SOME PRESENT NONLINEAR PROBLEMS OF THE ELECTRICAL AND AERONAUTICAL INDUSTRIES

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1. Introduction. The accelerated growth of research in the field of nonlinearity is due to different causes. The general advancement of science requires increasingly more precise expressions for the laws of science. Accurate nonlinear equations frequently depart from the linearized or postulated linear equations which have been previously used for approximate results. The quest for perfection and generalization and the love of difficult investigations by professional mathematicians play a large part in this growth. Another incentive is the increasingly exacting requirements of modern manufacturing. These requirements are born of the competitive necessity of producing ever improved machines and equipment in the most economical manner. The greatest incentive is necessity. Manufactured equipment and devices must be designed to work.

A nonlinear problem has been defined as "one which, when formulated mathematically, reduces to (one or) a system of differential, integral, or integro-differential equations such that at least one of the three quantities, a derivative, an integral, or a dependent variable, is involved transcendentally or algebraically to a power other than the first in at least one equation of the system or in at least one boundary condition of the system." Of course, in dealing with applied problems, a physical definition independent of all mathematical concepts is preferable, but such is difficult to formulate.

Nonlinear problems resolve themselves into two general types, continuous and discrete. The first type deals with the behavior of quantities in a field or in at least one continuous region of space and, more often than not, reduces mathematically to systems of nonlinear partial differential equations. Problems relating primarily to this field have been treated by Dr. Theodore von Kármán in his Josiah Willard Gibbs Memorial lecture. This paper is both a milestone and a beacon of progress in that it is an admirable exposition and inventory of the nonlinear problems of continuous fields and at the same time an inspiration and invitation to both the engineer and mathematician for further advancement in this difficult field. Among other subjects, the von Kármán lecture treated relaxation oscillations, subharmonic resonance, nonlinear problems in the theory of elasticity in which the hypotheses of (a) small deflections are abandoned, (b) Hooke's law no longer holds, plasticity, hydrodynam-
ics; and aerodynamics of (a) ideal fluids, (b) viscous fluids, and (c) compressible fluids. The bibliography of the paper contains 178 entries.

The second type of nonlinear problem is called discrete. Discrete nonlinear problems are characterized by the fact that they possess only a finite number of degrees of freedom. They are frequently reducible mathematically to systems of nonlinear ordinary (total) differential equations or to systems of nonlinear integral equations.

2. **Nature of industrial discrete nonlinear problems.** Solutions of nonlinear problems in industry are usually in the "small"; i.e., a solution of a system is not required for every magnitude whatsoever of the parameters involved. In the solution of such problems the greatest single body of theory contributing to nonlinear analysis of discrete systems is that which grew out of the attempts of the great French and German mathematicians of the last century to solve the problem of three bodies. While their objective, in its complete generality, was never realized, the pure mathematics developed (theory of differential equations, convergence, dominant functions, singularities, removable singularities, etc.) is today directly applicable in the study of nonlinear equations of electrical circuits, rotating electrical machines, and various nonlinear dynamical and aerodynamical devices. The two second largest bodies of theory are those of nonlinear integral equations as developed by T. Lalesco and others and the methods of Galerkin and Ritz along with the modifications of these techniques.

There are at least three salient characteristics of nonlinear engineering problems which distinguish them from purely theoretical problems. First, oscillograms, differential analyzer solutions, or other records frequently indicate the nature of the solution of the mathematical systems in question. Such mechanical or electrical solutions for the same system often differ so much among themselves that there is the risk of concluding erroneously that the solution is not unique. (For example, the differential equations which yield the two solutions represented in Figs. 3 and 4 also possess sinusoidal solutions. Yet the solutions are unique; i.e., the solution in (4) is identical with the sinusoidal solution.) Of course, there are systems which do not possess a unique solution. In general, even when a solution is unique it may have so many manifestations that it is often necessary to integrate the system to determine the effect of the various parameters involved. A second characteristic of industrial nonlinear problems is that frequently the methods of mathematics are not powerful enough to yield a complete solution of the problem in sufficiently simple form to be usable. Tricks and devices, born of physical concepts, must guide the mathematics if a usable solution is to be attained. The mathematics is surely necessary and it is just as surely not sufficient. The solution is mathematics plus. A third distinction of industrial nonlinear problems is the fact that the derivation of the equations of performance requires, in addition to a knowledge of mathematics, mathematical physics,

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7 For additional bibliographies see references 2 and 3 above, also the book, E. G. Keller, *Mathematics of modern engineering*, vol. II, Wiley and Sons, New York, 1942, pp. 303-304. These list a total of 302 papers; and these papers in turn possess bibliographies.
and engineering, inventive ability in thought. A system of equations may be an invention of the highest order. It is not always necessary to integrate a system of nonlinear equations. Often it is necessary only to determine under what conditions the physical system is stable. Of course, no single stability criterion exists for nonlinear systems such as exists for linear systems. When a solution of a nonlinear system cannot be obtained with sufficient rapidity or when it can be obtained but is worthless because the time consumed in applying it is too great, it may be possible to obtain the information desired by integrating a dominant and a "subordinate" system such that the solution of the actual problem is bounded or limited by the solutions of the dominant and subordinate system. The use of dominant and "subordinate" systems will be clear in the following problems.

