic functions; see [7]. Weighted spaces of analytic functions appear in the study of growth conditions of analytic functions and have been investigated in various articles since the work of Shields and Williams; see e.g. [3], [4], [10], [12], and the references therein.

Our notation for functional analysis is standard; see e.g. [11]. We recall that a sequence $(x_n)_n$ in a locally convex space E is a *Schauder basis* if every element $x \in E$ can be written in a unique way as $x = \sum_{n=1}^{\infty} u_n(x)x_n$ with $u_n : E \to \mathbb{K}, n \in \mathbb{N}$, continuous linear forms. We refer the reader to [9] for more information about Schauder bases in Banach spaces and to [8] for Schauder bases on locally convex spaces.

Let $e_n(z) = z^n$, $z \in \mathbb{D}$, for $n = 0, 1, 2, \ldots$ and $\Lambda = \{e_n : n = 0, 1, 2, \ldots\}$. The second author proved in [10] that Λ is not a Schauder basis for any $A_0^{-\mu}$ and in more general weighted Banach spaces of analytic functions. On the other hand, the monomials $(e_n)_n$ constitute a Schauder basis of the space $A^{-\infty}$. In fact associating each $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^{-\infty}$ to the sequence $(a_n)_n$ of Taylor coefficients defines a linear topological isomorphism from $A^{-\infty}$ into the strong dual s' of the Fréchet echelon space s of rapidly decreasing sequences.

The purpose of this note is to answer the following two questions:

Question 1. Are the monomials a Schauder basis of the spaces $A_+^{-\gamma}$ and $A_-^{-\gamma}$ for $\gamma \neq \infty$?

Question 2. Are there sequence space representations of the spaces $A_+^{-\gamma}$ for $0 \leq \gamma < \infty$ (resp. $A_-^{-\gamma}$, for $0 < \gamma < \infty$) as Köthe echelon (resp. Köthe co-echelon) spaces of order 0?

In connection with Question 2, recall that the Banach spaces $A_0^{-\mu}$ and $A^{-\mu}$ are isomorphic to c_0 and ℓ_∞ respectively [12], although the monomials are not a Schauder basis of them [10].

Question 1 is answered positively in Theorem 2.4, and Question 2 is dealt with in Section 3; see Theorem 3.2.

2. Monomial bases

The following lemma is easy to prove.

Lemma 2.1. *Let $\mu > 0$ and $N > 0$. The function $r^N(1-r)^\mu$, $0 \leq r \leq 1$, has a global maximum point at r if and only if $N = \mu r(1-r)^{-1}$.*

For $n > \mu > 0$ put $\rho_{n,\mu} = 1 - \frac{\mu}{n}$. Then $\rho_{n,\mu}$ is the global maximum point of $r^{n-\mu}(1-r)^\mu$.

Lemma 2.2. *Let $n \in \mathbb{N}$, $n > \mu$. Consider $f : \mathbb{D} \to \mathbb{C}$ analytic with $f(z) = \sum_{k=n}^{\infty} a_k z^k$. Then*

$$||f||_\mu = \sup_{\rho_{n,\mu} \leq r < 1} M_\infty(f, r)(1-r)^\mu.$$

Proof. Let $g(z) = z^{-n}f(z)$. Then, g can be regarded as an analytic function on $\mathbb{D}$ (with the natural extension to 0). We obtain, for $0 \leq r < \rho_{n,\mu}$,

$$\begin{aligned} M_\infty(f,r)(1-r)^\mu &= r^n M_\infty(g,r)(1-r)^\mu \\ &\leq \left(\frac{r}{\rho_{n,\mu}}\right)^n \left(\frac{1-r}{1-\rho_{n,\mu}}\right)^\mu \rho_{n,\mu}^n M_\infty(g,\rho_{n,\mu})(1-\rho_{n,\mu})^\mu \\ &\leq \left(\frac{r}{\rho_{n,\mu}}\right)^{n-\mu} \left(\frac{1-r}{1-\rho_{n,\mu}}\right)^\mu \rho_{n,\mu}^n M_\infty(g,\rho_{n,\mu})(1-\rho_{n,\mu})^\mu \\ &\leq M_\infty(f,\rho_{n,\mu})(1-\rho_{n,\mu})^\mu, \end{aligned}$$

where we have used the fact that $\rho_{n,\mu}$ is the global maximum point of $r^{n-\mu}(1-r)^\mu$. □

Proposition 2.3. Let $\mu_0 > 0$ and $\mu > \mu_0$. Then, for any $f \in A^{-\mu_0}$ the Taylor series of f converges to f with respect to $||\cdot||_\mu$.

