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ASYMPTOTIC VARIANCE OF THE NUMBER OF REAL ROOTS OF RANDOM POLYNOMIAL SYSTEMS

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Abstract. We obtain the asymptotic variance, as the degree goes to infinity, of the normalized number of real roots of a square Kostlan-Shub-Smale random polynomial system of any size. Our main tools are the Kac-Rice formula for the second factorial moment of the number of roots and a Hermite expansion of this random variable.

1. Introduction

The study of the roots of random polynomials is among the most important and popular topics in mathematics and in some areas of physics. For almost a century a considerable amount of literature about this problem has emerged from fields such as probability, geometry, algebraic geometry, algorithm complexity, quantum physics, etc. In spite of its rich history it is still an extremely active field.

There are several reasons that lead one to consider random polynomials and several ways to randomize them; see Bharucha-Reid and Sambandham [3].

The case of algebraic polynomials $P_d(t) = \sum_{j=1}^{d} a_j t^j$ with independent identically distributed coefficients was the first one to be extensively studied and was completely understood during the 1970s. If a_1 is centered, $\mathbb{P}(a_1 = 0) = 0$, and $\mathbb{E}\left(|a_1|^{2+\delta}\right) < \infty$ for some $\delta > 0$, then the asymptotic expectation and the asymptotic variance of the number of real roots of P_d, as the degree d tends to infinity, are of order $\log(d)$, and, once normalized, the number of real roots converges in distribution towards a centered Gaussian random variable. See the books by Farahmand [7] and Bharucha-Reid and Sambandham [3] and the references therein for the whole picture.

The case of systems of polynomial equations seems to be considerably harder and has received in consequence much less attention. The results in this direction are confined to the Shub-Smale model and some other invariant distributions. The ensemble of Shub-Smale random polynomials was introduced in the early 1990s by Kostlan [9]. Kostlan argues that this is the most natural distribution for a polynomial system. The exact expectation was obtained in the early 1990's by geometric means; see Edelman and Kostlan [5] for the one-dimensional case and Shub and Smale [18] for the multi-dimensional one. In 2004 and 2005 Azaïs and Wschebor [2] and Wschebor [19] obtained by probabilistic methods the asymptotic variance

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as the number of equations and variables tends to infinity. Recently, Dalmao [4] obtained the asymptotic variance and a CLT for the number of zeros as the degree d goes to infinity in the case of one equation in one variable. Letendre in [13] studied the asymptotic behavior of the volume of random real algebraic submanifolds. His results include the finiteness of the limit variance, when the degree tends to infinity, of the volume of the zero sets of Kostlan-Shub-Smale systems with strictly fewer equations than variables. Some results for the expectation and variance of related models are included in [2, 11, 12].

In the present paper we prove that as the degree goes to infinity, the asymptotic variance of the normalized number of real roots of a Kostlan-Shub-Smale square random system with m equations and m variables exists in $(0, \infty)$. We use Rice Formulas [1] to show the finiteness of the limit variance and Hermite expansions as in Kratz and León [10] to show that it is strictly positive. Furthermore, we strongly exploit the invariance under isometries of the distribution of the polynomials.

The reader may wonder, in view of the results mentioned above, if the normalized number of roots satisfies a CLT when the degree of the system tends to infinity. The answer is affirmative if $m = 1$ [4], but for the time being we cannot give an answer to this question for $m > 1$. The ingredients to prove a CLT for a nonlinear functional of a Gaussian process are: a) to write a representation in the Itô-Wiener chaos of the normalized functional, b) to demonstrate that each component verifies a CLT (Fourth Moment Theorem [16], [17]), and if the functional has an expansion involving infinitely many terms: c) to prove that the tail of the asymptotic variance tends uniformly (w.r.t. d) to zero. In the present case we lack a proof of c). For $m = 1$ the fact that the invariance by rotations is equivalent with the stationarity allows us to build a proof similar to the one made for the number of crossings of a stationary Gaussian process.

The rest of the paper is organized as follows. Section 2 sets the problem and presents the main result. Section 3 deals with the proof.

2. Main result

Consider a square system $\mathbf{P}$ of m polynomial equations in m variables with common degree $d > 1$. More precisely, let $\mathbf{P} = (P_1, \dots, P_m)$ with

$$P_\ell(t) = \sum_{|\boldsymbol{j}| \le d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}},$$

where

(1) $\boldsymbol{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$ and $|\boldsymbol{j}| = \sum_{k=1}^m j_k$;
(2) $a_{\boldsymbol{j}}^{(\ell)} = a_{j_1 \cdots j_m}^{(\ell)} \in \mathbb{R}$, $\ell = 1, \dots, m$, $|\boldsymbol{j}| \le d$;
(3) $t = (t_1, \dots, t_m)$ and $t^{\boldsymbol{j}} = \prod_{k=1}^m t_k^{j_k}$.

We say that $\mathbf{P}$ has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{\boldsymbol{j}}^{(\ell)}$ are independent centered normally distributed random variables with variances

$$\operatorname{Var}\left(a_{\boldsymbol{j}}^{(\ell)}\right) = \binom{d}{\boldsymbol{j}} = \frac{d!}{j_1! \dots j_m!(d - |\boldsymbol{j}|)!}.$$

We are interested in the number of real roots of $\mathbf{P}$, which we denote by $N_d^{\mathbf{P}}$. Shub and Smale [18] proved that $\mathbb{E}\left(N_d^{\mathbf{P}}\right) = d^{m/2}$. Our main result is the following.

Theorem 2.1. *Let* $\mathbf{P}$ *be a KSS random polynomial system with* m *equations,* m *variables, and degree* d*. Then, as* $d \to \infty$ *we have*

$$\lim_{d\to\infty} \frac{\operatorname{Var}(N_d^{\mathbf{P}})}{d^{m/2}} = V_\infty^2,$$

where $0 < V_\infty^2 < \infty$.

2.1. **Explicit expression of the variance.** Using the method of Section 12.1.2 of [1] an explicit expression for the limit variance can be given.

