A NEW METHOD OF INTEGRATION BY MEANS OF ORTHOGONALITY FOCI*

BY

A. A. POPOFF

Automechanical Institute, Moscow, U.S.S.R.

1. Introduction. This paper contains a new method of integration which is partly graphical, partly analytical.[1] It permits a simple determination of integrals of the form $\int \phi_i(x)\phi_k(x)dx$, where $\phi_i(x)$ is given graphically and $\phi_k(x)$ is given either graphically or analytically. The method requires the construction of certain diagrams, called scales, showing the abscissae of the centroids of certain areas associated with $\phi_k(x)$, and is based on some properties of the so-called orthogonality foci. Finally, the method is applied to interpolation, Fourier analysis, and the evaluation of Mohr integrals in the theory of structures.

2. Definite integrals. Let us consider the integral

$$T = \int_0^l \phi_i(x)\phi_k(x)dx. \qquad (2.1)$$

If rectangular cartesian coordinates x, y are introduced, the functions $\phi_i(x)$, $\phi_k(x)$ can be represented by curves, such as in Fig. 1. We now consider a distribution of mass along the curve $y = \phi_i(x)$, $0 \le x \le l$, the mass per unit length in the x-direction being $\phi_k(x)$. The centroid of this mass distribution we shall call the *orthogonality focus*.[2] We shall denote it by F_{ik}, and its coordinates by ρ_k, f_{ik} (neither ρ_k nor the total mass Ω_k of the system depend on $\phi_i(x)$). We have

$$\Omega_k = \int_0^l \phi_k(x)dx. \qquad (2.2)$$

Ω_k is also the area under the curve $y = \phi_k(x)$. Since T represents the mass moment of the mass distribution about the x-axis,
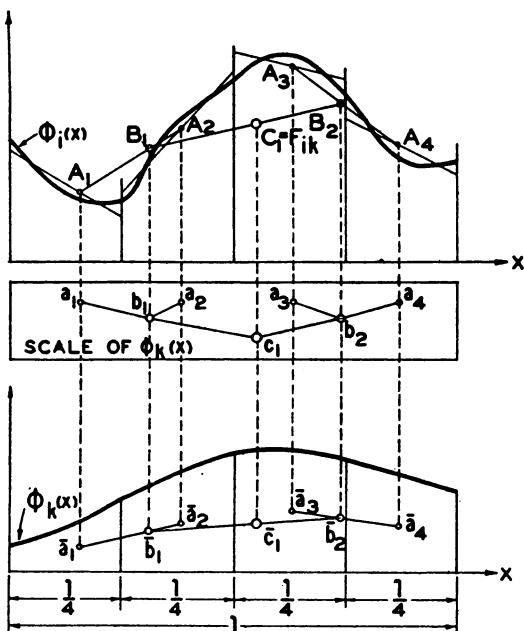

FIG. 1.

---

[1] This method was announced in the author's note entitled *A new method of graphical integration*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, (1943).

[2] This name is justified by the properties discussed in Section 4.

$$\rho_k = \frac{1}{\Omega_k} \int_0^l x\phi_k(x)dx, \qquad (2.3) \qquad\qquad f_{ik} = T/\Omega_k. \qquad (2.4)$$

ρ_k is also the abscissa of the centroid of the area under the curve $y = \phi_k(x)$.

The following lemmas can be verified easily:

(a) *If $\phi_i(x)$ is a linear function, its graph is a straight line and F_{ik} lies on this line; F_{ik} can thus be found immediately if ρ_k is known.*

(b) *If the interval $(0, l)$ is divided into two parts, the orthogonal foci of the two parts and of the whole are collinear.*

These two lemmas permit a graphical determination to any desired degree of accuracy of the point F_{ik} and hence of the integral T. The procedure is as follows:

(a) The interval $(0, l)$ is divided into 2^m equal intervals[3] $(0, l_1), (l_1, l_2), \cdots, (l_{2^m-1}, l)$.

