1. A surface $S$ of constant slope may be generated by a straight line $L$ sliding along a plane curve $C_0$ (say, in the $xy$ plane), maintaining a right angle with the tangent to $C_0$ and a constant angle $\theta$ with its binormal (i.e., with the $z$ axis). When a closed curve $C_0$ is chosen, the surface is an obvious generalization of a circular cone (see Fig. 1). Since "near-conical" shells occur often in practice, it may be of interest to discuss such effects as fall within the scope of the membrane theory of shells.

We introduce the following notations:

- $\mathbf{i}, \mathbf{j}, \mathbf{k}$, unit vectors in fixed rectangular directions $x, y, z$;
- $\lambda, \mu, \nu$, unit tangent, normal, and binormal of curve $C_0$;
- $t$, length along generators $L$;
- $s_t, \rho_t$, arc length and radius of curvature of a horizontal section $C_t$ of the surface $S$; subscripts 0 and 1 will designate corresponding quantities in the end sections $C_0$ and $C_1$ of the shell;
- $\vec{r} = \vec{r}_0$, vector equation of curve $C_0$;
- $\varphi$, angle between the positive $x$ axis and the outward normal of $C_0$;
- $E, \nu, G$, Young's modulus, Poisson's ratio, and shear modulus;
- $h$, thickness of shell having the surface $S$ for middle surface;
- $N_s, N_t$, normal forces per unit length of sections of the shell which are perpendicular to $s$- and $t$-directions respectively (Fig. 3);
- $N_{st}$, shearing force in $s$-direction per unit length of shell section perpendicular to $t$-direction;

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1 The author wishes to express his appreciation to Professor W. Prager for proposing the problem and for other valuable suggestions.

2 Non-circular cones (for which the generators meet in one point while their "slope" varies) have been considered recently by A. Pfüger, Z. angew. Math. Mech. 22, 99–116 (1942).

3 The fuselages of some aeroplanes, for instance, can be approximated by one or several shells of different slopes connected by stiff bulkheads. The construction of models is relatively simple because each portion forms a developable surface.
\( e_s, e_t, e_{st}, \) strains corresponding to \( N_s, N_t, \) and \( N_{st}, \) respectively.

We note some simple relationships:

\[
\frac{d\bar{r}}{ds_0} = \lambda; \quad \nu = \bar{k};
\]

\[
\lambda = -i \sin \varphi + j \cos \varphi; \quad \bar{\mu} = -i \cos \varphi - j \sin \varphi.
\]

Since \( \rho_0 = ds_0/d\varphi, \) we obtain from (1.2) the Frenet-Serret formulae for a plane curve:

\[
d\lambda/ds_0 = \bar{\mu}/\rho_0; \quad d\bar{\mu}/ds_0 = -\lambda/\rho_0.
\]

The vector equation of the surface of constant slope \( S \) has the form:

\[
\vec{R}(s_0, t) = \vec{r}(s_0) + t(\bar{\mu} \sin \theta + \nu \cos \theta).
\]

For a constant value of \( t, \) (1.4) is the vector equation of the horizontal section \( C_t. \) Then, \( \partial \vec{R}/\partial s_t \) is the unit vector tangent to \( C_t. \) Since \( \partial \vec{R}/\partial s_t = \lambda(\rho_0 - t \sin \theta) ds_0/\rho_0 ds_t, \) \( C_t \) is parallel to \( C_0 \) at corresponding points (see Fig. 2), and

\[
ds_t/ds_0 = (\rho_0 - t \sin \theta)/\rho_0.
\]

Hence, for corresponding points, the centers of curvature of \( C_0 \) and \( C_t \) coincide, and

\[
\rho_t = \rho_0 - t \sin \theta.
\]

If the shell is long, it may happen that at some point \( \rho_t = 0. \) At such a point the tangent to \( C_t \) ceases to turn continuously (see points \( P, P' \) in Fig. 2). We shall discuss only the portion of the shell where \( t \sin \theta < \rho_0, \) i.e., the open shell without the "tail edge."