For $k = 1, \dots, m$ let ξ_k, η_k be independent standard normal random vectors on $\mathbb{R}^k$. Let us define:

- $\bar\sigma^2(t) = 1 - \frac{t^2\exp(-t^2)}{1-\exp(-t^2)}$;
- $\bar\rho(t) = \frac{(1-t^2-\exp(-t^2))\exp(-t^2/2)}{1-(1+t^2)\exp(-t^2)}$;
- $m_{k,j} = \mathbb{E}\left(\|\xi_k\|^j\right) = 2^{j/2}\frac{\Gamma((j+k)/2)}{\Gamma(k/2)}$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^k$;
- for $k = 1, \dots, m-1$, $M_k(t) = \mathbb{E}\left[\|\xi_k\|\,\|\eta_k + \frac{e^{-t^2/2}}{(1-e^{-t^2})^{1/2}}\xi_k\|\right]$;
- for $k = m$, $M_m(t) = \mathbb{E}\left[\|\xi_m\|\,\|\eta_m + \frac{\bar\rho(t)}{(1-\bar\rho^2(t))^{1/2}}\xi_m\|\right]$.

Theorem 2.2. *We have*

$$V_\infty^2 = \frac{1}{2} + \frac{\kappa_m\kappa_{m-1}}{2(2\pi)^m}\cdot\int_0^\infty t^{m-1}\left[\frac{\bar\sigma^4(t)(1-\bar\rho^2(t))}{1-e^{-t^2}}\right]^{1/2}\left[\prod_{k=1}^m M_k(t) - \prod_{k=1}^m m_{k,1}^2\right] dt.$$

□

3. Proof

3.1. **Preliminaries.** It is customary and convenient to homogenize the polynomials, that is, to add an auxiliary variable t_0 and to multiply the monomial in P_ℓ corresponding to the index $\boldsymbol{j}$ by $t_0^{d-|\boldsymbol{j}|}$. Let $\mathbf{Y} = (Y_1, \dots, Y_m)$ denote the resulting vector of m homogeneous polynomials in $m+1$ real variables with common degree $d > 1$. We have

$$Y_\ell(t) = \sum_{|\boldsymbol{j}|=d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}, \quad \ell = 1, \dots, m,$$

where this time $\boldsymbol{j} = (j_0, \dots, j_m) \in \mathbb{N}^{m+1}$, $|\boldsymbol{j}| = \sum_{k=0}^m j_k$, $a_{\boldsymbol{j}}^{(\ell)} = a_{j_0\cdots j_m}^{(\ell)} \in \mathbb{R}$, $t = (t_0, \dots, t_m) \in \mathbb{R}^{m+1}$, and $t^{\boldsymbol{j}} = \prod_{k=0}^m t_k^{j_k}$.

Since $\mathbf{Y}$ is homogeneous, its roots consist of lines through 0 in $\mathbb{R}^{m+1}$. Then, it is easy to check that each root of $\mathbf{P}$ corresponds exactly to two (opposite) roots of $\mathbf{Y}$ on the unit sphere S^m of $\mathbb{R}^{m+1}$. Furthermore, one can prove that the subset of homogeneous polynomials $\mathbf{Y}$ with roots lying in the hyperplane $t_0 = 0$ has Lebesgue measure zero. Then, denoting by $N_d^{\mathbf{Y}}$ the number of roots of $\mathbf{Y}$ on S^m, we have $N_d^{\mathbf{P}} = N_d^{\mathbf{Y}}/2$ almost surely.

From now on we work with the homogenized version $\mathbf{Y}$. The standard multinomial formula shows that for all $s, t \in \mathbb{R}^{m+1}$ we have

$$r_d(s,t) := \mathbb{E}\left(Y_\ell(s)Y_\ell(t)\right) = \langle s, t\rangle^d,$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^{m+1}$. As a consequence, we see that the distribution of the system $\mathbf{Y}$ is invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$. For ease of notation we omit the dependence on d of $\mathbf{Y}$.

In the sequel we need to consider the derivative of Y_ℓ, $\ell = 1, \dots, m$. Since the parameter space is the sphere S^m, the derivative is taken in the sense of the sphere; that is, the spherical derivative $Y'_\ell(t)$ of $Y_\ell(t)$ is the orthogonal projection of the free gradient on the tangent space $t^\perp$ of S^m at t. The k-th component of $Y'_\ell(t)$ at a given basis of the tangent space is denoted by $Y'_{\ell k}(t)$.

The covariances between the derivatives and between the derivatives and the process are obtained via routine computations from the covariance of Y_ℓ. In particular, the invariance under isometries is preserved after derivation, and for each $t \in S^m$, $\mathbf{Y}(t)$ is independent from $\mathbf{Y}'(t) = (Y'_1(t), \dots, Y'_m(t))$.

3.2. **Finiteness of the limit variance.** In this section we prove that

$$\lim_{d\to\infty} \frac{\operatorname{Var}(N_d^{\mathbf{P}})}{d^{m/2}} < \infty.$$

Recall that $\mathbb{E}\,(N_d^{\mathbf{P}}) = d^{m/2}$. We write

$$\operatorname{Var}\big(N_d^{\mathbf{P}}\big) = \operatorname{Var}\left(\frac{N_d^{\mathbf{Y}}}{2}\right) = \frac{1}{4}\left[\mathbb{E}\left(N_d^{\mathbf{Y}}\left(N_d^{\mathbf{Y}} - 1\right)\right) - \mathbb{E}^{\,2}\big(N_d^{\mathbf{Y}}\big)\right] + \frac{d^{m/2}}{2}. \tag{3.1}$$

The quantity $\mathbb{E}\,(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}} - 1))$ is computed using the Rice formula [1, Theorem 6.3] and a localization argument:

$$\mathbb{E}\,(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}} - 1)) = \int_{(S^m)^2} \mathbb{E}\,[|\det \mathbf{Y}'(s) \det \mathbf{Y}'(t)|\,|\mathbf{Y}(s) = \mathbf{Y}(t) = 0] \cdot p_{\mathbf{Y}(s),\mathbf{Y}(t)}(0,0) ds dt.$$

Here ds and dt are the m-geometric measure on S^m, but we will use in other parts ds and dt for the Lebesgue measure.