Proof. Let P_n be the Dirichlet projections; i.e., $P_n f$ is the n'th partial sum of the Taylor series of f. It is well known that there is a universal constant $c \geq 1$ such that for every analytic function f, every n and every radius r have

$$M_\infty(P_n f, r) \leq c\log(n) M_\infty(f,r).$$

See e.g. [13].

We obtain, for $f \in A^{-\mu_0}$,

$$||f - P_n f||_{\mu_0} \leq c(1+\log(n))||f||_{\mu_0}.$$

If $f(z) = \sum_{k=0}^\infty a_k z^k$, then $(\mathrm{id} - P_n)f(z) = \sum_{k=n+1}^\infty a_k z^k$. For $\mu > \mu_0$ we apply Lemma 2.2 to get

$$\begin{aligned} ||(\mathrm{id}-P_n)f||_\mu &= \sup_{\rho_{n+1,\mu}\leq r<1} M_\infty((\mathrm{id}-P_n)f,r)(1-r)^\mu \\ &\leq \sup_{\rho_{n+1,\mu}\leq r<1} (1-r)^{\mu-\mu_0}||(\mathrm{id}-P_n)f||_{\mu_0} \\ &\leq c(1-\rho_{n+1,\mu})^{\mu-\mu_0}(1+\log(n))||f||_{\mu_0} \\ &= c\left(\frac{\mu}{n+1}\right)^{\mu-\mu_0}(1+\log(n))||f||_{\mu_0}. \end{aligned}$$

Since $\mu-\mu_0 > 0$ the right-hand side goes to 0 if $n \to \infty$. This proves the proposition. □

Theorem 2.4.

(i) Λ *is a Schauder basis of* $A_+^{-\gamma}$ *for any* $\gamma \geq 0$.

(ii) Λ *is a Schauder basis of* $A_-^{-\gamma}$ *for any* $\gamma > 0$.

Proof. (i) We have to prove that the Taylor series of every $f \in A_+^{-\gamma}$, $\gamma \geq 0$, converges in $A_+^{-\gamma}$ to f. Fix $\mu > \gamma$ and select μ_1 with $\gamma < \mu_1 < \mu$. Since $f \in A^{-\mu_1}$, we can apply Proposition 2.3 to conclude that the Taylor series of f converges in $A^{-\mu}$ to f. This implies the conclusion.

(ii) is a direct consequence of Proposition 2.3 and the properties of inductive limits. □

It is well known that the Korenblum space $A^{-\infty}$ is nuclear, since it is isomorphic to the nuclear (LB)-space s'. The following result is proved in [1].

Proposition 2.5. Each Fréchet space $A_+^{-\gamma}$ for $0 \le \gamma < \infty$ and each (LB)-space $A_-^{-\gamma}$ for $0 < \gamma < \infty$ fails to be nuclear.

This result is now a direct consequence of Theorem 2.4 and the Grothendieck–Pietsch criterion [11, Theorem 28.15]. We indicate the argument for $A_+^{-\gamma}$: If this Fréchet space is nuclear, given $\mu := \gamma + 1$, we can apply [11, Theorem 28.15] to find $\gamma < \nu < \mu$ such that $\sum_{n=1}^{\infty} \frac{||z^n||_\mu}{||z^n||_\nu} < \infty$. This implies by Lemma 2.1 that $\sum_{n=1}^{\infty} \frac{1}{n^{\mu-\nu}} < \infty$, a contradiction, since $0 < \mu - \nu < 1$.