2. An element of a shell of thickness \( h \) having the surface \( S \) for middle surface is shown in Fig. 3. According to the usual assumptions of the membrane theory of shells, the bending stresses as well as effects of curvature of \( S \) are disregarded and

one has \( N_{st} = N_{ts} \). The total forces acting on the faces \( hds_t \) and \( hdt \) of the element are respectively:

\[
\begin{align*}
- \{N_x(\mu \sin \theta + \nu \cos \theta) + N_x \lambda \} (\rho_0 - t \sin \theta) d\varphi, \quad (2.1a) \\
- \{N_x \lambda + N_x(\mu \sin \theta + \nu \cos \theta)\} dt. \quad (2.1b)
\end{align*}
\]

Let \( P = P_x + P_x (\mu \sin \theta + \nu \cos \theta) + P_x (\mu \cos \theta - \nu \sin \theta) \) represent the load per unit area of the surface. Then the condition of equilibrium of the element of the shell is:

\[
\frac{\partial}{\partial t} \left\{ [N_x(\mu \sin \theta + \nu \cos \theta) + N_x \lambda] (\rho_0 - t \sin \theta) \right\} dtd\varphi
\]

\[
+ \frac{\partial}{\partial \varphi} \left\{ N_x \lambda + N_x(\mu \sin \theta + \nu \cos \theta) \right\} dtd\varphi = (\rho_0 - t \sin \theta) \overline{P} dtd\varphi. \quad (2.2)
\]

Equating the components of these forces in the \( n, s, \) and \( t \) directions, we obtain three equations for the determination of the three stress components:

\[
N_s = (\rho_0 - t \sin \theta) P_n \sec \theta,
\]

\[
\frac{\partial}{\partial t} \left\{ N_{st}(\rho_0 - t \sin \theta) \right\} - N_{st} \sin \theta = (\rho_0 - t \sin \theta) P_s - \frac{\partial N_{st}}{\partial \varphi}, \quad (2.3)
\]

We proceed to solve equations (2.3) with the simplifying assumption that the load \( \overline{P} \) does not vary along the generators \( L \), and obtain:

\[
N_s = (\rho_0 - t \sin \theta) P_n \sec \theta,
\]

\[
N_{st} = \frac{f(\varphi) \sin \theta}{(\rho_0 - t \sin \theta)^2} - \frac{1}{3} \sec \theta(\rho_0 - t \sin \theta)(P_s - P'_s \sec \theta) + \frac{1}{3} \rho_0 P_n \csc \theta \sec \theta,
\]

\[
N_t = \frac{-1}{\rho_0 - t \sin \theta} \left[ \frac{f(\varphi)}{\rho_0 - t \sin \theta} - g(\varphi) \right] \quad (2.4)
\]

\[
+ \frac{t \csc \theta}{\rho_0 - t \sin \theta} \left[ \frac{1}{3} \rho_0 P_s - \frac{1}{3} \rho_0 P'_s \sec \theta - \frac{1}{3} \rho_0 P_n \csc \theta \sec \theta \right]
\]

\[
- \frac{1}{3} \csc \theta(\rho_0 - t \sin \theta) [P_t - P_n \tan \theta - \frac{1}{3} P_n \csc \theta \sec \theta + \frac{1}{3} P'_s \csc \theta],
\]

where \( f(\varphi) \) and \( g(\varphi) \) are arbitrary functions of \( \varphi \) and the prime denotes differentiation with respect to \( \varphi \). If the curve \( C_0 \) is closed the continuity of stresses demands that \( f \) and \( g' \) have a period of \( 2\pi \).

When the load on the shell is applied only through the end sections \( C_0 \) and \( C_1 \) the stress system becomes:

\[
N_s = 0; \quad N_{st} = \frac{f \sin \theta}{(\rho_0 - t \sin \theta)^2}; \quad N_t = \frac{-1}{(\rho_0 - t \sin \theta) \left[ \frac{f}{\rho_0 - t \sin \theta} - g \right]}.
\]

Substituting (2.5) into (2.1a) and integrating between 0 and \( 2\pi \), we obtain the resultant force \( \Phi_t \) acting on the section \( C_t \); the expression for \( \Phi_t \) simplifies readily by virtue of (1.2):

\( ^* \) In the case of cylindrical surfaces, \( \theta = 0 \) and integration of (2.3) leads to the special solution:

\[
N_s = \rho P_n; \quad N_{st} = f(s) + t(P_s - dN_s/ds); \quad N_t = g(s) - tdf/ds + tP_t + \rho/2(d^2 N_s/ds^2 - dP_s/ds); \quad \text{where} \quad f(s) \quad \text{and} \quad g(s)
\]

are arbitrary functions of the arc length \( s \). In this connection see pp. 66-76 of Flügge’s book.
\[
F_t = \int_0^{2\pi} \left\{ -\left[ \frac{f}{\rho_0 - t \sin \theta} (\bar{\mu} \sin \theta + \bar{v} \cos \theta) \right]' + g'(\bar{\mu} \sin \theta + \bar{v} \cos \theta) \right\} d\varphi
\]

\[
= -\sin \theta \left\{ i \int_0^{2\pi} g' \cos \varphi d\varphi + j \int_0^{2\pi} g' \sin \varphi d\varphi \right\} + k \cos \theta \{ g(2\pi) - g(0) \}. \tag{2.6}
\]