The following lemma allows us to reduce this integral to a one-dimensional one. The proof is a direct consequence of the co-area formula.

Lemma 3.1. *Let $\mathcal{H}$ be a measurable function defined on $\mathbb{R}$. Then, we have*

$$\begin{aligned}\int_{(S^m)^2} \mathcal{H}(\langle s,t\rangle)\, ds\, dt &= \kappa_m \kappa_{m-1} \int_0^\pi \sin(\psi)^{m-1} \mathcal{H}(\cos(\psi))\, d\psi \\ &= \frac{\kappa_m \kappa_{m-1}}{\sqrt{d}} \int_0^{\sqrt{d}\pi} \sin\left(\frac{z}{\sqrt{d}}\right)^{m-1} \mathcal{H}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right) dz,\end{aligned}$$

where κ_m is the m-geometric measure of S^m. □

Let $\{e_0, e_1, \dots, e_m\}$ be the canonical basis of $\mathbb{R}^{m+1}$. Because of the invariance of $\mathbf{Y}$ by isometries we can assume without loss of generality that

$$s = e_0, \quad t = \cos(\psi) e_0 + \sin(\psi) e_1. \tag{3.2}$$

For $s^\perp$ we choose as basis $\{e_1, \dots, e_m\}$ and $\{\sin(\psi)e_0 - \cos(\psi)e_1, e_2, \dots, e_m\}$ for $t^\perp$. Finally, take $\psi = z/\sqrt{d}$ and use Lemma 3.1. Hence,

$$\begin{aligned}&d^{-m/2}\mathbb{E}\,(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}} - 1)) \\ &\quad = \frac{\kappa_m \kappa_{m-1}}{(2\pi)^m \sqrt{d}} \int_0^{\sqrt{d}\pi} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \frac{d^{m/2}}{\left(1 - \cos^{2d}(\frac{z}{\sqrt{d}})\right)^{m/2}} \mathcal{E}\left(\frac{z}{\sqrt{d}}\right) dz,\end{aligned}$$

where $\mathcal{E}(z/\sqrt{d})$ is the conditional expectation written for s, t as in (3.2).

Now, we deal with the conditional expectation $\mathcal{E}(z/\sqrt{d})$. We introduce the following notation:

$$\begin{aligned}\mathcal{A}\left(\frac{z}{\sqrt{d}}\right) &= -\sqrt{d}\cos^{d-1}\left(\frac{z}{\sqrt{d}}\right)\sin\left(\frac{z}{\sqrt{d}}\right);\\ \mathcal{B}\left(\frac{z}{\sqrt{d}}\right) &= \cos^{d}\left(\frac{z}{\sqrt{d}}\right) - (d-1)\cos^{d-2}\left(\frac{z}{\sqrt{d}}\right)\sin^2\left(\frac{z}{\sqrt{d}}\right);\\ \mathcal{C}\left(\frac{z}{\sqrt{d}}\right) &= \cos^{d}\left(\frac{z}{\sqrt{d}}\right);\\ \mathcal{D}\left(\frac{z}{\sqrt{d}}\right) &= \cos^{d-1}\left(\frac{z}{\sqrt{d}}\right);\end{aligned} \tag{3.3}$$

and, omitting the $(z/\sqrt{d})$:

$$\sigma^2 = 1 - \frac{\mathcal{A}^2}{1-\mathcal{C}^2}, \qquad \rho = \frac{\mathcal{B}(1-\mathcal{C}^2) - \mathcal{A}^2\mathcal{C}}{1-\mathcal{C}^2-\mathcal{A}^2}.$$

Thus, the variance-covariance matrix of the vector $\left(Y_\ell(s), Y_\ell(t), \frac{Y_\ell'(s)}{\sqrt{d}}, \frac{Y_\ell'(t)}{\sqrt{d}}\right)$ at the given basis can be written in the form

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{12}^\top & I_m & A_{23} \\ \hline A_{13}^\top & A_{23}^\top & I_m \end{array}\right], \tag{3.4}$$

where I_m is the $m \times m$ identity matrix,

$$A_{11} = \begin{bmatrix} 1 & \mathcal{C} \\ \mathcal{C} & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\mathcal{A} & 0 & \cdots & 0 \end{bmatrix},\ A_{13} = \begin{bmatrix} \mathcal{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.5}$$

and A_{23} is the $m \times m$ diagonal matrix $\mathrm{diag}(\mathcal{B}, \mathcal{D}, \ldots, \mathcal{D})$.

Gaussian regression formulas (see [1, Proposition 1.2]) imply that the conditional distribution of the vector $\left(\frac{Y_\ell'(s)}{\sqrt{d}}, \frac{Y_\ell'(t)}{\sqrt{d}}\right)$ (conditioned on $\mathbf{Y}(s) = \mathbf{Y}(t) = 0$) is centered normal with variance-covariance matrix given by

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{12}^\top & B_{22} \end{array}\right], \tag{3.6}$$

with $B_{11} = B_{22} = \mathrm{diag}(\sigma^2, 1, \ldots, 1)$ and $B_{12} = \mathrm{diag}(\sigma^2\rho, \mathcal{D}, \ldots, \mathcal{D})$.