3. Sequence space representation

We recall the definition of Köthe echelon and co-echelon spaces of order infinity; see [5] and [11, Chapter 27]. A sequence $A = (a_k)_k$ of functions $a_k : \mathbb{N} \cup \{0\} \to]0, \infty)$ is called a *Köthe matrix* on $\mathbb{N}$ if $0 < a_k(j) \le a_{k+1}(j)$ for all $j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. The *Köthe echelon space of order infinity* associated to A is

$$\lambda_\infty(A) := \{x \in \mathbb{C}^{\mathbb{N}} : \sup_j a_k(j)|x_j| < \infty, \ \forall k \in \mathbb{N}\},$$

which is a Fréchet space relative to the increasing sequence of canonical seminorms

$$q_k^{(\infty)}(x) := \sup_j a_k(j)|x_j|, \qquad x \in \lambda_\infty(A), \quad k \in \mathbb{N}.$$

Then $\lambda_\infty(A) = \bigcap_{k \in \mathbb{N}} \ell_\infty(a_k)$. Here $\ell_\infty(a_k)$ is the usual weighted ℓ_∞ sequence space.

Given a decreasing sequence $V = (v_k)_k$ of strictly positive functions on $\mathbb{N} \cup \{0\}$, the Köthe co-echelon space of order infinity is $k_\infty(V) := \mathrm{ind}_k \ell_\infty(v_k)$, and it is endowed with the inductive limit topology. Then $k_\infty(V)$ is a regular (LB)-space [5].

Given $\mu \in]0, \infty[$ define $r_\mu(0) = s_\mu(0) := 1$ and

$$r_\mu(j) := \left(\frac{\mu}{2^n + \mu} \right)^\mu, \qquad j = 2^n, \ldots, 2^{n+1} - 1, \quad n = 0, 1, 2, \ldots,$$

and

$$s_\mu(j) := \left(\frac{\mu}{j + \mu} \right)^\mu, \qquad j = 1, 2, \ldots.$$

Lemma 3.1. *If $0 < \mu_2 < \mu_1$, then $r_{\mu_1}(j) \le r_{\mu_2}(j)$ and $s_{\mu_1}(j) \le s_{\mu_2}(j)$ for each $j = 0, 1, 2, \ldots$.*

Proof. It is enough to show that the function

$$f(x) = \left(\frac{x}{j + x} \right)^x = \exp\left(x \log(x) - x \log(j + x)\right), \qquad x > 0,$$

is decreasing. It is easily seen that $f'(x) \le 0$ if and only if

$$1 + \log(x) - \log(j + x) - \frac{x}{j + x} \le 0.$$

This inequality is valid for all $x > 0$ since $t \le e^{t-1}$ for each $t \in]0, 1[$ implies that

$$\frac{x}{j + x} \le \exp\left(\frac{x}{j + x} - 1 \right)$$

for all $x > 0$. □

Given $\gamma \geq 0$, put $\mu_k := \gamma + \frac{1}{k}, k \in \mathbb{N}$, and define $a_k(j) := s_{\mu_k}(j), k \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}$. Lemma 3.1 implies that $A_\gamma := (a_k)_k$ is a Köthe matrix. Analogously, for $\gamma > 0$, we set $\nu_k = \gamma - \frac{1}{k}$ with k large enough so that $\nu_k > 0$. Now, by Lemma 3.1 the sequence $V_\gamma := (v_k)_k$, $v_k(j) := s_{\nu_k}(j), k \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}$ is decreasing. Keeping this notation, we can state the main result of this section.

Theorem 3.2.

(i) *For each $\gamma \geq 0$ the Fréchet space $A_+^{-\gamma}$ is isomorphic to the Köthe echelon space $\lambda_\infty(A_\gamma)$.*

(ii) *For each $\gamma > 0$ the (LB)-space $A_-^{-\gamma}$ is isomorphic to the Köthe co-echelon space $k_\infty(V_\gamma)$.*

The proof of Theorem 3.2 is a consequence of the results presented below.

Firstly, we introduce, for a sequence $(x_j)_{j=0}^\infty$ of complex numbers, the norms

$$|||(x_j)|||_\mu = \sup\left(|x_0|, \sup_{n=0,1,2,\ldots}\left(\frac{\mu}{2^n + \mu}\right)^\mu \sup_{2^n \leq j < 2^{n+1}} |x_j|\right) = \sup_j r_\mu(j)|x_j|$$

and define

$$B_\gamma = \{(x_j) : |||(x_j)|||_\mu < \infty \ \text{ for all } \mu > \gamma\}.$$

We consider the locally convex topology on B_γ generated by the norms $|||\cdot|||_\mu$ for all $\mu > \gamma$. Finally put

$$C_\gamma = \{(x_j) : |||(x_j)|||_\mu < \infty \ \text{ for some } \mu < \gamma\}$$

endowed with the finest locally convex topology such that the embedding $J_\mu : \{(x_j) : |||(x_j)|||_\mu < \infty\} \to C_\gamma$ is continuous for all $\mu < \gamma$.