(b) Operation (a) divides the region under the curve $y = \phi_k(x)$ into 2^m regions. We find the centroids $\bar{a}_r$ $(r = 1, 2, \cdots, 2^m)$ of these 2^m regions,[4] then combine adjacent pairs of regions and find the centroids $\bar{b}_r$ $(r = 1, 2, \cdots, 2^{m-1})$ of the 2^{m-1} regions so formed, then combine adjacent pairs of these 2^{m-1} regions and find the centroids $\bar{c}_r$ $(r = 1, 2, \cdots, 2^{m-2})$ of the 2^{m-2} regions so formed, and so on. In the final stage, we find the centroid of the entire region under the curve $y = \phi_k(x)$. In the lower part of Fig. 1, we see these centroids in a case when $m = 2$.

(c) A diagram, called the *scale* of $\phi_k(x)$, is constructed. The middle part of Fig. 1 shows such a scale. It consists of points $a_r(r = 1, 2, \cdots, 2^m)$ vertically above $\bar{a}_r$ and all at the same level, points $b_r(r = 1, 2, \cdots, 2^{m-1})$ vertically above $\bar{b}_r$ and all at the same arbitrary level slightly below the points a_r, and so on.

(d) Operation (a) divides the curve $y = \phi_i(x)$ into 2^m parts. We replace each part by a segment of a straight line, as shown in the upper part of Fig. 1, and assume that the mass is distributed along these segments rather than along the curve.

(e) We determine the points $A_r(r = 1, 2, \cdots, 2^m)$ of intersection of these line segments with the verticals through the points a_r. We then determine the points $B_r(r = 1, 2, \cdots, 2^{m-1})$ of intersection of the straight lines joining adjacent pairs of points A_r with verticals through the points b_r. This process is repeated, until finally we arrive at the final point F_{ik}.

(f) f_{ik}, which is the ordinate of F_{ik}, is determined by measurement; the value T of the required integral then follows from (2.4).
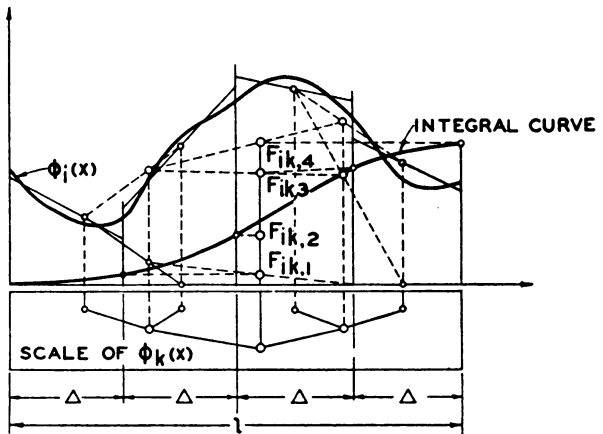

FIG. 2.

---

[3] Unequal intervals could also be used.

[4] It is to be noted that areas corresponding to negative values of $\phi_k(x)$ must be considered as corresponding to negative mass.

3. Indefinite integrals. The graphical construction of Section 2 can be applied to the indefinite integral $\int \phi_i(x)\phi_k(x)dx$, in the following manner (see Fig. 2). An interval $(0, l)$ on the x-axis is taken, and the construction of Section 2 is applied to the integral $\int_0^l \phi_{i,1}(x)\phi_k(x)dx$, where $\phi_{i,1}(x) = \phi_i(x)$ in $(0, l_1)$ and vanishes elsewhere. This yields a point $F_{ik,1}$ with ordinate $f_{ik,1}$. The point $(l_1, f_{ik,1})$ is then plotted. The construction of Section 2 is then applied to the integral $\int_0^l \phi_{i,2}(x)\phi_k(x)dx$, where $\phi_{i,2}(x) = \phi_i(x)$ in $(0, l_2)$ and vanishes elsewhere. This yields a point $F_{ik,2}$, and the point $(l_2, f_{ik,2})$ is plotted. In this way we obtain the sequence of points $(l_r, f_{ik,r})$, $(r = 1, 2, \cdots, 2^m)$. Since

$$f_{ik,r} = \frac{1}{\Omega_k} \int_0^l \phi_{i,r}(x)\phi_k(x)dx, \qquad (3.1)$$

the curve passing through these points is approximately the integral curve, except for the constant factor $1/\Omega_k$.