It is important to remark that if $A = (A_1\, A_2 \ldots A_m)$ is a matrix with column vectors A_j, it holds that $\det(A) = Q_m(A_1, A_2, \ldots, A_m)$ for a certain polynomial Q_m of degree m from $\mathbb{R}^{m^2}$ to $\mathbb{R}$. Using representation of Gaussian vectors from a standard one we can write

$$\mathcal{E}\left(\frac{z}{\sqrt{d}}\right) = \int_{(\mathbb{R}^{m^2})^2} \phi_{m^2}(\mathbf{x})\phi_{m^2}(\mathbf{y}) \left| Q_m\left(\begin{pmatrix} \sigma x_{11} \\ x_{12} \\ \cdot \\ x_{1m} \end{pmatrix}, \ldots, \begin{pmatrix} \sigma x_{m1} \\ x_{m2} \\ \cdot \\ x_{mm} \end{pmatrix}\right)\right|$$
$$\times \left| Q_m\left(\begin{pmatrix} \sigma(\rho x_{11} + \sqrt{1-\rho^2}y_{11}) \\ \mathcal{D}x_{12} + \sqrt{1-\mathcal{D}^2}y_{12} \\ \cdot \\ \mathcal{D}x_{1m} + \sqrt{1-\mathcal{D}^2}y_{1m} \end{pmatrix}, \ldots, \begin{pmatrix} \sigma(\rho x_{m1} + \sqrt{1-\rho^2}y_{m1}) \\ \mathcal{D}x_{m2} + \sqrt{1-\mathcal{D}^2}y_{m2} \\ \cdot \\ \mathcal{D}x_{mm} + \sqrt{1-\mathcal{D}^2}y_{mm} \end{pmatrix}\right)\right| d\mathbf{x}d\mathbf{y},$$

where ϕ_{m^2} is the standard normal density in $\mathbb{R}^{m^2}$. Because of the homogeneity of the determinant we have

$$\mathcal{E}\left(\frac{z}{\sqrt{d}}\right) = \sigma^2 \int_{(\mathbb{R}^{m^2})^2} Q_m(\mathbf{x})Q_m(\mathbf{z})\phi_{m^2}(\mathbf{x})\phi_{m^2}(\mathbf{y})d\mathbf{x}d\mathbf{y} =: \sigma^2 G(\rho, \mathcal{D}),$$

where $\mathbf{z} = \operatorname{diag}(\rho, \mathcal{D}, \ldots, \mathcal{D})\mathbf{x} + \operatorname{diag}(\sqrt{1-\rho^2}, \sqrt{1-\mathcal{D}^2}, \ldots, \sqrt{1-\mathcal{D}^2})\mathbf{y}$.

Now, we return to the expression of the variance in (3.1). We have

$$\begin{aligned} (3.7)\quad d^{-m/2}\operatorname{Var}\left(N_d^{\mathbf{P}}\right) &= \frac{1}{4d^{m/2}}\left[\mathbb{E}\left(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)\right) - \left(\mathbb{E}\left(N_d^{\mathbf{Y}}\right)\right)^2\right] + \frac{1}{2} \\ &= \frac{1}{2} + \frac{\kappa_m \kappa_{m-1}}{4(2\pi)^m} \int_0^{\sqrt{d}\pi} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) d^{(m-1)/2} \\ &\quad \times \left[\frac{\sigma^2(\frac{z}{\sqrt{d}})}{(1-\cos^{2d}(\frac{z}{\sqrt{d}}))^{m/2}} G\Big(\rho\Big(\frac{z}{\sqrt{d}}\Big), \mathcal{D}\Big(\frac{z}{\sqrt{d}}\Big)\Big) - G(0,0)\right] dz. \end{aligned}$$

The proof of the convergence of this integral is done in several steps.

In the rest of this section $\mathbf{C}$ denotes an unimportant constant; its value can change from one occurrence to another. It can depend on m, but recall that m is fixed.

Step 1 (Bounds for G).

- $G(\rho, \mathcal{D}) = \int_{(\mathbb{R}^{m^2})^2} Q_m(\mathbf{x})Q_m(\mathbf{z})\phi_{m^2}(\mathbf{x})\phi_{m^2}(\mathbf{y})d\mathbf{x}d\mathbf{y}$;
- $G(0,0) = \int_{(\mathbb{R}^{m^2})^2} Q_m(\mathbf{x})Q_m(\mathbf{y})\phi_{m^2}(\mathbf{x})\phi_{m^2}(\mathbf{y})d\mathbf{x}d\mathbf{y}$;
- $|\sqrt{1-\rho^2} - 1| \le \mathbf{C}|\rho|$; $|\sqrt{1-\mathcal{D}^2} - 1| \le \mathbf{C}|\mathcal{D}|$;
- $|Q_m(\mathbf{x})| \le \mathbf{C}(1 + \|\mathbf{x}\|_\infty)^m$;
- any partial derivative of $Q_m(\mathbf{w})$ is a polynomial of degree $m-1$ and thus it is bounded by $\mathbf{C}(1 + \|\mathbf{w}\|_\infty)^{m-1}$.

Applying these bounds to a point between $\mathbf{y}$ and $\mathbf{z}$, we get

$$\begin{aligned} |Q_m(\mathbf{z}) - Q_m(\mathbf{y})| &\le \mathbf{C}(1 + \|\mathbf{y}\|_\infty + \|\mathbf{z}\|_\infty)^{m-1}(|\rho| + |\mathcal{D}|) \\ &\le \mathbf{C}(1 + \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty)^{m-1}(|\rho| + |\mathcal{D}|) \end{aligned}$$

and

$$\begin{aligned} &|Q_m(\mathbf{x}) \cdot Q_m(\mathbf{z}) - Q_m(\mathbf{x}) \cdot Q_m(\mathbf{y})| \\ &\qquad \le \mathbf{C}(1 + \|\mathbf{x}\|_\infty)^m (1 + \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty)^{m-1}(|\rho| + |\mathcal{D}|). \end{aligned}$$

The finiteness of all the moments of the supremum of Gaussian random variables finally yields

$$|G(\rho, \mathcal{D}) - G(0,0)| \le \mathbf{C}(|\rho| + |\mathcal{D}|).$$

Step 2 (Point-wise convergence). It is a direct consequence of the expansions of sine and cosine functions. As d tends to infinity:

- $\mathcal{A}(\frac{z}{\sqrt{d}}) \to -z\exp(-z^2/2)$;
- $\mathcal{B}(\frac{z}{\sqrt{d}}) \to (1-z^2)\exp(-z^2/2)$;
- $\mathcal{C}(\frac{z}{\sqrt{d}})$ and $\mathcal{D}(\frac{z}{\sqrt{d}})$ tend to $\exp(-z^2/2)$;
- $\sigma^2(\frac{z}{\sqrt{d}}) \to \frac{1-(1+z^2)\exp(-z^2)}{1-\exp(-z^2)} = \bar{\sigma}^2(z)$;
- $\rho(\frac{z}{\sqrt{d}}) \to \frac{(1-t^2-\exp(-t^2))\exp(-t^2/2)}{1-(1+t^2)\exp(-t^2)} = \bar{\rho}(z)$,

being $\bar{\sigma}^2$ and $\bar{\rho}$ as in Subsection 2.1. This, in view of the continuity of the function G, implies the pointwise convergence of the integrand in (3.7).

