

## —NOTES—

A GEOMETRICAL INTERPRETATION OF THE  
RELAXATION METHOD\*By J. L. SYNGE (*The Ohio State University*)

Let  $a_{ij}, B_i$  ( $i, j = 1, 2, \dots, n$ ) be given constants such that  $a_{ij} = a_{ji}$  and  $\sum_{i,j=1}^n a_{ij}x_i x_j$  is a positive definite form. Consider the equations

$$\sum_{j=1}^n a_{ij}x_j - B_i = 0 \quad (i = 1, 2, \dots, n). \quad (1)$$

The solution is easily expressed as a set of quotients of determinants. However, as  $n$  increases, the task of calculating the determinants becomes excessively burdensome. The relaxation method<sup>1</sup> provides a set of easy steps by which the solution of (1) is approached. The method has been compactly described by Temple.<sup>2</sup>

The purpose of the present note is to give a geometrical description of the relaxation method. For the trivial case  $n = 2$  the geometrical description may be displayed accurately in a diagram. For  $n = 3$  a model may be visualized. For  $n > 3$  we pass beyond the region of simple concrete geometrical representation, but in many ways geometry in an  $n$ -space is closely analogous to geometry in 2-space or 3-space, and the geometrical description continues to serve as a general guide to procedure.

Let us regard  $x_i$  as rectangular Cartesian coordinates in a Euclidean  $n$ -space. Let us define

$$H(x) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}x_i x_j - \sum_{i=1}^n B_i x_i. \quad (2)$$

The equation  $H(x) = \text{const.}$  represents a family of ellipsoids  $E$ ; these ellipsoids have a common center, common directions for their principal axes, and common values for the ratios of their principal axes. They form, in fact, a family of similar and similarly situated ellipsoids.

The equations (1) represent a set of planes (i.e., flats of  $n-1$  dimensions). The point of intersection of these planes is the common center  $G$  of  $E$ . Thus the problem of solving (1) is the problem of finding the center of an ellipsoid when its equation is given.

It is important to note that  $H(x)$  takes a minimum value at  $G$ .  $H$  is constant over each ellipsoid, and increases steadily as we pass out from  $G$ .

It is not possible to define precisely what procedures are to be regarded as permissible. It is a question of ease of computation. Let us follow Southwell and consider an approach to  $G$  by steps each of which is parallel to one of the axes of coordinates  $x_i$ .

Fig. 1 shows Southwell's procedure. It is a schematic diagram in which the ellip-

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<sup>1</sup> R. V. Southwell, *Relaxation methods in engineering*, Oxford, 1940.

<sup>2</sup> G. Temple, *Proc. Roy. Soc. London A*, **169**, 476-500 (1939).

soids are represented by circles. (The ellipsoids can of course be transformed into concentric spheres by a linear transformation, which however destroys the orthogonality of perpendicular lines.) We start with an arbitrary point  $P_0$  (the zero approximation). Let  $E_0$  be the ellipsoid which passes through  $P_0$ . Through  $P_0$  we draw a straight line  $L$  parallel to one of the coordinate axes. Let  $Q_1$  be the second point in which  $L$  cuts  $E_0$ . Let  $P_1$  be the middle point of the chord  $P_0Q_1$ . Then  $P_1$  is the first approximation.

Since the ellipsoid is a convex surface,  $P_1$  lies inside  $E_0$  and so  $H(P_1) < H(P_0)$ . Moreover it is easy to see that  $P_0Q_1$  is tangent at  $P_1$  to the ellipsoid  $E_1$  which passes through  $P_1$ . Thus, of all points on the chord  $P_0Q_1$ , the point  $P_1$  gives the smallest value of  $H$ .

The process is repeated, starting from  $P_1$ . The second approximation  $P_2$  is the middle point of a chord  $P_1Q_2$  of  $E_1$ , drawn parallel to another of the coordinate axes. In this way we get a sequence of points  $P_0, P_1, \dots$ . The success of the method depends on the rapidity of the convergence of this sequence to  $G$ .

In one important respect the above procedure is incompletely defined. When we have reached  $P_m$ , in which of the directions defined by the coordinate axes are we to proceed in order to get  $P_{m+1}$ ? There are  $n$  coordinate axes. Of these one cannot be used, viz., that which gave the direction of the step  $P_{m-1}P_m$ . But, of the remaining  $n-1$  directions, which should we use?

Gaskell<sup>3</sup> has suggested the following plan. Write

$$C_i(x) = \sum_{j=1}^n a_{ij}x_j - B_i. \tag{3}$$

Having reached the point  $P_m$ , we calculate the quantities  $C_i(P_m)$ . Let  $C_k(P_m)$  be the greatest of these in absolute value. Then we choose for the step  $P_mP_{m+1}$  the direction of the axis of  $x_k$ .