The resultant moment \( M_t \) about the origin due to the forces on the section \( C_t \) is found similarly:

\[
M_t = \int_0^{2\pi} \bar{R} \times \left\{ -\left[ \frac{f}{\rho_0 - t \sin \theta} (\bar{\mu} \sin \theta + \bar{v} \cos \theta) \right]' + g'(\bar{\mu} \sin \theta + \bar{v} \cos \theta) \right\} d\varphi.
\]

It follows by integration by parts that

\[
M_t = \bar{R}(0) \times \bar{F}_t + k \sin \theta \int_0^{2\pi} f d\varphi + \cos \theta \int_0^{2\pi} (i \cos \varphi + j \sin \varphi) f d\varphi
\]

\[
- \int_0^{2\pi} \bar{R} \times \left\{ i \int_0^{2\pi} g' (\bar{\mu} \sin \theta + \bar{v} \cos \theta) d\varphi \right\} (\rho_0 - t \sin \theta) d\varphi. \tag{2.7}
\]

The results (2.6) and (2.7) will form the basis of analysis in later sections.

3. Let the vector of infinitesimal displacement be

\[
\bar{D} = u\bar{\lambda} + v(\bar{\mu} \sin \theta + \bar{v} \cos \theta) + w(\bar{\mu} \cos \theta - \bar{v} \sin \theta). \tag{3.1}
\]

The strains in the surface are given by the following scalar products between the rates of change of the displacement \( \bar{D} \) and the unit vectors in the \( t \) and \( s \) directions:

\[
e_{tt} = \frac{\partial \bar{R}}{\partial t} \cdot \frac{\partial \bar{D}}{\partial t}, \quad e_{ss} = \frac{\partial \bar{R}}{\partial s} \cdot \frac{\partial \bar{D}}{\partial s}, \quad e_{st} = \left\{ \frac{\partial \bar{R}}{\partial s} \cdot \frac{\partial \bar{D}}{\partial t} + \frac{\partial \bar{R}}{\partial t} \cdot \frac{\partial \bar{D}}{\partial s} \right\}. \tag{3.2}
\]

We evaluate (3.2) and substitute the results into Hooke's Law:

\[
\frac{1}{Eh} \left\{ N_t - vN_s \right\} = \frac{\partial u}{\partial t},
\]

\[
\frac{1}{Eh} \left\{ N_s - vN_t \right\} = \frac{1}{\rho_0 - t \sin \theta} \left\{ u' - (v \sin \theta + w \cos \theta) \right\}, \tag{3.3}
\]

\[
2(1 + v) \frac{1}{Eh} N_{st} = \frac{1}{\rho_0 - t \sin \theta} \left\{ (\rho_0 - t \sin \theta) \frac{\partial u}{\partial t} + u \sin \theta + v' \right\}.
\]

Equations (3.3) are easily integrated to yield expressions for the displacements:

\[
v = \frac{1}{Eh} \int^t (N_t - vN_s) dt + A(\varphi),
\]

\[
u = \frac{2(1 + v)}{Eh} (\rho_0 - t \sin \theta) \int^t \frac{N_{st}}{\rho_0 - t \sin \theta} dt
\]

\[
- \frac{(\rho_0 - t \sin \theta)}{Eh} \int^t \frac{1}{(\rho_0 - t \sin \theta)^2} \int^t (N_t' - vN_s') dt dt - A'(\varphi) \csc \theta + (\rho_0 - t \sin \theta)B(\varphi), \tag{3.4}
\]

\[
w = u' \sec \theta + v \tan \theta - \frac{1}{Eh} (\rho_0 - t \sin \theta)(N_s - vN_t) \sec \theta,
\]
where $A(\varphi)$ and $B(\varphi)$ are arbitrary functions. When the stresses have the form (2.5), the displacements can be expressed directly in terms of the functions $f$ and $g$:

$$v = \frac{\csc \theta}{Eh} \{ - f' \rho t^{-1} + \frac{1}{2} f' \rho t^{-2} - g' \ln \rho t \} + A,$$

$$u = \frac{\csc^2 \theta}{Eh} \{ (1 + v) f \rho t^{-1} \sin^2 \theta + \frac{1}{2} f'' \rho t^{-1} - \frac{1}{6} (f \rho'' + 3 f' \rho') \rho t^{-2} + \frac{1}{2} f' \rho^2 \rho t^{-3}$$