Step 3 (Symmetrization). We have $\mathcal{A}(\pi - z/\sqrt{d}) = (-1)^{d-1}\mathcal{A}(z/\sqrt{d})$, $\mathcal{B}(\pi - z/\sqrt{d}) = (-1)^d\mathcal{B}(z/\sqrt{d})$, $\mathcal{C}(\pi - z/\sqrt{d}) = (-1)^d\mathcal{C}(z/\sqrt{d})$, $\mathcal{D}(\pi - z/\sqrt{d}) = (-1)^{d-1}\mathcal{D}(z/\sqrt{d})$, $\sigma^2(\pi - z\sqrt{d}) = \sigma^2(z/\sqrt{d})$, and $\rho(\pi - z\sqrt{d}) = (-1)^d\rho(z/\sqrt{d})$. Hence, $B_{12}(\pi - z/\sqrt{d})$ in (3.6) becomes

$$\big((-1)^d\sigma^2(z/\sqrt{d})\rho(z/\sqrt{d}), (-1)^{d-1}\mathcal{D}(z/\sqrt{d}), \dots, (-1)^{d-1}\mathcal{D}(z/\sqrt{d})\big),$$

the rest being unchanged. This corresponds, for example, to performing some change of signs (depending on the parity of d) on the coordinates of $Y'_\ell(t)$. Gathering the different ℓ this may imply a change of sign in $\det(\mathbf{Y}'(t))$ that plays no role because of the absolute value. As a consequence

$$\mathcal{E}(\pi - z/\sqrt{d}) = \mathcal{E}(z/\sqrt{d}).$$

In conclusion, for the next step it suffices to dominate the integral in the r.h.s of (3.7) restricted to the interval $[0, \sqrt{d}\pi/2]$.

Step 4 (Domination). The following lemma gives bounds for the different terms.

Lemma 3.2. *There exists some constant α, $0 < \alpha \le 1/2$, and some integer d_0 such that for $\frac{z}{\sqrt{d}} \le \frac{\pi}{2}$ and $d > d_0$:*

- $\mathcal{C} \le \mathcal{D} \le \cos^{d-2}(\frac{z}{\sqrt{d}}) \le \exp(-\alpha z^2)$;
- $|\mathcal{A}| \le z\exp(-\alpha z^2)$;
- $|\mathcal{B}| \le (1+z^2)\exp(-\alpha z^2)$;
- *for* $z \ge z_0$, $1 - \mathcal{C}^2 \ge 1 - \mathcal{C}^2 - \mathcal{A}^2 \ge \mathbf{C} > 0$;
- $0 \le 1 - \sigma^2 \le \mathbf{C}\exp(-2\alpha z^2)$;
- $|\rho| \le \mathbf{C}(1+z^2)^2\exp(-2\alpha z^2)$.

Proof. We give the proof of the first item; the other cases are similar or easier. On $[0, \pi/2]$ there exists α_1, $0 < \alpha_1 < 1/2$, such that

$$\cos(\psi) \le 1 - \alpha_1\psi^2.$$

Thus,

$$\cos^{d-2}\left(\frac{z}{\sqrt{d}}\right) \le \left(1 - \frac{\alpha_1 z^2}{d}\right)^{d-2} \le \exp\left(-\frac{\alpha_1 z^2(d-2)}{d}\right) \le \exp\left(-\alpha z^2\right),$$

as soon as $\alpha < \alpha_1$ and d is big enough. □

We have to find a dominant and prove the convergence of the integral at zero and at infinity.

At zero, since the function G is bounded we have to give bounds for

$$\frac{d^{\frac{m-1}{2}}\sin^{m-1}\left(\frac{z}{\sqrt{d}}\right)\sigma^2(\frac{z}{\sqrt{d}})}{\left(1 - \cos^{2d}(\frac{z}{\sqrt{d}})\right)^{m/2}}.$$

Clearly, $d^{\frac{m-1}{2}} \sin^{m-1}(z/\sqrt{d}) \leq z^{m-1}$. Besides,

$$\frac{\sigma^2\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos^{2d}(\frac{z}{\sqrt{d}})\right)^{\frac{m}{2}}} = \frac{1-c_d^2(z)-c_d'^2(z)}{(1-c_d^2(z))^{\frac{m}{2}+1}},$$

where $c(z) = \mathcal{C}(z/\sqrt{d})$.

For the denominator, using Lemma 3.2, we have

$$1 - c_d^2(z) \geq \mathbf{C}(1-\exp(-2\alpha z^2)). \tag{3.8}$$

We turn now to the numerator. Let $X_d(.)$ be a formal Gaussian stationary process on the line with covariance c_d. Hence,

$$\begin{aligned}1 - c_d^2(z) - c_d'^2(z) &= \operatorname{Var}\big(X_d(z)|X_d(0), X_d'(0)\big)\\ &= \operatorname{Var}\big(X_d(z) - X_d(0) - zX_d'(0)|X_d(0), X_d'(0)\big)\\ &\leq \operatorname{Var}\big(X_d(z) - X_d(0) - zX_d'(0)\big) = z^4 \operatorname{Var}\Big(\int_0^1 (1-t)X_d''(ut)dt\Big),\end{aligned}$$

where we used the Taylor formula with the integral form of the remainder. The covariance function $\cos(z/\sqrt{d})$ corresponds to the spectral measure

$$\mu = \frac{1}{2}\big(\delta_{-d^{-1/2}} + \delta_{d^{-1/2}}\big);$$

see [1]. The spectral measure associated to $c_d(z) = \cos^d(z/\sqrt{d})$ is the d-th convolution of μ, and a direct computation shows that its fourth spectral moment exists and is bounded uniformly in d. As a consequence, $\operatorname{Var}(X_d''(t))$ is bounded uniformly in d, yielding that

$$1 - c_d^2(z) - c_d'^2(z) \leq \mathbf{C} z^4. \tag{3.9}$$

Using (3.8) and (3.9) we get the convergence at zero.