Since $s_\mu(j) \leq r_\mu(j) \leq 2^{\max(1,\mu)} s_\mu(j)$ for each $j = 0, 1, 2, \ldots$ it follows that $B_\gamma = \lambda_\infty(A_\gamma)$ and $C_\gamma = k_\infty(V_\gamma)$ algebraically and topologically. In order to complete the proof of Theorem 3.2, we must show that $A_+^{-\gamma}$ and B_γ, as well as $A_-^{-\gamma}$ and C_γ, are isomorphic.

To this end, given $f \in H(\mathbb{D})$ with $f(z) = \sum_{j=0}^\infty a_j z^j$, put $f_n(z) = \sum_{j=2^n}^{2^{n+1}-1} a_j z^j$. Define $(Tf)(0) = a_0$ and

$$(Tf)(j) = f_n(e^{i2\pi j/2^n}) \quad \text{ if } \quad 2^n \leq j \leq 2^{n+1} - 1$$

and $Tf = (\ (Tf)(j)\ )_{j=0}^\infty$.

The following technical result will be proved at the end of this section.

Lemma 3.3. *For each $0 < \mu_1 < \mu < \mu_2$ there are constants $d_1 > 0$ and $d_2 > 0$ such that the following hold*:

(i) $|||Tf|||_\mu \leq d_2 ||f||_{\mu_1}$ *for every $f \in H(\mathbb{D})$.*

(ii) *For each $x = (x_j)$ such that $|||x|||_\mu < \infty$ there is $f \in H(\mathbb{D})$ such that $Tf = x$ and $d_1||f||_{\mu_2} \leq |||x|||_\mu$.*

Proposition 3.4.

(a) $T|_{A_+^{-\gamma}}$ is an isomorphism between $A_+^{-\gamma}$ and B_γ.

(b) $T|_{A_-^{-\gamma}}$ is an isomorphism between $A_-^{-\gamma}$ and C_γ.

Proof. (a) Lemma 3.3(i) shows that T is well defined and continuous. On the other hand, part (ii) implies that T is bijective. For the injectivity observe that the values $f_n(e^{i2\pi j/2^n})$ are unique, since $f_n(z)/z^{2^n}$ is a polynomial of degree at most $2^n - 1$, and its value is taken at 2^n different points. See also the proof of Lemma 3.3 below. Finally, the estimate in Lemma 3.3(ii) shows that $T|_{A_+^{-\gamma}}$ is an isomorphism. The continuity of the inverse can also be deduced by the open mapping theorem for Fréchet spaces.

The proof for (b) is similar. □

Proposition 3.4 completes the proof of Theorem 3.2.

It remains to prove Lemma 3.3. Its proof is technical and requires several steps.

First we recall some basic facts from classical approximation theory. See [13] and [14]. Let, for $m \in \mathbb{N}$,

$$D_m(\varphi) = \sum_{j=-m}^{m} e^{ij\varphi}, \quad \varphi \in [0, 2\pi],$$

be the Dirichlet kernel and put

$$(P_m f)(re^{i\varphi}) = (D_m * f)(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} D_m(\varphi - \psi) f(re^{i\psi}) d\psi.$$

Then we obtain

$$(P_m f)(re^{i\varphi}) = \sum_{j=-m}^{m} a_j r^j e^{ij\varphi} \quad \text{provided that} \quad f(re^{i\varphi}) = \sum_{j=-\infty}^{\infty} a_j r^j e^{ij\varphi}.$$

Let, for $r > 0$, $M_1(f, r) = (2\pi)^{-1}\int_0^{2\pi} |f(re^{i\varphi})| d\varphi$. It is well known that

$$D_m \geq 0, \qquad M_1(D_m, 1) \leq c\log(m), \qquad M_q(P_m f, r) \leq c\log(m) M_q(f, r)$$

if $q \in \{1, \infty\}$. Here $c \geq 1$ is a constant independent of m.