3. Some representative discrete nonlinear problems of industry. In this paper a number of representative nonlinear systems are treated which illustrate the principles enumerated in the last section. These systems are either original, appearing here for the first time, or else of very recent date. Some of them pertain to electrical manufacturing, others to aircraft development. Although, as stated above, the derivation of the equations of a system is often more important than the solution, none of the equations considered are derived here. Some systems are derived in the literature and to these references are given. The derivations of the remaining ones can not be given for military reasons. These are viewed here merely as hypothetical nonlinear systems.

1. Nonlinear control circuits. As is well known, the volt-ampere characteristic of a nonlinear series circuit (Fig. 1) is represented by the curve in Fig. 2. Such circuits have numerous industrial applications due to their rugged mechanical simplicity and at the same time their electrical sensitivity. The characteristic in Fig. 2 displays the fact that there exists a so-called critical or resonant voltage $E_0$ at which the R.M.S. value of the current suddenly increases many fold. For a value of $E < E_0$ (see $E \sin \omega t$ in Fig. 1) the current is sinusoidal. For $E > E_0$ the current has the wave form displayed in Fig. 3.

![Fig. 1.](image1)

![Fig. 2.](image2)

![Fig. 3.](image3)

In industrial applications $E_0$ is prescribed. It is required to design a circuit which will be sensitive for this prescribed value of $E_0$. A simple slide rule formula is desired which will express $E_0$ as a function of the circuit parameters and of the nonlinear reactor employed. The equation of performance for the circuit in Fig. 1 is

$$L(i) \frac{di}{dt} + Ri + \frac{1}{C} \int i \, dt = -E \cos \omega(t - t_0).$$  (1)
For the range of interest the current $i$ is such that the saturation curve of the reactor is single-valued and represented by the equation

$$H = ki = x - a_3x^3 + a_6x^6.$$  \hspace{1cm} (2)

With $\theta = \omega t$, Eqs. (1) and (2) yield

$$M \frac{dx}{d\theta} + R(x - a_3x^3 + a_6x^6) + x_c \int (x - a_3x^3 + a_6x^6) d\theta = -Ek \cos(\theta - \theta_0),$$ \hspace{1cm} (3)

where, for a given $\omega$, $M$ and $x_c$ are constants. The integration of the nonlinear Eq. (3) and the development of $E_0$ as a function of the parameters of the physical problem are carried out elsewhere and need not be repeated here.\(^9\)

2. **Nonlinear transmission line phenomena.** If a series capacitor is employed in the primary side of a transmission line to improve the power factor, curious wave forms of voltage and current ensue. The system becomes unstable as far as possessing a periodic solution is concerned. This is to be expected since the maximum flux density attains a value close to that of the knee of the saturation curve if the transformers are operating efficiently. The addition of series capacitance is thus likely to create an unstable system. In this unstable system, the current and voltage taken on an indefinitely large number of wave forms such as shown in Fig. 4. Synchronous motors which require sinusoidal applied voltages cannot operate on currents and voltages of the type shown in Fig. 4.

If the capacitor of the system is shunted by a resistor as indicated in Fig. 5, then the equations of performance are

\[
\frac{d\psi}{d\theta} + R_1 i_1 + L \frac{di_1}{dt} + \frac{1}{C} \int (i_1 - i_2) dt = E \sin (\theta + \theta_0),
\]

\[
i_2 R_2 = \tilde{x}_c \int (i_1 - i_2) d\theta, \quad (\tilde{x}_c = 1/C),
\]

\[
\alpha i_1 = x - a_3 x^3 + a_5 x^5; \quad H = \alpha i; \quad x = B/B_0; \quad B_0 = dB/dH \text{ at } H = 0,
\]
or
\[
\alpha B_0 \frac{d^2 x}{d\theta^2} + \left[ \frac{\tilde{x}_c B_0 \alpha}{R_2} + R_1 (1 - 3a_3 x^2 + 5a_5 x^4) \right] \frac{dx}{d\theta} + \frac{\tilde{x}_c (R_1 + R_2)}{R_2} (x - a_3 x^3 + a_5 x^5)
\]

\[
= \frac{\alpha}{R_2} [R_2^2 + \tilde{x}_c^2]^{1/2} E \cos \left( \theta + \theta_0 - \tan^{-1} \frac{\tilde{x}_c}{R_2} \right).
\]

Now \(R_2\) must have the smallest possible value consistent with stability, since it represents a perpetual loss of power. There are ten parameters and two variables.

There are infinitely many values of the parameters for which the system is unstable and equally as many for which it is stable, i.e., for which the solutions are sinusoidal. A convenient slide rule formula is desired giving the above smallest value of \(R_2\) as functions of the other nine parameters of the system. The equations of the system are derived and solved elsewhere.\(^{10}\)

3. Nonlinear springs. It is sufficient to say that, in general, the differential equations involving nonlinear springs are integrable by hyperelliptic functions.\(^{11}\) If damping is large a combination of variation of parameters and hyperelliptic functions will usually afford sufficient accuracy.