At infinity, define

$$\begin{aligned}&\mathcal{H}\left(\sigma^2\left(\frac{z}{\sqrt{d}}\right), \mathcal{C}\left(\frac{z}{\sqrt{d}}\right), \rho\left(\frac{z}{\sqrt{d}}\right), \mathcal{D}\left(\frac{z}{\sqrt{d}}\right)\right)\\ &\qquad = \frac{\sigma^2(\frac{z}{\sqrt{d}})}{\left(1-\cos^{2d}(\frac{z}{\sqrt{d}})\right)^{m/2}} G\left(\rho\Big(\frac{z}{\sqrt{d}}\Big), \mathcal{D}\Big(\frac{z}{\sqrt{d}}\Big)\right) dz.\end{aligned}$$

Multiplication of bounded Lipschitz functions gives a Lipschitz function; thus

$$\begin{aligned}&\left|\mathcal{H}\Big(\sigma^2\Big(\frac{z}{\sqrt{d}}\Big), \mathcal{C}\Big(\frac{z}{\sqrt{d}}\Big), \rho\Big(\frac{z}{\sqrt{d}}\Big), \mathcal{D}\Big(\frac{z}{\sqrt{d}}\Big)\Big) - \mathcal{H}(1,0,0,0)\right|\\ &\qquad \leq \mathbf{C}\big(|\sigma^2 - 1| + |\mathcal{C}| + |\rho| + |\mathcal{D}|\big).\end{aligned}$$

The proof is achieved with Lemma 3.2.

3.3. **Positivity of the limit variance.**

3.3.1. *Hermite expansion of the number of real roots.* We introduce the Hermite polynomials $H_n(x)$ by $H_0(x) = 1$, $H_1(x) = x$, and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. The multi-dimensional versions are, for multi-indexes $\boldsymbol{\alpha} = (\alpha_\ell) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_{\ell,k}) \in \mathbb{N}^{m^2}$ and vectors $\mathbf{y} = (y_\ell) \in \mathbb{R}^m$ and $\mathbf{y}' = (y'_{\ell,k}) \in \mathbb{R}^{m^2}$,

$$\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{y}) = \prod_{\ell=1}^{m} H_{\alpha_\ell}(y_\ell), \qquad \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}') = \prod_{\ell,k=1}^{m} H_{\beta_{\ell,k}}(y'_{\ell,k}).$$

It is well known that the standardized Hermite polynomials $\{\frac{1}{\sqrt{n!}}H_n\}$, $\{\frac{1}{\sqrt{\boldsymbol{\alpha}!}}\mathbf{H}_{\boldsymbol{\alpha}}\}$, and $\{\frac{1}{\sqrt{\boldsymbol{\beta}!}}\overline{\mathbf{H}}_{\boldsymbol{\beta}}\}$ form orthonormal bases of the spaces $L^2(\mathbb{R}, \phi_1)$, $L^2(\mathbb{R}^m, \phi_m)$, and $L^2(\mathbb{R}^{m^2}, \phi_{m^2})$ respectively. Here, ϕ_j stands for the standard Gaussian measure on $\mathbb{R}^j$ $(j = 1, m, m^2)$ and $\boldsymbol{\alpha}! = \prod_{\ell=1}^{m} \alpha_\ell!$, $\boldsymbol{\beta}! = \prod_{\ell,k=1}^{m} \beta_{\ell,k}!$. See [16, 17] for a general picture of Hermite polynomials.

Before stating the Hermite expansion for the normalized number of roots of $\mathbf{Y}$ we need to introduce some coefficients. Let $f_{\boldsymbol{\beta}}$ $(\boldsymbol{\beta} \in \mathbb{R}^{m^2})$ be the coefficients in the Hermite basis of the function $f : \mathbb{R}^{m^2} \to \mathbb{R}$ such that $f(\mathbf{y}') = |\det(\mathbf{y}')|$. That is, $f(\mathbf{y}') = \sum_{\boldsymbol{\beta} \in \mathbb{R}^{m^2}} f_{\boldsymbol{\beta}} \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}')$ with

$$\begin{aligned} f_{\boldsymbol{\beta}} &= f_{(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m)} = \frac{1}{\boldsymbol{\beta}!}\int_{\mathbb{R}^{m^2}} |\det(\mathbf{y}')| \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}') \phi_{m^2}(\mathbf{y}') d\mathbf{y}' \\ &= \frac{1}{\boldsymbol{\beta}_1! \dots \boldsymbol{\beta}_m!}\int_{\mathbb{R}^{m^2}} |\det(\mathbf{y}')| \prod_{l=1}^{m} H_{\boldsymbol{\beta}_l}(\mathbf{y}'_l) \frac{\exp - \frac{||\mathbf{y}'_l||^2}{2}}{(2\pi)^{\frac{m}{2}}} d\mathbf{y}'_l, \end{aligned} \tag{3.10}$$

with $\boldsymbol{\beta}_l = (\beta_{l1}, \dots, \beta_{lm})$ and $\mathbf{y}'_l = (y'_{l1}, \dots, y'_{lm})$: $l = 1, \dots, m$.

Parseval's Theorem entails $||f||_2^2 = \sum_{q=0}^{\infty} \sum_{|\boldsymbol{\beta}|=q} f_{\boldsymbol{\beta}}^2 \boldsymbol{\beta}! < \infty$. Moreover, since the function f is even w.r.t. each column, the above coefficients are zero whenever $|\boldsymbol{\beta}_l|$ is odd for at least one $l = 1, \dots, m$.