$$+ \frac{1}{2} g' \rho \rho t^{-1} + g' (\ln \rho t + 1) \} - A' \csc \theta + \rho t B,$$

$$w = \frac{\sec \theta \csc^2 \theta}{Eh} \{ \sin^2 \theta \left[ - \frac{1}{2} f' \rho t^{-2} + 2 f' \rho t^{-1} + g' (\ln \rho t + v) \right] + \frac{1}{2} f'' \rho t^{-1}$$

$$- \frac{1}{6} (f \rho'' + 4 f' \rho'' + 6 f'' \rho') \rho t^{-2} + \frac{1}{2} (15 f' \rho'' + 10 f' \rho') \rho t^{-3}$$

$$- \frac{1}{2} g' \rho t^{-4} + \frac{1}{2} (g' \rho'' + 3 g' \rho') \rho t^{-1} - \frac{1}{2} g' \rho'^2 \rho t^{-2} + g' (\ln \rho t + 1) \}$$

$$- \tan \theta (A + A'' \csc^2 \theta) + B' \rho t \sec \theta + B \rho' \sec \theta.$$ 

Expressions for displacements $D_x, D_y, D_z$ in the $x, y, z$ (or any other) directions are best derived by taking a scalar product between a unit vector in the given direction and $\mathcal{D}$ of (3.1). For instance,

$$D_z = k \cdot \mathcal{D} = v \cos \theta - w \sin \theta. \quad (3.6)$$

4. The current literature on shells contains very little on the boundary conditions in the membrane theory of shells. We recall that local bending of the shell was disregarded according to the simplifying assumptions of the theory. Thus we cannot expect to satisfy all of the usual boundary conditions. For instance, we cannot ask that the heavy end bulkhead be considered rigid; in bending of the shell as a whole this would entail $\epsilon = N_t = 0$ in the end section which could consequently transmit no bending moment. By allowing deformations in the plane of the end sections we remove the restriction on $N_t$ and the problem of bending has a solution (see section 5). One has to decide in every particular problem which boundary conditions correspond more nearly to the assumption of no local bending.

A casual reader might be tempted to interpret the contribution of $A$ and $B$ to the displacements in (3.4) as that of rigid body motion since it is present when the stresses vanish. However, it is conceivable that a given state of stress induces inextensional displacements other than those of a rigid body as necessitated by the shape of the shell. Thus, in the case of a non-circular cylindrical shell under torsion, $A$ accounts for the warping of the cross-sections.6

In general, these inextensional deformations are accompanied by local bending stresses which must be small to be neglected in accordance with our assumptions. One would expect that no energy is expended in the inextensional deformations. The strain energy in shells loaded through the end-sections is

$$V = \frac{1}{2} k \int_0^{\ell} \int_0^{2\pi} \left( \frac{N_t^2}{E} + \frac{N_{tt}^2}{G} \right) \rho d\varphi dt, \quad (4.1)$$

6 Specifically, $A = (T/2A_0 G h) \int_0^\theta (\rho \rho \rho - (1/2A_0) \int_0^\theta (\rho \rho \rho \cos \varphi + \rho \rho \rho \sin \varphi) d\varphi \right \}. This expression is found by the method indicated in section 8.
or

\[ V = \frac{1}{2} \int_0^{2\pi} (N_{st} u + N_{st} v) \rho \, d\varphi \mid \text{at } t = t_1. \]  (4.2)

Substituting (2.5) and the contribution due to \( A \) and \( B \) into (4.2) and integrating by parts, we find that our expectation is verified. The accompanying local bending, however, absorbs energy and, therefore, places limitations on the inextensional displacements according to the principle of minimum strain energy. The minimum expenditure of energy in bending occurs when the inextensional displacements reduce to rigid body displacements.

One can easily verify that the most general functions \( A \) and \( B \) corresponding to rigid body displacements have the form

\[
A = -a_x \sin \theta \cos \varphi - a_y \sin \theta \sin \varphi + a_z \cos \theta + \alpha_z y_0 \cos \theta \\
- \alpha_y x_0 \cos \theta + \alpha_z \sin \theta (y_0 \cos \varphi - x_0 \sin \varphi),
\]

\[
B = \alpha_x \cot \theta \cos \varphi + \alpha_y \cot \theta \sin \varphi + \alpha_z,
\]

where \( a_x, a_y, a_z \) represent the infinitesimal translations in the \( x, y, z \) directions; \( \alpha_x, \alpha_y, \alpha_z \) the infinitesimal rotations about the \( x, y, z \) axes; and \( x_0, y_0 \) the coordinates in the base section \( C_0 \).