This procedure is called the liquidation of the greatest error, since we obtain  $C_k(P_{m+1}) = 0$ . It is interesting to see how this result fits into the geometrical discussion. The plane  $C_k(x) = 0$  is the plane through  $G$  conjugate to the direction of the axis  $x_k$ . The line  $P_mP_{m+1}$  is parallel to this axis and tangent at  $P_{m+1}$  to one of the ellipsoids,  $E_{m+1}$ . But the point of contact of a line with an ellipsoid lies on the central plane conjugate to the direction of the line. Hence  $P_{m+1}$  lies on  $C_k(x) = 0$ , i.e.,  $C_k(P_{m+1}) = 0$ .

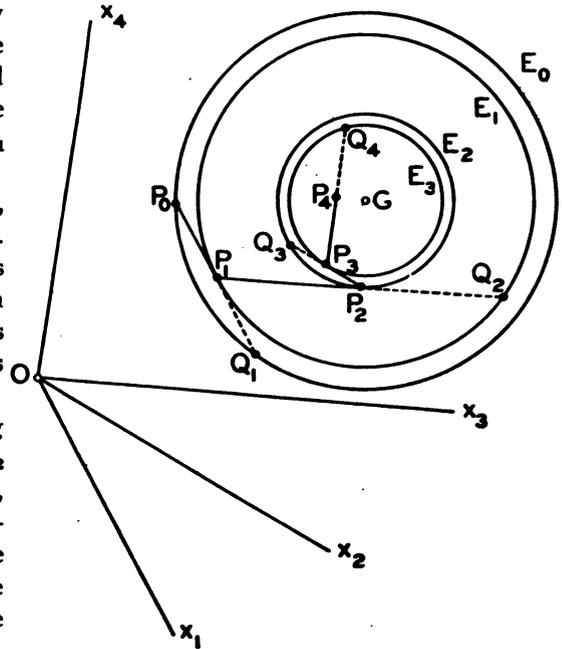


FIG. 1.

<sup>3</sup> R. E. Gaskell, Quarterly of Applied Mathematics, 1, 237-249 (1943).

But it may well be asked whether the quantities  $C_i$  themselves possess any deep significance. It is true that  $G$  satisfies  $C_i(x) = 0$ , but the quantity  $C_i(x)$  for a general point does not represent the perpendicular distance of that point from the plane  $C_i(x) = 0$ . This perpendicular distance is

$$f_i(x) = \frac{|C_i(x)|}{\left(\sum_{j=1}^n a_{ij}^2\right)^{1/2}}. \quad (4)$$

Should we not liquidate the greatest  $p_i$  rather than the greatest  $C_i$ ? Or is there a better plan than either?

The following plan is suggested. Having reached the point  $P_m$ , we have an option on  $n-1$  next points. Each of these points lies on an ellipsoid of the family  $E$ . Choose that point which lies on the innermost ellipsoid. This is equivalent to saying: Choose that point which gives the smallest value to  $H$ .

Now<sup>4</sup> for a step in the direction of the axis  $x_i$  the decrease in  $H$  is  $\frac{1}{2}C_i^2/a_{ii}$ . This is to be made as great as possible, and so we should pick the direction of the step  $P_m P_{m+1}$  according to the following rule: Proceed in the direction of the axis of  $x_k$  where  $C_k^2/a_{kk}$  is the greatest of the quantities  $C_i^2/a_{ii}$  ( $i=1, 2, \dots, n$ ).

Thus  $C_i^2/a_{ii}$  is made the criterion rather than Gaskell's  $C_i$ . The calculation of the former quantities involves slightly more computation, but this may be taken care of by making the initial transformation

$$x'_i = (a_{ii})^{1/2} x_i. \quad (5)$$

Then

$$H = \frac{1}{2} \sum_{i,j=1}^n a'_{ij} x'_i x'_j - \sum_{i=1}^n B'_i x'_i, \quad (6)$$

where

$$a'_{ii} = 1, \quad a'_{ij} = a_{ij}/(a_{ii}a_{jj})^{1/2}, \quad B'_i = B_i/(a_{ii})^{1/2}. \quad (7)$$

Now, with

$$C'_i(x') = \sum_{j=1}^n a'_{ij} x'_j - B'_i, \quad (8)$$

the criterion for the direction of the next displacement is  $C_i'^2$  or  $|C'_i|$ , the same as Gaskell's. Moreover the transformation from  $P_m$  to  $P_{m+1}$  is now extremely simple. It is<sup>5</sup>

$$\begin{aligned} P_m: & \quad x'_1, \dots, x'_k, \dots, x'_n; \\ P_{m+1}: & \quad x'_1, \dots, x'_k - C'_k(x'), \dots, x'_n. \end{aligned}$$

<sup>4</sup> R. E. Gaskell, loc. cit., Eq. (23).

<sup>5</sup> R. E. Gaskell, loc. cit.