TOOLING UP MATHEMATICS FOR ENGINEERING*

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It has often been said that one of the primary objectives of Mathematics is to furnish tools to physicists and engineers for solution of their problems. It is evident from the history of the mathematical sciences that many fundamental mathematical discoveries have been initiated by the urge for understanding nature's laws and many mathematical methods have been invented by men primarily interested in practical applications. However, every true mathematician will feel that a restriction of mathematical research to problems which have immediate applications would be unfair to the "Queen of Sciences." As a matter of fact, the devoted "minnesingers" of the Queen have often revolted against degradation of their mistress to the position of a "handmaiden" of her more practical minded and temporarily more prosperous sisters.

It is not difficult to understand the reasons for the controversial viewpoints of mathematicians and engineers. They have been pointed out more than once, by representatives of both professions.

The mathematician says to the engineer: I have built a building on a sound foundation: a system of theorems based on well defined postulates. I have delved into the analysis of the process of logical thinking to find out whether or not there are any statements which could be considered true or at least potentially true. I am interested in functional relations between entities which are well defined creations of my own mind and in methods which enable me to explore various aspects of such functional relations. If you find any of the concepts, logical processes or methods which I have developed useful for your daily work, I am certainly glad. All my results are at your disposal, but let me pursue my own objectives in my own way.

Says the engineer: Your great forbears, who were mathematicians long before you, talked a different language. Did not Leonhard Euler distribute his time between discoveries in pure mathematics and in the theory of engineering devices? The fundamentals of the theory of turbines, the theory of buckling of columns, the theory of driving piles into soil were contributions of Euler. The development of mathematical analysis cannot be separated from the development of physics and especially of mechanics. It is doubtful whether a human mind would ever have conceived the idea of differential equations without the urge to find a mathematical tool for the computation

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of the path of moving bodies. If one assumes that the motion is determined by certain fundamental mechanical or geometrical relations, which are valid at every instant of the motion, one naturally is led to the idea of the differential equation. Also, the calculus of variations was invented mainly for solution of physical problems; some of which were of teleological, some of practical nature. The eighteenth century and the first decades of the nineteenth were perhaps the period of the most glorious progress in mathematical science; at that time, there was no distinction between pure and applied mathematicians. The abstract minded mathematicians stepped in after the big job was done; they endeavored to fill certain logical gaps, to systematize and codify the abundance of methods and theorems which the giants of the foregoing period created by a combination of logical thinking and creative intuition.

The mathematician: It seems to me that you underestimate the importance of what you call systematization and codification. Don't you think that in order to assure the correct application of calculus and differential equations, there was an absolute necessity to define exactly what we mean by a limiting process; or, was it not absolutely necessary to give a real sense to such terms as infinitely small and infinitely large? You may remember that Galileo—whom you hardly can call an abstract or pure mathematician—pointed out the contradictions which are unavoidable if you try to apply the notions of equality and inequality to infinite quantities. He noticed that you can say either that the number of the integers is larger than the number of the squares, since every square is an integer, but not every integer is a square; or you can say with the same justification that there are as many squares as integers, since every number has a square. The notions of commensurability, denumerability, the logical analysis of the continuum, the theory of sets, and in more recent times, topology, were fundamental steps in the development of the human mind. Many of these developments were conceived independently of any conscious physical applications. But even for the sake of applications, it was necessary to improve the foundations of our own house, that is to improve the logical structure of mathematics. Without exact analysis of the conditions for the convergence of series (the conditions which allow carrying out the processes of differentiation and integration), nobody could feel safe in handling series. It is not correct that the tendency to seek for a solid foundation of the new discoveries began after the men endowed with imagination and intuition did the big job. D'Alembert already demanded that the calculus be founded on the methods of limits. Cauchy, Legendre and Gauss certainly were among the creative mathematical geniuses in your sense; they effectively contributed to the transition from intuition to rigor. In the second half of the nineteenth century this development continued toward the great goal that the mathematicians of that age—perhaps optimistically—considered as perfect logic or absolute rigor. However, in addition to the clarification of the fundamentals, that period also opened

new paths for applied mathematics. You mentioned, for example, differential equations. Don't you believe that the theory of functions of complex variables, the classification of differential equations according to their singularities, and the investigation of these singularities, all developed in the period that you call the period of codification, were most important steps in building up the very branch of mathematics from which you engineers derive so much benefit? These theories changed the primitive way of finding solutions of differential equations by trial into a systematic method of mastering the whole field.