Instead of imposing conditions on the displacements, one may prescribe a sensible distribution of stresses at the boundary. We note that by (2.5) the state of stress in the whole shell is determined as soon as the stresses \( N_t \) and \( N_{st} \) are given at one end-section. Thus two different stress distributions which are statically equivalent over an end-section will determine distinctly different stress distributions in the rest of the shell.\(^7\)

5. We shall study first the effects of taper\(^8\) as exhibited in a conical shell of circular cross-section; later, we shall discuss the influence of a variable radius of curvature \( \rho_t \) of the section \( C_t \).

Let \( M \) represent the bending moment (causing tension for \( x_t > 0 \)) applied to the shell through the end-sections \( C_0 \) and \( C_1 \). We shall try to satisfy the conditions that the end-sections (bulkheads) remain plane, i.e.,

\[
D_z = 0 \text{ for } t = 0; \quad D_z = \beta (x_0 - t_1 \sin \theta) \text{ for } t = t_1; \]  (5.1)

where \( \beta \) is the (undetermined) angle of bending, and that the displacements due to \( A \) and \( B \) reduce to rigid body displacements (4.3). By virtue of (3.6), (3.5), and (4.3), we obtain for the first of conditions (5.1)

\[
- \frac{\sec \theta \csc \theta}{E h} \left\{ \frac{1}{2} (2f'(1 + \sin^2 \theta) + f'') \\
+ (g' + g'') \ln r + \nu g' \sin^2 \theta + g'' \right\} - \alpha_y r \cos \varphi = 0. \]  (5.2)

\(^7\) This is the price that has to be paid for the simplifications due to the assumptions of the membrane theory. A "disturbance" of the state of stress on one end-section (the difference between the equivalent stress distributions) "propagates" itself along the generators without "dying out." The general theory of thin shells would lead to differential equations of higher order; for these one can find solutions representing disturbances that die out with the distance from the end-section.

\(^8\) All the results of sections 5-9 simplify to the corresponding expressions for a cylinder as the taper approaches zero.
By symmetry, the functions $g'$ and $f'$ are odd in $x_0$; let their Fourier expansions read

$$g' = \sum_{n=0}^{\infty} (2n + 1)a_{2n+1} \cos (2n + 1)\varphi, \quad f' = \sum_{n=0}^{\infty} (2n + 1)b_{2n+1} \cos (2n + 1)\varphi. \quad (5.3)$$

Since the resultant force $F_0$ on $C_0$ must vanish, one concludes from (2.6) that $a_1 = 0$. Equation (2.7) yields $M_t = (1/\pi) \alpha_1 \cos \theta$ or $b_1 = -(1/\pi) M \sec \theta$. It follows from the coefficient of $\cos \varphi$ in (5.2) that

$$\alpha_\varphi = (1/2\pi Eh r^2) \sec^2 \theta \csc (1 + 2 \sin^2 \theta) M. \quad (5.4)$$

Substituting (5.4) and (5.3) into the second of conditions (5.1) and equating coefficients of $\cos \varphi$ in the two members, we obtain

$$\beta = \frac{M \sin \theta (2 + \csc^2 \theta)}{2\pi Eh \cos^2 \theta} \left\{ \frac{1}{(r - t \sin \theta)^2} - \frac{1}{r^2} \right\}. \quad (5.5)$$

For the coefficients $a_{2n+1}$ and $b_{2n+1}$, $n > 0$, one obtains a system of two homogeneous equations with a non-vanishing determinant. Therefore $a_{2n+1} = b_{2n+1} = 0$, $n > 0$, and

$$N_\dot{t} = \frac{M \sec \theta \cos \varphi}{\pi(r - t \sin \theta)^2}, \quad N_{\dot{at}} = -\frac{M \tan \theta \sin \varphi}{\pi(r - t \sin \theta)^2}; \quad (5.6)$$

$$D_\dot{t} = \frac{M \sin \theta (2 + \csc^2 \theta)}{2\pi Eh \cos^2 \theta} \left\{ \frac{1}{(r - t \sin \theta)^2} - \frac{1}{r^2} \right\} x_\dot{t}. \quad (5.7)$$