Figure 2 shows this construction for the same functions $\phi_i(x)$, $\phi_k(x)$ considered in Fig. 1, m again having the value 2.

4. Some properties of orthogonality foci. (a) If the x-axis passes through the point F_{ik}, then $f_{ik} = 0$, and by (2.4)

$$\int_0^l \phi_i(x)\phi_k(x)dx = 0, \qquad (4.1)$$

i.e., $\phi_i(x)$ and $\phi_k(x)$ are orthogonal. It is for this reason that F_{ik} is called the *orthogonality* focus.

(b) Let us set $\phi_i(x) \equiv \phi_k(x)$. Then, from (2.4),

$$f_{kk} = \frac{1}{\Omega_k} \int_0^l [\phi_k(x)]^2 dx = 2f_k \qquad (4.2)$$

where f_k is the ordinate of the centroid of the region under the curve $y = \phi_k(x)$, $(0 \leq x \leq l)$. We shall now prove the following theorem. *The curve $y = h\phi_k(x)$, where h is a constant, has the least mean square deviation from the curve $y = \phi_i(x)$ when*

$$h = f_{ik}/f_{kk}. \qquad (4.3)$$

To prove this, we note that the mean square deviation is a minimum when

$$\frac{d}{dh} \int_0^l [\phi_i(x) - h\phi_k(x)]^2 dx = 0,$$

i.e., when

$$\int_0^l [\phi_i(x) - h\phi_k(x)]\phi_k(x)dx = 0,$$

or

$$h = \int_0^l \phi_i(x)\phi_k(x)dx \bigg/ \int_0^l [\phi_k(x)]^2 dx = f_{ik}/f_{kk}.$$

(c) Let us draw the horizontal line β through the point F_{ik} (Fig. 3), and then rotate β about F_{ik} through an angle α to a new position β'. A new curve $y=\phi_i'(x)$ is constructed such that the vertical distance from β' to points on this curve is equal to the vertical distance from β to points on the curve $y=\phi_i(x)$. We shall now prove that

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx. \tag{4.4}$$

We have $\phi_i'(x)=\phi_i(x)+(\rho_k-x)\tan\alpha$, where, it is recalled, ρ_k is the abscissa of F_{ik}. Thus

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx + \tan\alpha \int_0^l (\rho_k - x)\phi_k(x)dx.$$

The last integral vanishes, by the definition of ρ_k, and the desired result is obtained.

It is to be noted that the two curves have a common point I, about which the curve is "rotated."

(d) Let us consider the case when $\phi_i(x)$ is linear in each of the intervals $(0, l_1)$ (l_1, l), so that its graph is a broken line abcd (Fig. 4). We shall now show that, *if ab is rotated about a point M on ab to a new position a'b', then F_{ik} is unchanged if cd is rotated about a certain point N on cd in such a way that bc=b'c'.* The points M, N are called conjugate foci. To prove this theorem, we use Fig. 4, in which A_1, A_2, A_1', A_2' are points leading to the determination of F_{ik}, following the procedure laid down in operation (e) of Section 2. From the three pairs

FIG. 3.

of similar triangles Mbb' and MA_1A_1', Ncc' and NA_2A_2', $F_{ik}A_1A_1'$ and $F_{ik}A_2A_2'$, we have
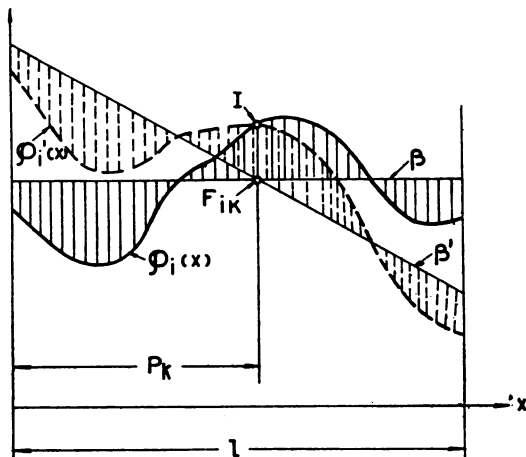
$$\frac{bb'}{A_1A_1'} = \frac{\beta_1}{\beta_1 - \gamma_1}, \qquad \frac{cc'}{A_2A_2'} = \frac{\beta_2}{\gamma_2 - \beta_2}, \qquad \frac{A_1A_1'}{A_2A_2'} = \frac{\alpha_1}{\alpha_2}.$$