4. Electric locomotive oscillations. Experience classifies the five oscillatory motions of an electric locomotive as pitch, roll, plunge, nose, and rear-end lash. The last two are especially important because their pronounced existence in a locomotive produces a tendency to derail. Considered superficially, characteristic oscillations of an electric locomotive would seem to be similar to those of an ordinary vehicle such as an automobile, but experimental data and observation indicate the existence of dangerous

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nose and rear-end lash which are not common to an automobile. If the tendency to nose exists in an electric locomotive and if the locomotive noses for a speed \( V_0 \), then it will nose for all speeds greater than \( V_0 \). Consequently, nosing is not a resonance phenomenon and cannot be avoided by running at a slightly different speed. It might be supposed that nosing is due to the coning of the wheels or to the staggering of the rails or to a combination of these two possible causes. Such causes, however, would produce resonance frequencies for definite discrete values of \( V \) instead of instability for all values of \( V \) exceeding \( V_0 \). Rails on European railroads are not staggered and yet electric locomotive nosing still persists. The tendency to nose and the violence of the oscillation increase with the \textit{weight} and \textit{power} of the locomotive.

In seeking the source of the phenomenon, consider first an elementary experiment. Let a miniature set of driving wheels and axle be constructed from two rubber paste bottle stoppers and a lead pencil. If the miniature drivers are forced down against two rulers as rails, if a torque is applied tending to rotate the wheels, and if further in the forward motion slight lateral motion is permissible then an oscillating torque will be experienced tending to rotate the axle about a line through the center of axle and perpendicular to the plane of the track. The creepage forces between the rubber wheels and the rails produce an oscillatory torque.

The weight of an electric locomotive is so great that it effectively rolls on elastic wheels on elastic rails. Making use of this fact and whatever additional facts are necessary the equations of motion\(^{12}\) can be shown to be

\[
M\ddot{x}_0 = 0,
\]

\[
M\ddot{y}_0 = -F_2 - f_2 - 2f\left(\frac{\dot{y}_2}{V} - \zeta\right) - 2f\left(\frac{\dot{y}_1}{V} - \zeta\right) - F_1 - f_1 - 2f\left(\frac{\dot{y}_1}{V} - \zeta\right),
\]

\[
M\ddot{z}_0 + \lambda_1(z_0 - b_1\eta) + \lambda_2(z_0 - c\xi + b_2\eta) + \lambda_2(z_0 - c\xi + b_2\eta) + k_1^2\ddot{z}_0 = 0,
\]

\[
\lambda_1\ddot{x} + \lambda_2(z_0 - c\xi + b_2\eta) - \lambda_2(z_0 - c\xi + b_2\eta) + k_2^2\ddot{x}
\]

\[
= -b_3(F_1 + F_2 + f_1 + f_2) - 2b_3\frac{f}{V}(\dot{y}_1 + \dot{y} + \dot{y}_2) + 6b_5\ddot{z},
\]

\[
B\ddot{\eta} - \lambda_1b_1(z_0 - b_1\eta) + \lambda_2b_2(z_0 + c\xi + b_2\eta) + \lambda_2b_2(z_0 - c\xi + b_2\eta) + k_3^2\ddot{\eta} = 0,
\]

\[
C\ddot{\eta} = -d_3(F_1 - F_2) - d_4(f_1 - f_2) - \frac{2fd_3}{V}(\dot{y}_1 - \dot{y}_2) - \frac{6fb^2}{V}\ddot{\eta}
\]

\[
= \frac{2fb}{r}(\ddot{y} + \dot{y}_1 + \dot{y}_2) + F_1(\dot{y}_1).
\]

To the accuracy required, the flange forces are given by

\[
F_1 = H\left(\frac{y_1}{\delta_1}\right)^3 + I\left(\frac{y_1}{\delta_1}\right)^5 + J\left(\frac{y_1}{\delta_1}\right)^7 + \cdots,
\]

\[
F_2 = H\left(\frac{y_2}{\delta_1}\right)^3 + I\left(\frac{y_2}{\delta_1}\right)^5 + J\left(\frac{y_2}{\delta_1}\right)^7 + \cdots.
\]

where the constants $H, I, J, j, i, h$ are determined from force curves, and $\delta_1$ and $\delta_2$ are lengths shown in Fig. 6.

The variables $y_1, y_2, y_3, y_4,$ and $\bar{y}$ are eliminated from (5) by means of the relations

\[ y_1 = y_0 + b_5 \xi + d_3 \xi, \quad y_2 = y_0 + b_6 \xi - d_3 \xi, \]
\[ y_3 = y_0 + h_1 \xi + d_4 \xi, \quad y_4 = y_0 + h_1 \xi - d_4 \xi, \]
\[ \bar{y} = y_0 + b_5 \xi, \]

where $b_5, d_3, d_4, h_1$ and $h_2$ are lengths defined in reference 12.

If in (5) $F_1 = F_2 = f_1 = f_2 = 0$, then the equations are linear; and the solution can be written down at once. This solution is either stable or unstable as indicated by the roots of the characteristic equation. The nature of the roots are, of course, a function of $V$, the operating speed of the locomotive. Even if the locomotive is unstable with vanishing flange forces, it is stable with non-vanishing flange forces. In this case the locomotive is operating roughly and damaging the track needlessly.