If we designate by $I_1$ the moment of inertia, $\pi(r - t \sin \theta)^2$, of $C_1$ about the neutral axis, we can write $N_\dot{t} = (1/I_1) M x_\dot{t} \sec \theta$. Essentially the stresses in the $z$ direction follow the classical beam formula; the influence of taper is manifested by the presence of the $x$ components of stresses $N_t$ which have to be balanced by $N_{\dot{at}}$. From (5.7) we see that all sections $C_t$ remain plane. The rate of change of the angle of bending increases as the shell grows narrower:

$$\frac{d\beta}{dz} = \frac{M(1 + 2 \sin^2 \theta)}{\pi Eh \cos^2 \theta(r - t \sin \theta)^3} = \frac{M(1 + 2 \sin^2 \theta)}{EhI_1 \cos^2 \theta}. \quad (5.8)$$

Further effects of taper are apparent in the other displacements:

$$v = \frac{M \tan \theta}{2\pi Eh} \cos \varphi \left\{ \frac{2 \csc^2 \theta}{r - t \sin \theta} - \frac{1}{r} (2 \csc^2 \theta - \nu) \right\}, \quad (5.9)$$

$$u = \frac{M \sec \theta}{2\pi Eh} \sin \varphi \left\{ \frac{\csc^2 \theta - 2 - 2\nu}{r - t \sin \theta} - \frac{1}{r} (2 \csc^2 \theta - \nu) \right\} + \frac{1}{r^2} (r - t \sin \theta)(\csc^2 \theta + 2), \quad (5.10)$$

$$w = \frac{M \sec^2 \theta}{2\pi Eh} \cos \varphi \left\{ \frac{\csc^2 \theta - 4}{r - t \sin \theta} - \frac{1}{r} \cos^2 \theta(2 \csc^2 \theta - \nu) \right\} + \frac{1}{r^2} (r - t \sin \theta)(\csc^2 \theta + 2), \quad (5.11)$$

$$D_{\dot{z}} = \frac{M \sec \theta}{2\pi Eh} \left\{ \frac{2 - \csc^2 \theta + 2\nu \sin^2 \varphi}{r - t \sin \theta} + \frac{1}{r} (2 \csc^2 \theta - \nu) \right\} - \frac{1}{r^2} (r - t \sin \theta)(\csc^2 \theta + 2). \quad (5.12)$$
If we take for $X$, the fictitious displacement of the axis of the cone, the average of $D_x$ over $C_i$ (by analogy with a cylinder or prism), we obtain for the slope of the deformed axis

$$\frac{dX}{dz} = \frac{M \sin \theta}{2\pi Eh \cos^2 \theta} \left\{ \frac{2 + \nu - \csc^2 \theta}{(r - t \sin \theta)^2} + \frac{1}{r^2} \right\}. \quad (5.13)$$

Comparison with equation (5.7) shows that the axis is not perpendicular to the sections $C_i$ as one might expect. Nor is the increment in slope equal to $\beta$, the angle between the end sections. In fact, for $\csc^2 \theta = 2 + \nu$, a large taper, the axis remains altogether straight despite the angle between $C_0$ and $C_1$. This is due to a slipping effect caused by an interplay of the shearing forces $N_{st}$ and the $x$ components of $N_i$. Finally, let us check (5.5) by the customary application of Castigliano's Principle, $\partial V/\partial M = \beta$. Substituting (5.6), (5.9), and (5.10) into (4.2), we have

$$V = \frac{M^2 \sin \theta}{4\pi Eh \cos^2 \theta} \frac{2\nu}{r^2}, \quad (5.14)$$

$$V = \frac{M^2 \sin \theta(2 + \csc^2 \theta + 2\nu)}{4\pi Eh \cos^2 \theta} \left\{ \frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2} \right\}, \quad (5.15)$$

$$\beta = \frac{M \sin \theta(2 + \csc^2 \theta + 2\nu)}{2\pi Eh \cos^2 \theta} \left\{ \frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2} \right\}. \quad (5.16)$$

The discrepancy between (5.5) and (5.16) is negligible in practical applications, but is interesting theoretically. It springs from a loose interpretation of Castigliano's Principle above, which is strictly true only for a concentrated couple $M$. Since $M$ is distributed over the end sections, it does work not only in bending the shell but also in deforming the end-sections within their planes, as seen from (5.14). When the end-sections are alike as in a cylinder or prism, as much energy is spent in the deformation of one end as is gained at the other end; then, Castigliano's Principle holds even for a distributed moment. But to obtain the correct angle of bending in the case of a cone, one must deduct from the total strain energy (5.15) the net energy absorbed in the plane deformation of $C_0$ and $C_1$, namely

$$\frac{M^2 \nu \sin \theta}{2\pi Eh \cos^2 \theta} \left\{ \frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2} \right\}.$$

6. We derive easily the expressions for the stresses in a cone twisted by a torque $T$ by making either $D_x = 0$ or $N_i = 0$ at $C_0$ and $C_1$ and using (2.6) and (2.7),

$$N_{st} = \frac{T}{2\pi(r - t \sin \theta)^2}, \quad N_t = 0. \quad (6.1)$$

From the displacements or the strain energy we obtain the total angle of twist $\gamma$ and the angle of twist per unit length of the cone.

