AN APPLICATION OF ORTHOGONAL MOMENTS TO PROBLEMS IN STATICALLY INDETERMINATE STRUCTURES*

BY W. M. KINCAID AND V. MORKOVIN
Brown University

1. Numerous methods have been devised for determining moment profiles in statically indeterminate structures. Most of these can be classified as either approximate or exact. The approximate methods are usually simple to apply, but when employing them it is necessary to specify numerical values for the dimensions and stiffnesses (or their ratios) of the structure involved. The exact methods consist in solving systems of linear equations obtained by setting up relations between generalized displacements or by the equivalent means of using Castigliano’s Principle. As the degree of indeterminacy of the structure increases, the solution of the equations becomes more laborious. Most of the methods aim at reducing such labor by a suitable choice of unknowns. This paper is an attempt in that direction.

2. Consider a structure whose degree of statical indeterminacy is \( N \). Denote by \( M \) the true moment profile in the structure under a given load, and by \( M_0 \) the moment profile under the same load when the structure has been made statically determinate by removing \( N \) constraints (the so-called basic structure). The effect of the removed constraints may be replaced by the combined effect of \( N \) unknown generalized forces (couples and forces) \( X_1, X_2, \ldots, X_N \). The generalized displacements of the loaded structure at the points of application and in the directions of \( X_1, X_2, \ldots, X_N \), are assumed to be known and will be denoted by \( \delta_1, \delta_2, \ldots, \delta_N \). (If \( X_i \) is a couple, \( \delta_i \) is a rotation; if \( X_i \) is an ordinary force, \( \delta_i \) is an ordinary displacement.) Let \( M_i \) be the moment profile obtained when the force corresponding to \( X_i = 1, X_j = 0 \) for \( j \neq i \), acts on the (unloaded) basic structure. Then the moment profile \( X_i M_i \) represents the effect of the \( i \)th constraint. Superposing the moment profiles due to the load and the constraints yields the true moment profile \( M \):

\[
M = M_0 + \sum_{i=1}^{N} X_i M_i. \tag{1}
\]

The unknown quantities \( X_i \) in (1) are to be determined by means of Castigliano’s Principle.

Disregarding as usual the contributions of shearing stresses and axial forces, we get for the total strain energy \( U \) the following expression:

\[
U = \frac{1}{2} \int M^2 dx'. \tag{2}
\]

---

* Received July 1, 1943.

1 By the moment profile of a structure under a given load we understand the magnitude of the bending moment as a function of position; the graph of this function is commonly called the bending moment diagram.
ORTHOGONAL MOMENTS IN INDETERMINATE STRUCTURES 335

where \( dx' = dx/EI \), \( E \) is Young’s modulus, \( I \) is the moment of inertia of any cross-section about the neutral axis, and the integral is taken over the entire structure. Castigliano’s Principle states that

\[
\frac{\partial U}{\partial X_i} = \delta_i \quad (i = 1, 2, \ldots, N),
\]

or, by virtue of (1) and (2),

\[
M_0M_idx' + \sum_{j=1}^{N} X_j \int M_iM_jdx' = \delta_i \quad (i = 1, 2, \ldots, N).
\]

It will be observed that (1) and (3) are still valid if each \( X_i \) represents a set of generalized forces (rather than a single force) acting simultaneously at different points. (See for instance Fig. 2d.) In such a case, each \( \delta_i \) would be made up of the generalized displacements corresponding to the given set of generalized forces.

Previous attacks upon the problem have essentially consisted in choosing the points of application and lines of action of the unknown forces \( X_i \) so as to make the system of equations (3) as simple as possible. Thus the moment profiles \( M_i \) were completely determined. We propose to reverse this procedure by specifying the moment profiles \( M_i \) first. Let us choose these profiles so that

\[
\int M_iM_jdx' = 0 \quad (i, j = 1, 2, \ldots, N; i \neq j).
\]

Then each of the equations (3) will contain only one unknown, and we get at once

\[
X_i = \frac{\delta_i - \int M_0M_idx'}{\int M_i^2dx'} \quad (i = 1, 2, \ldots, N).
\]

The true moment profile \( M \) may now be obtained by substituting these values \( X_i \) into (1).

It will be recognized that equations (4) require that the moment profiles \( M_i \) form an orthogonal system over the structure. Such a system can always be constructed by the standard orthogonalization process from the original set of moment profiles (or any similar set of linearly independent moment profiles).\(^2\) Therefore, the system of orthogonalized moment profiles and the corresponding generalized forces which appear in (5) will consist of linear combinations of the original sets of \( M_i \) and \( X_i \), respectively. In most practical applications, the exact form of these relations, as well as the physical interpretation of the generalized forces \( X_i \), is immaterial; only the orthogonalized moment profiles \( M_i \) are needed. The simple example that follows will illustrate the notions introduced in the preceding discussion.