The following lemma is essentially known. Since we do not have a precise reference we insert a proof which is a modification of the proof of [14, II E 9].

Lemma 3.5. *There is a universal constant $c \geq 1$ such that, for any f with $f(z) = \sum_{j=2^n}^{2^{n+1}-1} a_j z^j$, we have*

$$\sup_{j=1,\dots,2^n} |f(e^{i2\pi\, j/2^n})| \leq M_\infty(f, 1) \leq cn^2 \sup_{j=1,\dots,2^n} |f(e^{i2\pi\, j/2^n})|.$$

Proof. Let $\varphi_j = 2\pi j/2^n$, $j = 1, \dots, 2^n$. For functions g of the form $g(\varphi) = \sum_{k=-2^n}^{2^n} b_k \exp(ik\varphi)$ we have, since $\sum_{j=1}^{2^n} \exp(i2\pi kj/2^n) = 0$ for $k \neq 0$,

$$\frac{1}{2^n}\sum_{j=1}^{2^n} g(\varphi_j) = b_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\varphi) d\varphi. \tag{3.1}$$

We claim that

$$\frac{1}{2^n}\sum_{j=1}^{2^n} |g(\varphi_j)| \leq cn\frac{1}{2\pi}\int_0^{2\pi} |g(\varphi)|, d\varphi \tag{3.2}$$

where $c \geq 1$ is a universal constant. Indeed, we have $D_{2^n} * g = g$, and hence, using (3.1), we conclude that

$$\begin{aligned}
\frac{1}{2^n}\sum_{j=1}^{2^n}|g(\varphi_j)| &= \frac{1}{2^n}\sum_{j=1}^{2^n}|\frac{1}{2\pi}\int_0^{2\pi} D_{2^n}(\varphi_j-\psi)g(\psi)d\psi| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2^n}\sum_{j=1}^{2^n} D_{2^n}(\varphi_j-\psi)|g(\psi)|d\psi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2^n} D_{2^n}(\varphi-\psi)d\varphi|g(\psi)|d\psi \\
&\leq cn\frac{1}{2\pi}\int_0^{2\pi}|g(\psi)|d\psi.
\end{aligned}$$

Now take f as in the statement and put

$$g(e^{i\varphi}) = e^{-i3\cdot 2^{n-1}\varphi}f(e^{i\varphi}) = \sum_{j=-2^{n-1}}^{2^{n-1}-1} a_{j+3\cdot 2^{n-1}}e^{ij\varphi}.$$

We use that $l\cdot g$ is a trigonometric polynomial of degree 2^n if l is a trigonometric polynomial of degree 2^{n-1}.

For each $\varepsilon > 0$, we choose $h \in L_1(\partial\mathbb{D})$ such that $M_1(h,1) = 1$ and $\frac{1}{1+\varepsilon}M_\infty(g,1) \leq \frac{1}{2\pi}|\int_0^{2\pi} h(e^{i\varphi})g(e^{i\varphi})d\varphi|$. Then, using (3.2), we get

$$\begin{aligned}
\frac{1}{1+\varepsilon}M_\infty(f,1) &= \frac{1}{1+\varepsilon}M_\infty(g,1) \\
&\leq \frac{1}{2\pi}|\int_0^{2\pi} h(e^{i\varphi})g(e^{i\varphi})d\varphi| \\
&= \frac{1}{2\pi}|\int_0^{2\pi} (D_{2^{n-1}}h)(e^{i\varphi})g(e^{i\varphi})d\varphi| \\
&= |\frac{1}{2^n}\sum_{j=1}^{2^n}(D_{2^{n-1}}h)(e^{i\varphi_j})g(e^{i\varphi_j})| \\
&\leq \frac{1}{2^n}\sum_{j=1}^{2^n}|(D_{2^{n-1}}h)(e^{i\varphi_j})|\cdot|g(e^{i\varphi_j})| \\
&\leq cn\frac{1}{2\pi}\int_0^{2\pi}|(D_{2^{n-1}}h)(e^{i\varphi})|d\varphi\sup_j|g(e^{i\varphi_j})| \\
&\leq c^2n^2M_1(h,1)\sup_j|g(e^{i\varphi_j})| \\
&= c^2n^2\sup_j|f(e^{i\varphi_j})|,
\end{aligned}$$

where the second equality follows from the restriction of the degree of g and the usual orthonormality relations.