Here, the effects of taper as manifested in (6.1) and (6.2) are not unexpected.

More interesting is the case of a cone supported at $C_0$ and bent by a force $R$ (in the $x$ direction) distributed over $C_t$. We learn from (2.6) that the function $f(\phi)$ and hence the shear stress $N_{st}$ do not actually contribute to the resultant $R$ acting on any section $C_t$. Expressions (2.6) and (2.7) show that the term in $\cos \phi$ of $g'$ alone influences the resultant force as well as moment on $C_t$. We superpose a state of stress given by (5.6) with $M = R \cot \theta (r - t_1 \sin \theta)$ in order to bring the moment across $C_t$ to zero, and obtain the final result

$$N_t = -\frac{R(t_1 - t) \cos \phi}{\pi(r - t \sin \theta)^2}, \quad N_{st} = -\frac{R(r - t_1 \sin \theta) \sin \phi}{\pi(r - t \sin \theta)^2}.$$  

7. Let us now consider shells with non-circular cross-sections $C_t$. The coordinates of points on $C_t$ are expressed in terms of $\rho_t$ and $\phi$

$$x_t = x_t(0) - \int_0^\phi \rho_t \sin \phi \, d\phi, \quad y_t = \int_0^\phi \rho_t \cos \phi \, d\phi.$$  

It is clear from (7.1) that $\rho_t$ cannot contain any terms in $\cos \phi$ or $\sin \phi$ if the shell is closed. If only cosine terms appear in the Fourier expansion

$$\rho_t = r_t - \sum_{n=2}^\infty r_n \cos n\phi,$$  

the section $C_t$ is symmetric with respect to the $x$ axis. The simple section, for which $r_n = 0$ if $n \neq 3$, approximates the cross-section of many a fuselage:

$$x_t = r_t \cos \phi + \frac{1}{2} r_3 \cos 2\phi - \frac{1}{3} r_3 \cos 4\phi; \quad x_t(0) = r_t + \frac{r_3}{8}, \quad x_t(\pi) = -r_t + \frac{r_3}{8};$$

$$y_t = r_t \sin \phi - \frac{1}{2} r_3 \sin 2\phi - \frac{1}{3} r_3 \sin 4\phi; \quad y_t(\pi/2) = r_t.$$  

The neutral axis of the sections coincides with the $y$ axis (i.e., is independent of $t$) if $x_t$ contains no constant term and if

$$\int_0^{2\pi} x_0 \rho_0 d\phi = \frac{\pi}{2} \sum_{n=2}^\infty \frac{r_n (r_{n+1} - r_{n-1})}{n} = 0.$$  

In bending, only sections satisfying (7.4) will be considered.

8. The stresses in a shell of constant slope under torsion are determined from the conditions that the load is applied in such a manner that only shearing stresses are generated at the end-sections. The conditions $N_t = 0$ at $t = 0$ and $t = t_1$ yield $f = k \rho_0 (\rho_0 - t_1 \sin \theta)$ and $g = k (\rho_0 - t_1 \sin \theta)$. Substituting the expression for $f$ into $N_{st}$, we find the torque $T$ on $C_0$

$$T = \int_0^{2\pi} \vec{R} \times N_{st} \rho_0 d\phi = k \sin \theta \left\{ \oint \vec{R} \times \frac{d\vec{R}}{ds_0} ds_0 - t_1 \sin \theta \int_0^{2\pi} \vec{R} \times \lambda d\phi \right\},$$
which reduces to

\[ T = k \sin \theta \{ 2A_0 - t_1L_0 \sin \theta \} = k \sin \theta \{ A_0 + A_1 - \pi t_1^2 \sin^2 \theta \}. \] (8.1)

Here the \( A \)'s represent the areas of the sections and \( L_0 \) is the length of \( C_0 \), all quantities easily measurable. Then,

\[ N_{st} = \frac{T}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{\rho \rho_1}{\rho^2}, \] (8.2)

\[ N_{t} = -\frac{T}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{t}{\rho_1} \left( \frac{\rho_1}{\rho} \right) = -\frac{T}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{t(t_1 - t)\rho'}{\rho^3}. \] (8.3)

The effect of the variable radius of curvature of \( C_t \) is observed in the expression for \( N_{st} \); tensile stresses increase directly with \( \rho' \) and inversely with \( \rho^3 \).