3. Let us consider the triply indeterminate bent ABCD (Fig. 1), which has undergone a vertical displacement \( d \) and a rotation \( \theta \) at D (as a result of a settlement of

the foundation). The member AB is subjected to a uniform horizontal pressure $p$. The stiffnesses $EI$ of the members AB and CD are assumed to be equal. In this and succeeding examples, the positive direction will be taken as downward in vertical members and toward the right in horizontal members.

The solution will be carried out as follows: (1) the construction of an orthogonal system will be shown, and (2) this system will be used to obtain the desired moment profiles.

Figures 2a, 2b, and 2c show three self-equilibrating systems of generalized forces and the corresponding moment profiles; it is obvious that any set of reactions (in equilibrium) can be built up by superposing these three in the right proportions.

We note that the moment profiles $M_{2a}$ and $M_{2b}$ are symmetric with respect to a vertical axis through the midpoint of BC and would be orthogonal to any antisymmetric profile. Therefore we replace $M_{2c}$ by $M_{2a}$ (Fig. 2d). It remains to replace $M_{2a}$ by a linear combination of $M_{2a}$ and $M_{2b}$, say $\alpha M_{2a} + \beta M_{2b}$, that will be orthogonal to $M_{2b}$. The orthogonality condition reads

$$0 = \alpha \int M_{2a} M_{2b} dx' + \beta \int M_{2b}^2 dx' = \alpha (l' + h') + \beta (l' + 2h'), \quad (6)$$

where $l' = l/EI_{BC}$, $h' = h/EI_{AB}$. Eq. (6) will be satisfied if $\alpha = \kappa + 2$ and $\beta = -(\kappa + 1)$, where $\kappa = l'/h'$. The resulting moment profile $M_{2e}$ is shown in Fig. 2e. The moment profiles $M_{2b}$, $M_{2d}$, and $M_{2e}$ form an orthogonal system over the structure. From the manner in which this system was derived, we see that it is independent of the loading and the choice of a basic structure. This is true not only with respect to this particular structure but in general.

We choose a convenient basic structure and find the moment profile $M_0$ due to the load (see Fig. 3). Eq. (1) now takes the form

$$M = M_0 + X_{2b} M_{2b} + X_{2d} M_{2d} + X_{2e} M_{2e}. \quad (7)$$
Clearly, the quantities $X$ may best be interpreted as mere coefficients by which the generalized forces in Figs. 2b, 2d, and 2e must be multiplied in order to yield, upon superposition, the correct reactions at the supports. As for the interpretation of $\delta_{2b}$, $\delta_{2d}$, and $\delta_{2e}$, we recall that these quantities are equal to the partial derivatives, with respect to the corresponding quantities $X$, of the work done by the reactions. For instance,

$$\delta_{2d} = \frac{\partial}{\partial X_{2d}} \left[ \{X_{2d} - X_{2b} + (\kappa + 1)X_{2e}\} \theta - \frac{2X_{2d}}{l} d \right] = \theta - \frac{2d}{l}.$$ 

Similarly $\delta_{2b} = -\theta$ and $\delta_{2e} = (\kappa + 1)\theta$.

Next, we evaluate the following important quantities:

$$\int M_2^2 \ dx' = h'(\kappa + 2), \quad \int M_2^2 \ dx' = \frac{1}{3} h'(\kappa + 6),$$

$$\int M_2 d \ dx' = \frac{1}{3} h'(\kappa + 2)(2\kappa + 1);$$

$$\int M_0M_2^2 dx' = \int M_0M_2 d dx' = -\frac{1}{6} \phi h^2 h', \quad \int M_0M_2^2 dx' = \frac{1}{24} \phi h^2 h'(3\kappa + 2).$$

Thus the equations (5) specialize to

$$X_{2b} = \frac{-6\theta + \phi h^2 h'}{6h'(\kappa + 2)}, \quad X_{2d} = \frac{6\theta - 12d/l + \phi h^2 h'}{2h'(\kappa + 6)},$$

$$X_{2e} = \frac{24(\kappa + 1)\theta - (3\kappa + 2)\phi h^2 h'}{8h'(\kappa + 2)(2\kappa + 1)}.$$ (10)

To complete the solution we have only to substitute (10) into (7) and tabulate the values of $M$ at A, B, C, and D.

4. In the present method, the problem of solving the system (3) is replaced by that of constructing a set of orthogonal moment profiles. Once such a set is known for a given structure, the solution $M$ corresponding to any loading is obtained merely by evaluating the right-hand member of (5) and substituting into (1). This is an advantage of the present method over others in which the complete system (3) has to be solved anew every time the load is altered. Thus, for important structures, it may be worthwhile to construct an orthogonal set, even in cases where the orthogonalization itself is fairly complicated.