Since ε is arbitrary, this proves the right-hand side inequality of the statement. The left-hand side is trivial. □

Completion of the proof of Lemma 3.3. We consider $r_{\mu,n} = 1 - \mu/(2^n + \mu)$ for given $\mu > 0$. The function $r^{2^n}(1-r)^\mu$ attains its maximum at $r_{\mu,n}$. Let $f(z) = \sum_{j=0}^\infty a_j z^j \in H(\mathbb{D})$ and $f_n(z) = \sum_{j=2^n}^{2^{n+1}-1} a_j z^j$. It suffices to consider the case $f(0) = a_0 = 0$. Put $g_n(z) = \sum_{j=0}^{2^n-1} a_{j+2^n} z^j$. We obtain, for $r < r_{\mu,n}$,

$$\begin{aligned} M_\infty(f_n, r)(1-r)^\mu &\leq \frac{r^{2^n}(1-r)^\mu}{r_{\mu,n}^{2^n}(1-r_{\mu,n})^\mu} M_\infty(g_n, r) r_{\mu,n}^{2^n}(1-r_{\mu,n})^\mu \\ &\leq M_\infty(g_n, r_{\mu,n}) r_{\mu,n}^{2^n}(1-r_{\mu,n})^\mu \\ &\leq M_\infty(f_n, r_{\mu,n})(1-r_{\mu,n})^\mu \\ &\leq M_\infty(f_n, 1)(1-r_{\mu,n})^\mu. \end{aligned}$$

We have for $r_{\mu,n} < s < 1$,

$$M_\infty(f_n, s)(1-s)^\mu \leq M_\infty(f_n, 1)(1-r_{\mu,n})^\mu,$$

and combining this with the previous estimate yields

$$||f_n||_\mu \leq M_\infty(f_n, 1)(1-r_{\mu,n})^\mu. \tag{3.3}$$

Moreover we have, by [10, Lemma 3.1(a)],

$$\begin{aligned} M_\infty(f_n, 1) &\leq \left(\frac{1}{r_{\mu,n}}\right)^{2^{n+1}} M_\infty(f_n, r_{\mu,n}) \\ &= (1 + \frac{\mu}{2^n})^{2^{n+1}} M_\infty(f_n, r_{\mu,n}) \\ &\leq e^{2\mu} M_\infty(f_n, r_{\mu,n}). \end{aligned} \tag{3.4}$$

Now let $\mu_1 < \mu < \mu_2$. In view of (3.4) we have

$$\begin{aligned} &\sup_n \frac{\mu^\mu}{(2^n+\mu)^\mu} \sup_{2^n \leq j < 2^{n+1}} |f_n(e^{i2\pi j/2^n})| \\ &\leq \sup_n \frac{\mu^\mu}{(2^n+\mu)^\mu} M_\infty(f_n, 1) \\ &\leq c_1 \sup_n \frac{\mu^\mu}{\mu_1^{\mu_1}} \frac{(2^n+\mu_1)^{\mu_1}}{(2^n+\mu)^\mu} \frac{\mu_1^{\mu_1}}{(2^n+\mu_1)^{\mu_1}} M_\infty(f_n, r_{\mu_1,n}) \\ &\leq c_1 \sup_n \delta_n ||f_n||_{\mu_1} \\ &= c_1 \sup_n \delta_n ||(P_{2^{n+1}-1} - P_{2^n-1})f||_{\mu_1} \\ &\leq c_1 c_2 \sup_n \delta_n n ||f||_{\mu_1}, \end{aligned}$$

where

$$\delta_n = \frac{\mu^\mu}{\mu_1^{\mu_1}} \frac{(2^n+\mu_1)^{\mu_1}}{(2^n+\mu)^\mu}$$

and c_1, c_2 are universal constants. Since $\mu > \mu_1$ we obtain $\sup_n \delta_n n < \infty$. This proves part (i).