The expression (4.2) for strain energy takes the form

\[ V = \frac{k^2 \csc \theta}{2Eh} \left\{ t_1(1 + \nu) \sin^2 \theta (L_0 + L_1) + \frac{t_1^3 \sin^2 \theta}{12} \int_0^{2\pi} \left( \frac{\rho'}{\rho_1} \right)^2 \sin \frac{\nu \theta}{2} \right\} \left( \frac{\rho_1}{\rho_0} \right)^n 
- \frac{t_1 \sin \theta}{2} \int_0^{2\pi} \rho'^2 \left( \frac{1}{\rho_1} + \frac{1}{\rho_0} \right) d\varphi - \int_0^{2\pi} \rho'^2 \ln \left( \frac{\rho_1}{\rho_0} \right) d\varphi \right\}, \] (8.4)

and the angle of twist is

\[ \gamma = \frac{T}{Eh(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)^2} \left[ t_1(1 + \nu)(L_0 + L_1) + \csc^2 \theta \int_0^{2\pi} \left\{ \frac{t_1^3 \sin^2 \theta}{12} \rho'^2 \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_0^2} \right) 
- \frac{t_1 \sin \theta}{2} \rho'^2 \left( \frac{1}{\rho_1} + \frac{1}{\rho_0} \right) - \rho'^2 \ln \left( \frac{\rho_1}{\rho_0} \right) \right\} d\varphi \right]. \] (8.5)

The quantity in the braces is of the order of \( \sin^5 \theta \). Also, each of its terms contains the factor \( \rho'^2 \). In the common case of small taper and nearly circular shell we may use as a good approximation

\[ \gamma_{app} = \frac{Tt_1(L_0 + L_1)}{2Gh(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)^2}. \] (8.6)

Neglecting the terms in the braces of (8.5) is equivalent to disregarding the effect of the stress \( N_{st} \); see (4.1).

The inextensional displacements given by \( A \) and \( B \) in (3.5) can be determined from the twist of the end-sections (centers of twist along \( z \) axis)

\[ u = 0, \quad t = 0; \quad u = \gamma(x_1 \cos \varphi + y_1 \sin \varphi), \quad t = t_1. \] (8.7)

These displacements include warping. The actual process of solving (8.7) is quite tedious even when a definite section is given.

9. We conclude with a short discussion of stresses in a general shell of constant slope bent by couples \( M \) as in section 5. We assume that the moments at the end-sec-

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tions $C_0$ and $C_1$ are applied in such a manner that the stress $N_t$ at these sections is proportional to the distance from the neutral axis:

$$N_t = e_0 x_0 \quad \text{for} \quad t = 0; \quad N_t = e_1 x_1 \quad \text{for} \quad t = t_1.$$  \hfill (9.1)

Conditions (9.1) and the fact that the moments across $C_0$ and $C_1$ are alike lead us to the following expressions:

$$f = \frac{M}{t_1 \sin \theta \cos \theta} \left( \frac{Q_0}{I_0} - \frac{Q_1}{I_1} \right); \quad g = \frac{M}{t_1 \sin \theta \cos \theta} \left( \frac{\rho_0 Q_0}{I_0} - \frac{\rho_1 Q_1}{I_1} \right),$$ \hfill (9.2)

where the $I$'s are the moments of inertia about the neutral axis of the full respective sections and $Q_i = \int_0^\varphi x_i \rho_i d\varphi$ the variable first moment (about the same axis) of the section included between 0 and $\varphi$. The expressions for the stresses themselves read:

$$N_{st} = \frac{M}{t_1 \rho_i^2 \cos \theta} \left( \frac{Q_0}{I_0} - \frac{Q_1}{I_1} \right),$$ \hfill (9.3)

$$N_t = -\frac{M}{t_1 \cos \theta} \frac{(t_1 - t) \sin \theta \rho'}{\rho_i^2} \left( \frac{Q_0}{I_0} - \frac{Q_1}{I_1} \right)$$

$$+ \frac{M}{t_1 I_0 \cos \theta} \frac{x_0 \rho_0^2}{\rho_i^2} + \frac{M t}{t_1 \cos \theta} \frac{x_1 \rho_1^2}{I_1 \rho_i^2}.$$ \hfill (9.4)

The corresponding expressions for strain energy and displacements are very cumbersome and can hardly be useful in practical applications.