In practical applications, then, it is not necessary to integrate the nonlinear system (5). As a check on the validity of the theory, however, it is necessary to integrate the nonlinear system and compare the predicted motion with actual motion as determined by runs on a test track. Evidently the solution of (5) for $F_1 = F_2 = f_1 = f_2 = 0$ cannot be used as a generating solution for the case of the non-vanishing of the flange forces because the stability or unstability of this generating solution is carried over into the complete solution.

\[ f_1 = h \left( \frac{y_3}{\delta_2} \right)^3 + i \left( \frac{y_2}{\delta_2} \right)^5 + j \left( \frac{y_8}{\delta_2} \right)^7 + \cdots, \]
\[ f_2 = h \left( \frac{y_4}{\delta_2} \right)^3 + i \left( \frac{y_4}{\delta_2} \right)^5 + j \left( \frac{y_7}{\delta_2} \right)^7 + \cdots, \]

Fig. 6.

Since nosing and rear end lash are the two motions of most importance, it is sufficient to resort to elementary means. Consequently, $F_i$ and $f_i$ [Eqs. (6)] are replaced by segments of straight lines as shown in Fig. 6, and the second and sixth equations of (5) are solved for $y_0$ and $\xi$ by operational methods (general operational methods where both initial charges and initial currents exist must be used.) Since the flange
forces are taken as functions with discontinuous slopes, the system of differential
equations and its characteristic equation change as the flange forces, as functions
of \( y_1, y_2, y_3, y_4 \), change at points \( a, b, c, d, e, f \) shown in Fig. 6. The first set of initial
conditions are chosen by trial and error in such a way that the resulting motion is
periodic in \( y_0 \) and \( \xi \). The solution for a complete cycle is sufficient. The check of the
theory is the approximate agreement of computed and test periods.

The equations of dynamic braking are

\[
\frac{dI}{dt} = \frac{(E - IR)}{L} \frac{[(rs_0/s)^2 + x_dx_q]}{[(rs_0/s)^2 + x_d 'x_q]} - \frac{\mu_0}{s^4 \left(\frac{(rs_0/s)^2}{2} + x_d 'x_q\right)} I^3 \left(\frac{x_q^2}{[(rs_0/s)^2 + x_d 'x_q]} + (rs_0/s)^2\right)^3
\]

\[
\frac{ds}{dt} = -\frac{\mu_1 I^2}{s} \frac{x_q^2 + (rs_0/s)^2}{[x_d 'x_q + (rs_0/s)^2]^2},
\]

where

\[
\mu_0 = \frac{2KP_r^2 x_q^2 (x_d - x_d ') s_0^2}{JI_0^2}, \quad \mu_1 = \frac{KP_r}{JI_0^2},
\]

\( I \) being the field current, \( s \) the rotor speed, \( t \) the time in seconds, all other symbols
being constant parameters. It is desired to obtain an expression for the time of
stopping of the rotor as a function of the parameters of the machine.

The last term in the right member of the first of Eqs. (7) has in all cases a magni-
tude of approximately ten per cent of its predecessor. Thus a solution as a power series
in a parameter which vanishes with \( \mu_0 \) is to be expected. Neglecting the term con-
taining \( \mu_0 \) in (7) and dividing the first equation by the second, we obtain a solution
of the resulting equation immediately. However, this solution is an implicit function
of \( I \) and \( s \) and such that it is solvable explicitly for either \( I \) or \( s \) only as a slowly con-
vergent series. An attempted solution by the method of variation of parameters is
equally cumbersome.

It is known from oscillograms, however, that both \( I \) and \( s \) are monotone decreas-
ing functions of the time for the interval within which (7) is valid. Change of de-
pendent variables by

\[ I = \frac{E}{R} + I_1 e^{-\nu} \quad \text{and} \quad s = s_0 e^{-s} \]

yields

\[
\frac{dy}{dt} = \frac{R(A^2 - A_0^2)}{L} + \frac{\mu_0}{[A_0^2 + (re^t)^2] [A^2 + (re^t)^2]^3} \frac{[E/R + I_1 e^{-\nu}]^3 e^{\nu + t}}{I_0 s_0^2},
\]

\[
\frac{ds}{dt} = \frac{[(E/R) + I_1 e^{-\nu}]^2 [x_q^2 + (re^t)^2]}{[A^2 + (re^t)^2]^2} e^{2s},
\]

where

\[ A_0^2 = x_d 'x_q, \quad A^2 = x_d x_q, \quad A > A_0. \]

The number of revolutions before the rotor of the machine comes to rest is

\[
N = \frac{1}{2\pi} \int_0^\infty s \, dt = \frac{1}{2\pi} \int_0^\infty s_0 e^{-s} \, dt.
\]
Now it is sufficient for practical purposes to set an upper limit to $N$ as given by (9) provided the upper limit is sufficiently small and provided the results display the effect of each parameter of the system. To accomplish this (8) may be replaced by a simpler system of equations. Evidently,

$$[A_0^2 + (re^s)^2] \leq (A_0^2 + r^2)e^{2s}, \quad [A^2 + (re^s)^2] \leq (A^2 + r^2)e^{2s} \quad (10)$$

for $s \geq 0$. Employing (10) in (7) and integrating, we have

$$y = \frac{R}{L} t + \frac{R(A^2 - A_0^2)}{L(A_0^2 + r^2)} \int_0^t e^{2s} dt
\quad + \frac{1}{s_0 I_1} \frac{(r^2 + x_0^2)\mu_0}{(A_0^2 + r^2)(A^2 + r^2)^3} \int_0^t \left[ \frac{E}{R} + I_1 e^{-v} \right]^3 e^{-4s} dt, \quad (11)$$

where the instantaneous values of $y$ and $z$ as given by (11) are always less than those given by the solution of (7) for $0 < t < \infty$.