To introduce the next coefficients let us consider first the coefficients in the Hermite basis in $L^2(\mathbb{R}, \phi_1)$ for the Dirac delta $\delta_0(x)$. They are $b_{2j} = \frac{1}{\sqrt{2\pi}}(-\frac{1}{2})^j \frac{1}{j!}$ and zero for odd indices [10]. Introducing now the distribution $\prod_{j=1}^{m} \delta_0(y_j)$ and denoting as $b_{\boldsymbol{\alpha}}$ its coefficients it holds that

$$b_{\boldsymbol{\alpha}} = \frac{1}{[\frac{\boldsymbol{\alpha}}{2}]!} \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{2} \right]^{[\frac{\alpha_j}{2}]} \tag{3.11}$$

or $b_{\boldsymbol{\alpha}} = 0$ if at least one index α_j is odd.

Since the formulas for the covariances of Hermite polynomials work in a neater way when the underlying random variables are standardized, we define the standardized derivative as

$$\overline{Y}'_\ell(t) := \frac{Y'_\ell(t)}{\sqrt{d}} \qquad \text{and} \qquad \overline{\mathbf{Y}}'(t) := (\overline{Y}'_1(t), \dots, \overline{Y}'_m(t)),$$

where $Y'_\ell(t)$ denotes the spherical derivative of Y_ℓ at $t \in S^m$. As said above, the k-th component of $\overline{Y}'_\ell(t)$ in a given basis is denoted by $\overline{Y}'_{\ell k}(t)$.

Proposition 3.3. *With the same notation as above, we have, in the L^2 sense, that*

$$\bar{N}_d := \frac{N_d^{\mathbf{Y}} - 2d^{m/2}}{2d^{m/4}} = \sum_{q=1}^{\infty} I_{q,d},$$

where

$$I_{q,d} = \frac{d^{m/4}}{2} \int_{S^m} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'(t)) dt,$$

with $\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{N}^m \times \mathbb{N}^{m^2}$ and $|\boldsymbol{\gamma}| = |\alpha| + |\beta|$ and $c_{\boldsymbol{\gamma}} = b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}}$.

Remark 3.4. Hermite polynomials' properties imply that for $q \neq q'$,

$$\mathbb{E}\left(I_{q,d} I_{q',d}\right) = 0.$$

Remark 3.5. The main difficulty in order to obtain a CLT relies on the bound of the variance of the tail $\sum_{q \geq Q} I_{q,d}$ because of the degeneracy of the covariances of $(\mathbf{Y}, \overline{\mathbf{Y}})$ near the diagonal $\{(s,t) \in S^m \times S^m : s = t\}$. Besides, on the sphere, finding a convenient re-scaling as in the one-dimensional case [4] is a difficult issue.

Proposition 3.3 is a direct consequence of the following lemma.

Lemma 3.6. *For $\varepsilon > 0$ define*

$$N_\varepsilon := \int_{S^m} |\det(\mathbf{Y}'(t))| \, \delta_\varepsilon(\mathbf{Y}(t)) dt,$$

where $\delta_\varepsilon(\mathbf{y}) := \prod_{\ell=1}^m \frac{1}{2\varepsilon} \mathbf{1}_{\{|y_\ell| < \varepsilon\}}$ for $\mathbf{y} = (y_1, \ldots, y_m)$ and $\mathbf{Y}'$ is the spherical derivative of $\mathbf{Y}$. Then, we have the following:

1. *For $\mathbf{v} \in \mathbb{R}^m$, let $N_d^{\mathbf{Y}}(\mathbf{v})$ denote the number of real roots in S^m of the equation $\mathbf{Y}(t) = \mathbf{v}$. Then, $N_d^{\mathbf{Y}}(\mathbf{v})$ is bounded above by $2d^m$ almost surely.*
2. *$N_\varepsilon \to N_d^{\mathbf{Y}}$ almost surely and in the L^2 sense as $\varepsilon \to 0$.*
3. *The random variable $N_d^{\mathbf{Y}}$ admits a Hermite expansion.*

Proof. Since the paths of $\mathbf{Y}$ are smooth, Proposition 6.5 of [1] implies that for every $\mathbf{v} \in \mathbb{R}^m$ almost surely there is no point $t \in S^m$ such that $\mathbf{Y}(t) = \mathbf{v}$ and the spherical gradient is singular. Using the local inversion theorem, this implies that the roots of $\mathbf{Y} = \mathbf{v}$ are isolated and by compactness they are finitely many. As a consequence, $N_d^{\mathbf{Y}}(\mathbf{v})$ is well defined and a.s. finite. Moreover, for every $t \in \mathbb{R}^{m+1}$ such that $Y(t) = \mathbf{v}$, $\|t\| = 1$, we have that the set $\{Y_1'(t), \ldots, Y_m'(t), t\}$ is almost surely linearly independent in $\mathbb{R}^{m+1}$. This implies that $N_d^{\mathbf{Y}}(\mathbf{v})$ is uniformly bounded by the Bézout number $2d^m$, concluding (1) (see for example Milnor [15, Lemma 1, p. 275]).

By the inverse function theorem, a.s. for every regular value $\mathbf{v} \in \mathbb{R}^m$, $N_d^{\mathbf{Y}}(\cdot)$ is locally constant in a neighborhood of $\mathbf{v}$. Furthermore, by the Area Formula (see Federer [8] or [1, Proposition 6.1]), for small $\varepsilon > 0$ we have

$$N_\varepsilon = \frac{1}{(2\varepsilon)^m} \int_{[-\varepsilon,\varepsilon]^m} N_d^{\mathbf{Y}}(\mathbf{v}) \, d\mathbf{v}, \quad \text{a.s.} \tag{3.12}$$

Hence,

$$N_d^{\mathbf{Y}}(0) = \lim_{\varepsilon \to 0} N_\varepsilon, \quad \text{a.s.} \tag{3.13}$$

From (1) and (3.12) we have $N_\varepsilon \leq 2d^m$ a.s. Then, the convergence in (3.13) also happens in L^2.