The engineer: I agree, especially with what you say about complex variables. Indeed, the conformal transformation is one of the most powerful and most elegant methods for the solution of innumerable physical and engineering problems. I also agree with you on the fundamental importance of the analysis of singularities. In fact, our graphical and numerical methods necessarily fail or become awkward near the irregular points and we have to take recourse to analytical methods. However, you mathematicians unfortunately are somewhat like a physician who is less interested in the laws of normal functioning of the human body than in its diseases, or like the psychologist who instead of investigating the laws of normal mental processes concentrates his attention on the pathological aberrations of the human mind. We have to deal in most cases with "sound functions" and would like to have efficient methods to determine with fair accuracy their behavior in certain definite cases.

Answers the mathematician: Can you not apply the general methods that we developed for the solution of differential and integral equations? If the solutions are given by "sound functions," as you please to call them, I do not see any great difficulty nor do I see what more you expect us to do.

The engineer: Your general theorems deal mostly with the existence of solutions and the convergence of your methods of solution. You may recall the wisecrack of Heaviside: "According to the mathematicians this series is divergent; therefore, we may be able to do something useful with it." You people spend much time and much wit to show the existence of solutions whose existence often is evident to us for obvious physical reasons. You seldom take the pains to find and discuss the actual solutions. If you do so, then you restrict yourself mostly to simple cases, as for example, problems involving bodies of simple geometrical shapes. I refer to the so-called special functions. I concede that a great many such functions were investigated by mathematicians. Their values have been tabulated, their developments in series and their representations by definite integrals have been worked out in great detail. Unfortunately, such functions have only a restricted field of application in engineering. The physicist in his search for fundamental laws may choose specimens of simple geometrical shapes for his experimentation. The engineer has to deal directly with structures of complicated shapes; he

cannot give to a structure a simple geometrical form just because the stress distribution in such a structure can be calculated by special functions. Furthermore, most special functions are applicable only to linear problems. In the past, physicists and engineers often linearized their problems for simplicity's sake. Mathematicians liked this simplification because it furnished a beautiful hunting ground for the application of elegant mathematical methods. Unfortunately, as engineering science progressed, the need for more exact information and the necessity to get nearer and nearer to physical reality, forces us to grapple with many nonlinear problems.

The mathematician: Well, many modern mathematicians are extremely interested in non-linear problems. It seems your primary need is the development of appropriate methods of approximation. However, you are not right in your criticism of our proofs of existence. Many proofs of existence in modern mathematics go far beyond the limits of intuition. Then, too, I understand you engineers have good success with various iteration methods. Now, if we want to prove for example the existence of a solution of a boundary value problem, very often we use the iteration method. In other words, we really construct a sequence of approximate solutions exactly as you do. The whole difference is that we prove and you only assume that the process of iteration leads to a unique solution. Also, your so-called "energy method" used for the solution of your problems in elasticity and structures appears to me closely related to the direct methods of the calculus of variations, i.e., to methods which try to construct directly the minimizing function for given boundary values, without referring to the Euler-Lagrange differential equation. It seems to me that after all there are many common elements in pure analysis and applied mathematics.

The engineer: I shall not deny that; as a matter of fact, I have always felt that analysis is the backbone of applied mathematics. However, if you really start to apply analysis to actual cases you will see that there is a long way from the general idea of a method of approximation to a successful application of the same method. There is, for example, the question of available time and manpower. For certain types of work, we have ingenious mechanical or electrical devices such as the differential analyzer or electric computers. However, in most cases we have to do the computation without such help. Then it is not sufficient to know that the process of approximation converges. We have to find out which method requires the least time for a given degree of approximation; we have to have a fair estimate of the improvement of accuracy by successive steps. All such practical questions require difficult mathematical considerations. I think we definitely need mathematicians who help us to refine and, if you wish to say so, criticize and systematize our intuitive methods. In fact, successful applications of mathematics to engineering require the close cooperation of mathematicians and engineers. It is by no means a routine job to recognize the underlying common

mathematical relations in apparently very different fields. The mathematician who intends to do applied mathematical research has to have a pretty good sense for the physical processes involved. On the other hand, the engineer has to go into the fundamentals of analysis to a considerable depth in order to use the mathematical tools properly. An arbitrary assembly of machine tools does not constitute an efficient machine shop. We know there are powerful machine tools in your mathematical arsenal. The task before us is to know how to adapt and apply them.

The mathematician: I think you've got something there. To carry your analogy further, in order to get the solution of engineering problems into production, you need some kind of tool designers. These are the real applied mathematicians. Their original backgrounds may differ; they may come from pure mathematics, from physics or from engineering, but their common aim is to "tool up" mathematics for engineering.