Since $bb' = cc'$, a value for A_1A_1'/A_2A_2' can be determined from the first two equations. Substitution of this value in the third equation yields

$$\alpha_1\gamma_2(1/\beta_2) + \alpha_2\gamma_1(1/\beta_1) = \alpha_1 + \alpha_2. \tag{4.5}$$

Thus β_2 is uniquely determined by β_1; hence N is uniquely determined by M.

It is easily seen that, if rotations of the above type are carried out about conjugate foci, and if $\phi_i'(x)$ denotes the function the graph of which is a'b'c'd', then

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx. \tag{4.6}$$
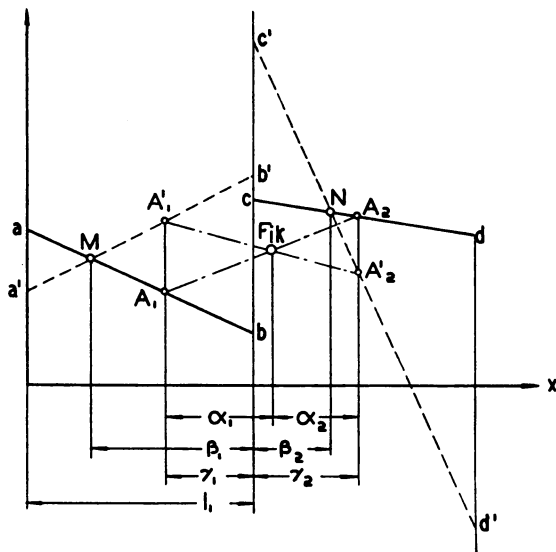
FIG. 4.

(e) Let us denote by F_{ik} and F_{il} the orthogonal foci of the function $\phi_i(x)$ represented by the broken line *abcd* with respect to two functions $\phi_k(x)$ and $\phi_l(x)$. We shall now show that, *there is a unique pair of conjugate foci M (on ab) and N (on cd) such that rotations of ab about M and cd about N (with bc = b'c') leave both F_{ik} and F_{il} unchanged.* This follows from the fact that F_{ik} is unchanged if β_1 and β_2 (Fig. 4) satisfy (4.5), and F_{il} is unchanged if they satisfy a second relation of the same form as (4.5). Since both these relations are linear in $1/\beta_1$, $1/\beta_2$, they can be solved for unique values of β_1 and β_2.

It is easily seen that, if rotations of the above type are carried out about such conjugate foci, and if $\phi_i'(x)$ denotes the function the graph of which is displaced position of *abcd*, then (4.6) holds and also

$$\int_0^l \phi_i'(x)\phi_l(x)dx = \int_0^l \phi_i(x)\phi_l(x)dx.$$

Conjugate foci can be used widely in graphical computations dealing with statically indeterminate structures.

5. Application of orthogonality foci to the interpolation of curves. Let us consider the application of orthogonality foci to the following problem. We are given two functions or curves $\phi_i(x)$ and $\phi_k(x)$. It is required to find a straight line $y = A + Bx$ such that the integral

$$U = \int_0^l [\phi_i - (A + Bx)]^2 \phi_k dx, \tag{5.1}$$

will have the least possible value. We shall now show that *U has the least possible value when the straight line $y = A + Bx$ passes through the orthogonality foci F_{ik}, F_{il}, where $\phi_l = x\phi_k$.*

We set $\partial U/\partial A = \partial U/\partial B = 0$, to obtain the equations

$$\left. \begin{aligned} A\int_0^l \phi_k dx + B\int_0^l x\phi_k dx &= \int_0^l \phi_i\phi_k dx, \\ A\int_0^l x\phi_k dx + B\int_0^l x^2\phi_k dx &= \int_0^l x\phi_i\phi_k dx. \end{aligned} \right\} \tag{5.2}$$