The system (11) is of the form

$$u_k(t) = \phi_k(t) + \int_0^t K_k[l, \xi, u_1(\xi), \cdots, u_n(\xi)] d\xi \quad (k = 1, 2, \cdots, n)$$

which is Lalesco's system of nonlinear integral equations. The solution of this is the limit of the sequences

$$u_k^{(0)} = \phi_k(t),$$

$$u_k^{(1)} = \phi_k(t) + \int_0^t K_k[l, \xi, \phi_1(\xi), \cdots, \phi_n(\xi)] d\xi \quad (k = 1, 2, \cdots, n),$$

$$\cdots \cdots \cdots \cdots$$

In the present application $\phi_1(t) = Rt/L$ and $\phi_2(t) = 0$. For small synchronous machines the second approximations $u_1^{(1)}$ and $u_2^{(2)}$ give values of $y$ and $z$ such that $N$ in (9) is in error by five per cent. The integration in (9) is carried out numerically. Because of bearing friction and other decelerating factors not included in (7) the upper limit in (9) is finite.

6. A double-valued nonlinear problem. Consider the integration of the equation

$$I\dot{\theta} + \beta \dot{\theta} + \mu [k_1 \theta + k_2 \tan^{-1} k_3(\theta \pm a)] = 0. \quad (12)$$

This equation was derived ingeniously by W. W. Beman to express an important phenomenon in aerodynamics. The quantities $I$, $\beta$, $\mu$, $k_1$, $k_2$, $k_3$, and $a$ are all positive numbers and the plus or minus sign in $(\theta \pm a)$ is used according as $\theta < 0$ or $\theta > 0$.

Evidently, for a particular amplitude of $\theta$, Eq. (12) possesses a periodic solution. The period and amplitude of this solution are desired. Eq. (12) in the normal form is

$$\theta = \theta_1, \quad \dot{\theta}_1 = - (\mu/I) [k_1 \theta + k_2 \tan^{-1} k_3(\theta \pm a)] - \beta \dot{\theta}_1/I. \quad (13)$$

An integral of (13) for $\beta = 0$ is

$$\theta_1^2 = c - \frac{2\mu}{I} \left\{ \frac{k_1 \theta^2}{2} + k_2 \left[ (\theta \pm a) \tan^{-1} k_3(\theta \pm a) - \frac{1}{2k_3} \log (1 + k_3^3(\theta_1 \pm a)^2) \right] \right\}$$
or
\[
\theta_1 = \pm \sqrt{c - f(\theta)},
\]
where \( c = f(\theta_0) \) and where \( \theta_0 \) is the maximum positive displacement for \( t = t_0 \). If (14) is used as an equation of change of variable, the method of variation of parameters yields
\[
\frac{\partial \theta_1}{\partial c} \frac{dc}{dt} + \frac{\partial \theta_1}{\partial t} = - \frac{\mu}{I} \left[ k_1 \theta + k_2 \tan^{-1} k_3 (\theta \pm \alpha) \right] - \frac{\beta}{I} \theta_1;
\]
whence
\[
\dot{c} = - \frac{2\beta \theta_1^2}{I},
\]
or
\[
c = - \frac{2\beta}{I} \int_{t_0}^{t} \theta_1^2 dt + d = - \frac{2\beta}{I} \int_{\theta_0}^{\theta} \theta d\theta + d,
\]
where \( d \) is an arbitrary constant. From the last equation
\[
\frac{dc}{d\theta} = - \frac{2\beta}{I} \theta_1 = - \frac{2\beta}{I} \left( \pm \sqrt{c - f(\theta)} \right).
\]

To determine the signs in (16) it is evident from (15) that \( c \) is a decreasing function of the time. Consequently, for \( \dot{\theta} < 0 \)
\[
\frac{dc}{d\theta} = \frac{dc}{dt} \frac{dt}{d\theta} = \frac{dc}{dt} \left( - \sqrt{c - f(\theta)} \right).
\]
Thus Eq. (16) is
\[
\frac{dc}{d\theta} = \frac{2\beta}{I} \sqrt{c - f(\theta)}
\]
according as \( \theta_1 < 0 \) or \( \theta_1 > 0 \).