On the other hand, with Lemma 3.5 and (3.3) applied to μ_2 we obtain

$$\begin{aligned} ||f||_{\mu_2} &\leq \sum_{n=0}^{\infty} ||f_n||_{\mu_2} \\ &\leq \sum_{n=0}^{\infty}(1-r_{\mu_2,n})^{\mu_2} M_\infty(f_n,1) \\ &\leq c\sum_{n=0}^{\infty} \frac{\mu_2^{\mu_2}}{\mu^\mu}\frac{(2^n+\mu)^\mu}{(2^n+\mu_2)^{\mu_2}}\frac{\mu^\mu}{(2^n+\mu)^\mu} n^2 \sup_{2^n\leq j<2^{n+1}} |f_n(e^{i2\pi j/2^n})| \\ &\leq d\sup_n \frac{\mu^\mu}{(2^n+\mu)^\mu} \sup_{2^n\leq j<2^{n+1}} |f_n(e^{i2\pi j/2^n})|, \end{aligned}$$

where

$$d = c\sum_{n=0}^{\infty} \frac{\mu_2^{\mu_2}}{\mu^\mu}\frac{(2^n+\mu)^\mu}{(2^n+\mu_2)^{\mu_2}} n^2.$$

Since $\mu_2 > \mu$ this series converges.

Because dim $\{f_n : f \in A_+^{-\gamma}\} = 2^n$ = number of the elements $\exp(i2\pi j/2^n)$ if $j = 2^n, \ldots, 2^{n+1}-1$, given $x = (x_j)$, the polynomials f_n with $f_n(e^{i2\pi j/2^n}) = x_j$ if $2^n \leq j \leq 2^{n+1}-1$ are uniquely defined. Consequently, the estimates above imply statement (ii).

The proof of Lemma 3.3 is now complete.

References

[1] Angela A. Albanese, José Bonet, and Werner J. Ricker, *The Cesàro operator on Korenblum type spaces of analytic functions*, Collect. Math. **69** (2018), no. 2, 263–281, DOI 10.1007/s13348-017-0205-7. MR3783155

[2] Carlos A. Berenstein and Roger Gay, *Complex variables*, An introduction, Graduate Texts in Mathematics, vol. 125, Springer-Verlag, New York, 1991. MR1107514

[3] Klaus D. Bierstedt, José Bonet, and Antonio Galbis, *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J. **40** (1993), no. 2, 271–297, DOI 10.1307/mmj/1029004753. MR1226832

[4] Klaus D. Bierstedt, José Bonet, and Jari Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), no. 2, 137–168. MR1488148

[5] Klaus-D. Bierstedt, Reinhold G. Meise, and William H. Summers, *Köthe sets and Köthe sequence spaces*, Functional analysis, holomorphy and approximation theory (Rio de Janeiro, 1980), North-Holland Math. Stud., vol. 71, North-Holland, Amsterdam-New York, 1982, pp. 27–91. MR691159

[6] Boris Korenblum, *An extension of the Nevanlinna theory*, Acta Math. **135** (1975), no. 3-4, 187–219, DOI 10.1007/BF02392019. MR0425124

[7] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000. MR1758653

[8] Hans Jarchow, *Locally convex spaces*, Mathematische Leitfäden. [Mathematical Textbooks], B. G. Teubner, Stuttgart, 1981. MR632257

[9] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. I*, Sequence spaces; Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92, Springer-Verlag, Berlin-New York, 1977. MR0500056

[10] Wolfgang Lusky, *On the Fourier series of unbounded harmonic functions*, J. London Math. Soc. (2) **61** (2000), no. 2, 568–580, DOI 10.1112/S0024610799008443. MR1760680

[11] Reinhold Meise and Dietmar Vogt, *Introduction to functional analysis*, translated from the German by M. S. Ramanujan and revised by the authors, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997. MR1483073

[12] A. L. Shields and D. L. Williams, *Bonded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. **162** (1971), 287–302, DOI 10.2307/1995754. MR0283559

[13] Alberto Torchinsky, *Real-variable methods in harmonic analysis*, Pure and Applied Mathematics, vol. 123, Academic Press, Inc., Orlando, FL, 1986. MR869816

[14] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Studies in Advanced Mathematics, vol. 25, Cambridge University Press, Cambridge, 1991. MR1144277

Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain
Email address: jbonet@mat.upv.es

Institut für Mathematik, Universität Paderborn, D-33098 Paderborn, Germany
Email address: lusky@uni-paderborn.de

Department of Mathematics and Statistics, P.O. Box 68, University of Helsinki, 00014 Helsinki, Finland
Email address: jari.taskinen@helsinki.fi