Fortunately, the rather lengthy standard orthogonalization process (see footnote 2) seldom needs to be used in its entirety, as the preceding example indicates. Shortcuts involving the use of symmetry, antisymmetry, and other characteristics of the

\[\text{FIG 4.}\]
structure are usually available. For instance, in the case of the triply indeterminate frame in Fig. 4 (with $I_{AB} = I_{CD}$ and $I_{BC} = I_{AD}$), we obtain the $M_i$'s immediately from symmetry considerations (Fig. 5).

![Fig. 5.](image)

The evaluation of integrals $\int M_0 M_0 \, dx'$ in (5) presents few difficulties, because the moments $M_i$ always vary linearly over any member and $M_0$ can be made to vanish over all but a few members by a suitable choice of the basic structure. Furthermore, there exist tables of such integrals for the common forms of moment profiles (trapezoidal, parabolic, etc.).

Similarly, the remaining operations can be systematized so that even a person with little mathematical training can perform them. This is particularly true when numerical values of the stiffnesses $EI$ are known; then substituting (5) into (1) reduces to taking scalar products on a computing machine.

5. We conclude with an example illustrating an efficient arrangement of the work. Consider the four-legged bent shown in Fig. 6; assume that $I_{AB} = I_{CD} = I_{EF} = I_{GH}$ and $I_{BC} = I_{CF} = I_{FG}$, and denote by $k$ the ratio $I'/h' = I_{AB}/h_{BC}$, as before.

![Fig. 6.](image)

![Fig. 7.](image)

Since the structure is statically indeterminate of the ninth degree, we obtain nine orthogonal moment profiles $M_1, M_2, \ldots, M_9$, whose values at different points of the structure are given in the first ten columns of Table I. (We understand by $M_i(CB)$ the value of $M_i$ at the end of $C$ of the member $BC$, and give a similar meaning to $M_i(AB), M_i(BC), \text{etc.}$) It will be observed that $M_1, M_2, M_3, M_7, M_8$, and $M_9$ are essentially reproductions of the moment profiles $M_{2b}, M_{2a}$, and $M_{2e}$ found for the two-legged bent in section 3. Considerations of symmetry are helpful in constructing $M_4, M_5$, and $M_6$. The integrals $\int M_0^2 \, dx'$ are now computed by means of the tables referred to in footnote 5, and are given in the eleventh column of our table.

The parts of the table so far discussed can be used to find the moment profile due to any loading of the bent. Suppose the member $BC$ is subjected to a set of vertical

---

loads and moments. By selecting the basic structure as indicated in Fig. 7, we confine the moment profile \( M_o \) to BC.

Defining

\[
P_i = \frac{1}{l^2} \int_{BC} M_o x \, dx, \quad P_r = \frac{1}{l^2} \int_{BC} M_o (l - x) \, dx,
\]

we see at once that

\[
\int M_0 M_i \, dx' = l' \left[ P_r M_i(BC) + P_t M_i(CB) \right] \quad (i = 1, 2, \cdots, 9). \tag{11}
\]

The values \( X_i \) (twelfth column) are now obtained by dividing the negatives of the right members of (11) by the corresponding values in the eleventh column.

To find the value of the true moment profile at (say) A, we multiply each term \( M_i(AB) \) (first column) by the corresponding \( X_i \) and add the results, and similarly for other points. It would ordinarily not be necessary to carry out the algebraic simplification of these sums, but this has been done for the sake of compactness:

\[
M(AB) = \frac{\kappa}{ST} \left\{ (4\kappa^3 - 170\kappa^2 - 414\kappa - 225)P_r - 2(\kappa + 1)(10\kappa^2 - 6\kappa - 3)P_t \right\},
\]

\[
M(BC) = \frac{\kappa}{ST} \left\{ -(64\kappa^3 + 628\kappa^2 + 1188\kappa + 585)P_r + 2(\kappa + 1)(16\kappa^2 + 102\kappa + 51)P_t \right\},
\]

\[
M(CB) = \frac{1}{ST} \left\{ 2\kappa(16\kappa^3 + 118\kappa^2 + 131\kappa + 51)P_r - 4(\kappa + 1)(16\kappa^2 + 141\kappa^2 + 195\kappa + 72)P_t \right\},
\]

\[
M(DC) = \frac{\kappa}{ST} \left\{ (20\kappa^3 + 96\kappa^2 + 233\kappa + 135)P_r + 2(-2\kappa^3 + 51\kappa^2 + 91\kappa + 39)P_t \right\},
\]