For the integration of (17) it is sufficient to replace \( \sqrt{c - f(\theta)} \) by \( k \left[ c - f(\theta) \right] \) where \( k \) is determined graphically by
\[
\int \sqrt{c - f(\theta)} d\theta = k \int \left[ c - f(\theta) \right] d\theta,
\]
\( c = f(\theta_0) \) or \( c = f(\theta'') \) according as \( \theta_0 \leq \theta \leq \theta'' \) or \( \theta'' \leq \theta \leq \theta_0'' \), and \( \theta_0, \theta', \) and \( \theta'' \) are shown in Fig. 7. The curve in Fig. 7 is the solution (14). With this replacement and simple integration
\[
c = e^{2\beta k (\theta - \theta_0)/I} \left[ c_0 - \frac{2\beta k}{I} \int_{\theta_0}^{\theta} e^{-2\beta k \theta'/I} f(\theta) d\theta \right] \quad (\theta_0 \leq \theta \leq \theta'),
\]
\[
c = e^{-2\beta k (\theta - \theta'')/I} \left[ c_0' - \frac{2\beta k}{I} \int_{\theta''}^{\theta} e^{2\beta k \theta'/I} f(\theta) d\theta \right] \quad (\theta'' \leq \theta \leq \theta''),
\]
where \( c_0 = f(\theta_0) \) and \( c'_0 = f(\theta''') \). The above values of \( c \) are substituted in Eq. (14). The
solution is periodic when \( \theta_0 \) is chosen such that \( \theta_0' \) turns out to be equal to \( \theta_0 \). The period of the motion is then given by

\[
P = 2 \int_{\theta_0}^{\theta_0'} \frac{d\theta}{\theta_1}.\tag{19}
\]

The numerical integration of (19) presents no difficulty at the limits \( \theta_0 \) and \( \theta_0' \) since \( \theta_1 \) in the vicinity of \( \theta_0 \) and \( \theta_0' \) can be replaced by an integrable function \( f_0 \) such that the limit of \( (f_0/\theta_1) = 1 \) at \( \theta_1 = \theta_0 \) and \( \theta_1 = \theta_0' \).

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7. A nonlinear problem of two oleo-pneumatically coupled masses one of which is subject to impact. It can be shown without difficulty that the nonlinear differential equations of motion of \( m_1 \) and \( m_2 \) shown in Fig. 8 are

\[
m_1\ddot{s}_1 - w_1 - \frac{\rho_0 S}{\left[1 - \frac{(s_2 - s_1)}{D}\right]^{1/2}} - \frac{\rho(S - s_m)^3(s_2 - s_1)^2}{2gc^2[A(r)]^2} + k_2s_1 + f(s_1) = 0,
\]

\[
m_2\ddot{s}_2 - nw_2 + \frac{\rho_0 S}{\left[1 - \frac{(s_2 - s_1)}{D}\right]^{1/2}} + \frac{\rho(S - s_m)^3(s_2 - s_1)^2}{2gc^2[A(r)]^2} = 0,
\]

where

\[
[A(r)]^2 = \pi^2\left\{R^2 - \left[r_i + b_i(s_2 - s_1)\right]^2\right\}, \quad f(s_1) = (K_0 + k_0s_1),
\]

and \( n = (1 - m) \), \( 0 \leq m \leq 1 \). In Eqs. (20) and (21), \( s_1, s_2, \) and \( t \) are the dependent and independent variables, the remaining symbols being constants.

A solution of (20) is desired for the initial conditions \( s_1(0) = s_2(0) = v_0 \). The time \( t \) is counted from the instant when the lower end of the spring is in contact with a fixed horizontal surface. For suitable values of \( R, r_i, \) and \( b_i \) a graph of \([A(r)]^2\) is either Fig. 9a or 9b.
Even in the case where $A(r) = \pi R$ the following methods, (a) power series solutions in the time, (b) numerical integration, (c) successive integrations, (d) Galerkin's, (e) curvature, and (f) expansions in power series in parameters, become so laborious that they fail for practical purposes. Lord Rayleigh\textsuperscript{13} has given methods of handling differential equations linear in all terms except containing a damping coefficient which depends on the square of the velocity. The velocity term is supposed small. In Eq. (20) the velocity term is small or large dependent upon the stage of the motion.

Consider the curves shown in Figs. 10, 11, and 12. It is evident that the forces relevant to these curves can be approximated by the arc $Q_0Q_1$ and the small number of secant lines $Q_1Q_2$, $R_0R_1$, $R_1R_2$, and $R_2R_3$. The location of $Q_1$, $R_0$, $R_1$, $R_2$, and $R_3$ will become evident from physical conditions presently discussed.

For the first interval of motion replace Eq. (20) by the equations

$$s_1 + K_0 + k_0s_1 - \alpha [A_0 + m_0(s_2 - s_1)] - g = \beta (s_2 - s_1)^2,$$

$$s_2 + \gamma [A_0 + m_0(s_2 - s_1)] - ng = - \delta (s_2 - s_1)^2,$$

where

$$\alpha = \frac{p_0S}{m_1}, \quad \beta = \frac{\rho(S - s_m)^3}{2w_1e^2\pi^2R^2}, \quad \gamma = \frac{p_0S}{m_2}, \quad \delta = \frac{\rho(S - s_m)^2}{2w_2e^2\pi^2R^2}.$$

and \( A_0, m_0, K_0, \) and \( k_0 \) are shown in the figures. The ordinate of \( R_3 \) represents the value of \((1 - \xi)^{-1.20}\) when the system is at rest under the force of gravity. The points \( R_2 \) and \( R_1 \) are located such that no ordinate on the secant lines exceeds the corresponding ordinate on the arc by more than ten per cent.