This convergence allows us getting a Hermite expansion. We have

$$\delta_\varepsilon(\mathbf{y}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^m} b^\varepsilon_{\boldsymbol{\alpha}} \mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{y}),$$

$$\left|\det\left(\frac{\mathbf{y}'}{\sqrt{d}}\right)\right| = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{m^2}} f_{\boldsymbol{\beta}} \overline{\mathbf{H}}_{\boldsymbol{\beta}}\left(\frac{\mathbf{y}'}{\sqrt{d}}\right),$$

where $b^\varepsilon_{\boldsymbol{\alpha}}$ are the Hermite coefficients of $\delta_\varepsilon(\mathbf{y})$ and the $f_{\boldsymbol{\beta}}$ have already been defined. Furthermore, we know that $\lim_{\varepsilon\to 0} b^\varepsilon_{\boldsymbol{\alpha}} = b_{\boldsymbol{\alpha}}$. Now, taking limit and regrouping terms we get as in Estrade and León [6] that

$$N_d = d^{m/2} \sum_{q=0}^{\infty} \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q} b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} \int_{S^m} \mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'(t))dt.$$

This concludes the proof. □

3.3.2. $V_\infty > 0$. To prove that $V_\infty > 0$ we use the Hermite expansion. In fact,

$$V^2_\infty = \lim_{d\to\infty} \sum_{q=2}^{\infty} \mathrm{Var}(I_{q,d}) \geq \lim_{d\to\infty} \mathrm{Var}(I_{2,d}).$$

By Proposition 3.3, we have

$$I_{2,d} = \frac{d^{m/4}}{2} \sum_{|\boldsymbol{\gamma}|=2} c_{\boldsymbol{\gamma}} \int_{S^m} H_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) H_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'(t))dt.$$

The coefficients $c_{\boldsymbol{\gamma}} = b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}}$ vanish for any odd α_ℓ and $|\boldsymbol{\beta}_\ell|$. Thus, the only possibilities to satisfy the condition $|\boldsymbol{\gamma}| = 2$ are that either only one of the indices is 2 and the rest vanish or $\beta_{\ell,k} = \beta_{\ell,k'} = 1$ for some $k \neq k'$ and the rest vanish. Hence,

$$I_{2,d} = \frac{d^{m/4}}{2} \int_{S^m} \left[\sum_{\ell=1}^{m} \left(b_2 b_0^{m-1} f_{(0,\ldots,0)} H_2(Y_\ell(t)) + b_0^m \tilde{f}_{\ell 12} H_2(\overline{Y}'_{\ell,1}(t)) \right) \right.$$
$$\left. + \sum_{k=2}^{m} b_0^m \tilde{f}_{\ell k2} H_2(\overline{Y}'_{\ell,k}(t)) + \sum_{k\neq k'} b_0^m \tilde{f}_{\ell kk'1} H_1(\overline{Y}'_{\ell,k}(t)) H_1(\overline{Y}'_{\ell,k'}(t)) \right] dt,$$

where $\tilde{f}_{\ell k2} = f_{(0,\ldots,\beta_{\ell k},0,\ldots,0)}$, $\beta_{\ell k} = 2$ and $\tilde{f}_{\ell kk'1} = f_{(0,\ldots,\beta_{\ell k},\ldots,\beta_{\ell k'}0,\ldots,0)}$, $\beta_{\ell k} = \beta_{\ell k'} = 1$. By (3.4)-(3.5) the variables in different sums are orthogonal when evaluated at $s, t \in S^m$. Now, by Mehler's formula, $\mathbb{E}\,(H_2(\xi)H_2(\eta)) = 2(\mathbb{E}\,(\xi\eta))^2 \geq 0$ for jointly normal variables ξ, η. Hence, bounding the sum of the variances by one convenient term, we have

$$\begin{aligned}
\mathrm{Var}(I_{2,d}) &\geq \mathrm{Var}\left(\frac{d^{m/4}}{2} b_0^m \tilde{f}_{\ell 22} \int_{S^m} H_2(\overline{Y}'_{\ell 2}(t))dt \right) \\
&= \frac{d^{m/2}}{2} (b_0^m \tilde{f}_{\ell 22})^2 \int_{(S^m)^2} (\mathbb{E}\,\overline{Y}'_{\ell,2}(s)\overline{Y}'_{\ell,2}(t))^2 dsdt \\
&= (b_0^m \tilde{f}_{\ell 22})^2 \frac{d^{m/2}}{2} \int_{(S^m)^2} \left(\langle s,t\rangle^d - (d-1)\langle s,t\rangle^{d-2} \sqrt{1-\langle s,t\rangle^2} \right)^2 dsdt,
\end{aligned}$$

where the last equality is a consequence of (3.3).

The integral tends to a positive limit as can be seen using Lemma 3.1 and the scaling $t = z/\sqrt{d}$ as in Section 3.2.

Finally, by (3.11) $b_0 \neq 0$. Besides, by the symmetry of the function $f(\cdot) = |\det(\cdot)|$ and (3.10), $\tilde{f}_{\ell k 2} = \tilde{f}_{\ell k' 2}$ for all ℓ, k, k'. Therefore, adding up (3.10) w.r.t. ℓ and k, we get

$$\tilde{f}_{\ell 22} = \frac{1}{m^2}\left(\mathbb{E}\left(|\det(\mathbf{y}')|\|\mathbf{y}'\|_F^2\right) - m^2\mathbb{E}\left(|\det(\mathbf{y}')|\right)\right),$$

with $\|\cdot\|_F$ being Frobenius's norm and $\mathbf{y}'$ an $m \times m$ standard Gaussian matrix. Straightforward computations using polar coordinates show that $\tilde{f}_{\ell 22} > 0$ for all $m \geq 1$. This concludes the proof of the claim $V_\infty > 0$.

Note added in proof

Some time after this article was submitted the authors became aware that Letendre and Puchol in [14] extended our main result to the case of systems of r equations and m variables ($r \leq m$). The methods used in the proof of the finiteness of the limit variance are rather different from ours, but their proof of its positivity is influenced by ours.

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