\[
M(CF) = \frac{1}{ST} \left\{ \kappa(140\kappa^2 + 412\kappa + 237)P_r - 2(86\kappa^3 + 355\kappa^2 + 411\kappa + 144)P_t \right\},
\]

\[
M(FC) = \frac{1}{ST} \left\{ -\kappa(124\kappa^2 + 248\kappa + 93)P_r + 2(70\kappa^3 + 179\kappa^2 + 144\kappa + 36)P_t \right\},
\]

\[
M(EF) = \frac{1}{ST} \left\{ \kappa(36\kappa^3 + 260\kappa^2 + 377\kappa + 135)P_r - 2\kappa(18\kappa^3 + 125\kappa^2 + 176\kappa + 69)P_t \right\},
\]

\[
M(GF) = \frac{1}{ST} \left\{ -\kappa(108\kappa^2 + 208\kappa + 135)P_r + 2\kappa(54\kappa^2 + 73\kappa + 21)P_t \right\},
\]

\[
M(FG) = \frac{1}{ST} \left\{ 2\kappa(54\kappa^2 + 73\kappa + 21)P_r + 2(-54\kappa^3 - 38\kappa^2 + 51\kappa + 36)P_t \right\},
\]

\[
M(HG) = \frac{1}{ST} \left\{ \kappa(36\kappa^3 + 198\kappa^2 + 284\kappa + 135)P_r - 2\kappa(18\kappa^3 + 90\kappa^2 + 104\kappa + 33)P_t \right\}.
\]

The arrangement given here is especially adapted to the use of computing machines in case numerical values of \( \kappa \) and the other quantities appearing are known.\(^6\)

\(^6\) After the manuscript of this paper had been completed, our attention was drawn to a work along similar lines by S. Müller (S. Müller, Zur Berechnung mehrfach statisch unbestimmter Tragwerke, Zentralblatt der Bauverw. 1907, p. 23). Müller introduces a "system of forces \( X_i \)" instead of single forces, and reduces the system of equations to the diagonal form. However, his point of view and emphasis are quite different from ours, and his procedure is more lengthy than that presented here.
### Table I

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>$\xi+1$</td>
<td>1</td>
<td>1</td>
<td>$-(\xi+1)$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$h'(\xi+2)/(2\xi+1)$</td>
<td>3\xi(P_\xi+P_1)/(\xi+2)(2\xi+1)</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$h'(\xi+2)/(\xi+2)$</td>
<td>$h'(\xi+2)$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$h'(\xi+6)/3$</td>
<td>$3\xi(P_\xi-P_1)/(\xi+6)$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>$-\xi(3\xi+7)$</td>
<td>$-11\xi$</td>
<td>$13\xi+12$</td>
<td>$-\xi(\xi-5)$</td>
<td>$-11\xi$</td>
<td>$-13\xi+12$</td>
<td>$-11\xi$</td>
<td>$-11\xi$</td>
<td>$-11\xi$</td>
<td>$h'\xi S/3$</td>
<td>$h'\xi R$</td>
</tr>
<tr>
<td>$M_5$</td>
<td>$2\xi^2+8\xi+5$</td>
<td>$6\xi+5$</td>
<td>$-2(3\xi+2)$</td>
<td>$2\xi^2+9\xi+5$</td>
<td>$2\xi+1$</td>
<td>$-2(3\xi+2)$</td>
<td>$6\xi+5$</td>
<td>$-2(3\xi+2)$</td>
<td>$6\xi+5$</td>
<td>$h'T R/3$</td>
<td>$3\xi[(6\xi+5)P_\xi-2(3\xi+2)P_1]/RT$</td>
</tr>
<tr>
<td>$M_6$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$-\xi$</td>
<td>$x+1$</td>
<td>1</td>
<td>1</td>
<td>$(x+1)$</td>
<td>$(x+1)$</td>
<td>$h'\xi+2)/(2\xi+1)$</td>
<td>0</td>
</tr>
<tr>
<td>$M_7$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$h'\xi +2$</td>
<td>0</td>
</tr>
<tr>
<td>$M_8$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$x+1$</td>
<td>$x+1$</td>
<td>$x+1$</td>
<td>$x+1$</td>
<td>$x+1$</td>
<td>$h'(\xi+6)/3$</td>
<td>0</td>
</tr>
</tbody>
</table>

$R = 4\xi^2+31\xi+20,$
$S = 4\xi^2+41\xi+36,$
$T = 4\xi^2+10\xi+5,$
$\lambda = 2(2\xi+1)(\xi+6)$

$M_4(BC) = M_4(FC) - M_4(FG),$  
$M_4(CD) = M_4(CB) - M_4(CF),$  
$M_4(BA) = -M_4(BC),$  
$M_4(GH) = M_4(GF)$