The solution of Eq. (20) is now broken up into two time intervals. We reduce Eq. (21) to the normal form by the substitutions

\[
s_1 = \xi_1 + a_1, \quad s_2 = \xi_2 + a_2, \quad \xi_1 = \xi_3, \quad \xi_2 = \xi_4,
\]

where \( a_1 \) and \( a_2 \) are constants such that no constant term remains in the resulting differential equations. Then the equations are

\[
\begin{align*}
\dot{\xi}_1 &= \xi_3, \\
\dot{\xi}_2 &= \xi_4, \\
\dot{\xi}_3 &= - (am_0 + k_0)\xi_1 + am_0\xi_2 + \beta(\xi_4 - \xi_3)^2, \\
\dot{\xi}_4 &= \gamma m_0\xi_1 - \gamma m_0\xi_2 - \delta(\xi_4 - \xi_3)^2,
\end{align*}
\]

and

\[
\begin{align*}
a_1 &= \gamma[g + (\alpha A_0 - K_0)] - \alpha[yA_0 - ng]/\gamma k_0, \\
a_2 &= \{m_0[y(g + \alpha A_0 - K_0) - \alpha(yA_0 - ng)] - k_0(yA_0 - ng)\}/m_0\gamma k_0,
\end{align*}
\]

and \( \alpha A_0 = K_0 \) in order that \( \xi_1 \) may not be positive in its initial motion. That is, the origin of time is taken to be the instant at which the upward force of the spring \( S \) is equal to the downward force due to gas pressure on \( m_1 \).

The general solution of (23) (with squared terms suppressed) is

\[
\begin{align*}
\xi_1 &= A_1 \sin \omega_1 t + A_2 \cos \omega_1 t + A_3 \sin \omega_2 t + A_4 \cos \omega_2 t, \\
\xi_2 &= b_1 A_1 \sin \omega_1 t + b_2 A_2 \cos \omega_1 t + b_3 A_3 \sin \omega_2 t + b_4 A_4 \cos \omega_2 t, \\
\xi_3 &= \omega_1 A_1 \cos \omega_1 t - \omega_1 A_2 \sin \omega_1 t + \omega_2 A_3 \cos \omega_2 t - \omega_2 A_4 \sin \omega_2 t, \\
\xi_4 &= b_1 \omega_1 A_1 \cos \omega_1 t - b_2 \omega_1 A_2 \sin \omega_1 t + b_3 \omega_2 A_3 \cos \omega_2 t - b_4 \omega_2 A_4 \sin \omega_2 t,
\end{align*}
\]

where \( \omega_1 \) and \( \omega_2 \) are the roots of the characteristic equation and

\[
\begin{align*}
b_1 &= (am_0 + k_0 - \omega_1^2)/am_0, \quad b_2 = (am_0 + k_0 - \omega_2^2)/am_0.
\end{align*}
\]

The nonlinear terms in (22) are taken into account by the method of variation of parameters. Employing (24) as equations of change of variables and remembering that (24) satisfies

\[
\frac{\partial \xi_1}{\partial t} = \xi_3, \quad \frac{\partial \xi_2}{\partial t} = \xi_4,
\]

\[
\frac{\partial \xi_3}{\partial t} = - (am_0 + k_0)\xi_1 + am_0\xi_2, \quad \frac{\partial \xi_4}{\partial t} = \gamma m_0\xi_1 - \gamma m_0\xi_2
\]

we have

\[
\begin{align*}
\frac{\partial \xi_1}{\partial A_1} \dot{A}_1 + \frac{\partial \xi_1}{\partial A_2} \dot{A}_2 + \frac{\partial \xi_1}{\partial A_3} \dot{A}_3 + \frac{\partial \xi_1}{\partial A_4} \dot{A}_4 &= 0, \\
\frac{\partial \xi_2}{\partial A_1} \dot{A}_1 + \frac{\partial \xi_2}{\partial A_2} \dot{A}_2 + \frac{\partial \xi_2}{\partial A_3} \dot{A}_3 + \frac{\partial \xi_2}{\partial A_4} \dot{A}_4 &= 0
\end{align*}
\]
\[
\frac{\partial \xi_3}{\partial A_1} A_1 + \frac{\partial \xi_3}{\partial A_2} A_2 + \frac{\partial \xi_3}{\partial A_3} A_3 + \frac{\partial \xi_3}{\partial A_4} A_4 = \beta (\xi_4 - \xi_3)^2, \tag{25'}
\]
\[
\frac{\partial \xi_4}{\partial A_1} A_1 + \frac{\partial \xi_4}{\partial A_2} A_2 + \frac{\partial \xi_4}{\partial A_3} A_3 + \frac{\partial \xi_4}{\partial A_4} A_4 = -\delta (\xi_4 - \xi_3)^2.
\]

The solutions of (25), after some rather lengthy trigonometric manipulations, are

\[
A_1 = \frac{\delta + b_0 \beta}{\omega_1 (b_2 - b_1)} (\xi_4 - \xi_3)^2 \cos \omega_1 t, \quad A_2 = -\frac{\delta + b_0 \beta}{\omega_1 (b_2 - b_1)} (\xi_4 - \xi_3)^2 \sin \omega_1 t,
\]
\[
A_3 = -\frac{\delta + b_1 \beta}{\omega_2 (b_2 - b_1)} (\xi_4 - \xi_3)^2 \cos \omega_2 t, \quad A_4 = \frac{\delta + b_1 \beta}{\omega_2 (b_2 - b_1)} (\xi_4 - \xi_3)^2 \sin \omega_2 t. \tag{26}
\]

The solution of (26) is obtained with sufficient approximation by using a device common in celestial mechanics; i.e., for small values of the time, the \(A_i\) entering (26) through \(\xi_3\) and \(\xi_4\) may be considered constants having the values obtained by the solution of (24) for \(\xi_1 = -a_1, \xi_2 = -a_2, \xi_3 = \xi_4 = v_0\) at \(t=0\). Thus the solution of (26), to the accuracy required, is reduced to quadratures. Moreover, since the interval for which this solution is valid is small \((0 \leq t \leq 0.01)\) the trigonometric functions involved may be expanded as power series in \(t\) before the quadratures are performed. The solution of (25) is

\[
A_i = C_i + f_i(t) \quad (i = 1, \ldots, 4), \tag{27}
\]

where \(f_i(0) = 0\). The substitution of (27) in (24) gives the complete solution for \(0 \leq t \leq t_1\), where \(\omega_2 t_1 < \frac{1}{2}\) and \(\omega_2 > \omega_1\). The values of \(C_i = A_i\) as determined above.

The value of \(\left[\xi_i(t_1) - \xi_3(t_1)\right]^2\) locates the point \(Q_i\) in Fig. 10. The ordinate of \(Q_2\) is \(v_0^2\). The ordinate of \(R_1\) is given by \(s_2(t_1) - s_1(t_1)\). The ordinate of \(R_4\) is the value of \((1-\xi)^{-1.2}\) when the air chamber is decreased to 0.7 of its initial value.
In the solution for the second interval \( t_1 \leq t < t_2 \) of motion it is sufficient to replace the arcs \( R_1R_2R_3R_4 \) and \( Q_1Q_2 \) by the secant lines \( R_1R_4 \) and \( Q_1Q_2 \). The quantity \( (s_2 - s_1)^2 \) on the interval \( Q_1Q_2 \) may be written

\[
(s_2 - s_1)^2 = -B_0 + n_0(s_2 - s_1),
\]

and (22) becomes

\[
(p^2 + \alpha m_0 + k_0)s_1 - \alpha m_0 s_2 = g + \alpha A_0 - \beta B_0 + \beta n_0(s_2 - s_1) - K_0,
\]

\[
-\gamma m_0 s_1 + (p^2 + \gamma m_0)s_2 = n g - \gamma A_0 + \delta B_0 - \delta n_0(s_2 - s_1),
\]

where any additional constants are shown in the figures. At the new origin of time for (29), \( s_1(0) = s_2(0) = 0 \) and \( s_1(t_2) = s_1(t_1) = 0, s_2(0) = s_2(t_1) = 0 \).

The proper determination of the constants \( b_1 \) and \( b_2 \) in the substitution \( s_1 = \xi_1 + b_1, s_2 = \xi_2 + b_2 \) in (29) yields

\[
(p^2 + \beta n_0 p + \alpha m_0 + k_0)\xi_1 - (\beta n_0 p + \alpha m_0)\xi_2 = 0,
\]

\[
- (\delta n_0 p + \gamma m_0)\xi_1 + (p^2 + \delta n_0 p + \gamma m_0)\xi_2 = 0.
\]

While the characteristic equation of (30) is of the fourth degree, yet its roots are widely separated in practical cases and quickly found by Graeffe's method.

The values of \( s_1 = \xi_1 + b_1, s_2 = \xi_2 + b_2 \) as given by the solution of (30) do not yield the equilibrium positions of \( m_1 \) and \( m_2 \), because when \( (s_2 - s_1)^2 \) becomes small the relation (28) and Eqs. (29) are no longer valid. This is no defect of the solution because its purpose is the determination of the maximum accelerations acting on \( m_1 \) and \( m_2 \). These maxima occur in the interval \( 0 \leq t \leq t_2 \). The equilibrium positions of \( m_1 \) and \( m_2 \) are determined from static considerations.

A point of special interest is the determination of the effects of the factor \( n g \) upon the solution. The above solution is constructed with this in mind.

The roots of the characteristic equation of (30) have special physical significance. In practical cases these are usually one or two pairs of complex roots. If there are four complex roots, one pair gives a high frequency oscillation of moderate magnitude for \( m_1 \). This is to be avoided.

If \( [A(r)]^2 \) is given by the graph shown in Fig. 9b the above method is still applicable. The solution is very sensitive with respect to \( [A(r)]^2 \). Of course, the intervals of solution will exceed two in number, but in each interval the value of \( (s_2 - s_1)^2 \) will be given by the ordinates of the arc \( Q_0Q_1 \) or the secant \( Q_1Q_2 \).

The most complicated process involved in solving (20) is the solution of a quartic equation.

4. Concluding remarks. The seven problems presented above are representative of the nonlinear discrete problems of industry in so far as one nonlinear problem can represent a group the members of which differ greatly. No bibliography is given for the reason stated in footnote 7.

Methods of handling industrial nonlinear problems of continuous systems arising in industry are reserved for a subsequent paper.