Let us consider the functions $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2$. We have $\Omega_0 = l$, $\Omega_1 = \frac{1}{2}l^2$, $\Omega_2 = \frac{1}{3}l^3$. If the orthogonality foci of ϕ_k with ϕ_0, ϕ_1, ϕ_2 are denoted by $F_{k0}(\rho_0, f_{k0})$, $F_{k1}(\rho_1, f_{k1})$, $F_{k2}(\rho_2, f_{k2})$, respectively, and the orthogonality foci of ϕ_i with ϕ_k and ϕ_l are denoted by $F_{ik}(\rho_k, f_{ik})$, $F_{il}(\rho_l, F_{il})$, respectively, then

$$\int_0^l \phi_k dx = f_{k0}l, \qquad \int_0^l \phi_k x dx = \tfrac{1}{2}f_{k1}l^2, \qquad \int_0^l \phi_k x^2 dx = \tfrac{1}{3}f_{k2}l^3,$$

$$\int_0^l \phi_i \phi_k dx = f_{ik} \int_0^l \phi_k dx = f_{ik}f_{k0}l,$$

$$\int_0^l \phi_i \phi_l dx = f_{il} \int_0^l \phi_k x dx = \tfrac{1}{2}f_{il}f_{k1}l^2. \qquad (5.3)$$

Thus (5.2) can be written in the form

$$A + B \frac{f_{k1}l}{2f_{k0}} = f_{ik}, \qquad A + B \frac{2f_{k2}l}{3f_{k1}} = f_{il}. \qquad (5.4)$$

Since ρ_k, ρ_l are abscissas of orthogonality foci, by (2.3) we have

$$\rho_k = \int_0^l x\phi_k dx \bigg/ \int_0^l \phi_k dx = \frac{f_{k1}l}{f_{k0}},$$

$$\rho_l = \int_0^l x^2\phi_k dx \bigg/ \int_0^l x\phi_k dx = \frac{2f_{k2}l}{3f_{k1}}. \qquad (5.5)$$

Thus (5.5) take the form

$$A + B\rho_k = f_{ik}, \qquad A + B\rho_l = f_{il},$$

whence it follows that A and B must be such that the straight line $y = A + Bx$ passes through the orthogonality foci F_{ik}, F_{il}.

In order to construct the straight line $y = A + Bx$ which is such that U has the least possible value, we can proceed as follows:

(a) Scales are constructed for $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = x^2$. These are as shown in Fig. 5 when the interval $(0, l)$ is divided into 8 equal parts ($m = 3$).

(b) A scale is constructed for ϕ_k. If the function ϕ_k is given, this can be done analytically. In any event, it can be done graphically using the scales in Fig. 5, since it involves integrals of the forms $\int \phi_k dx$, $\int \phi_k x dx$.
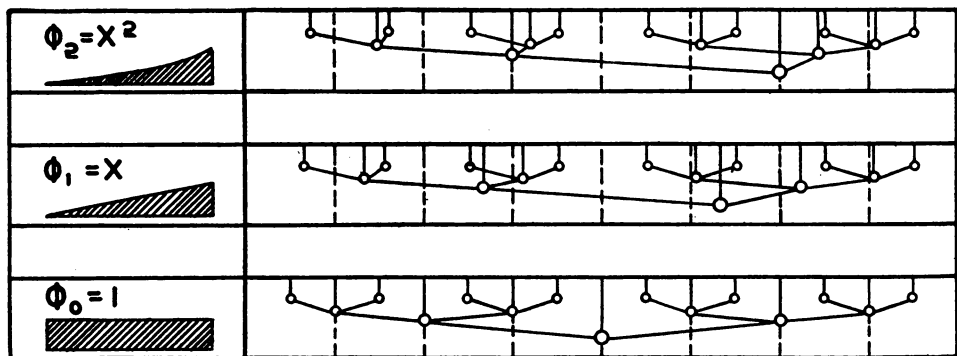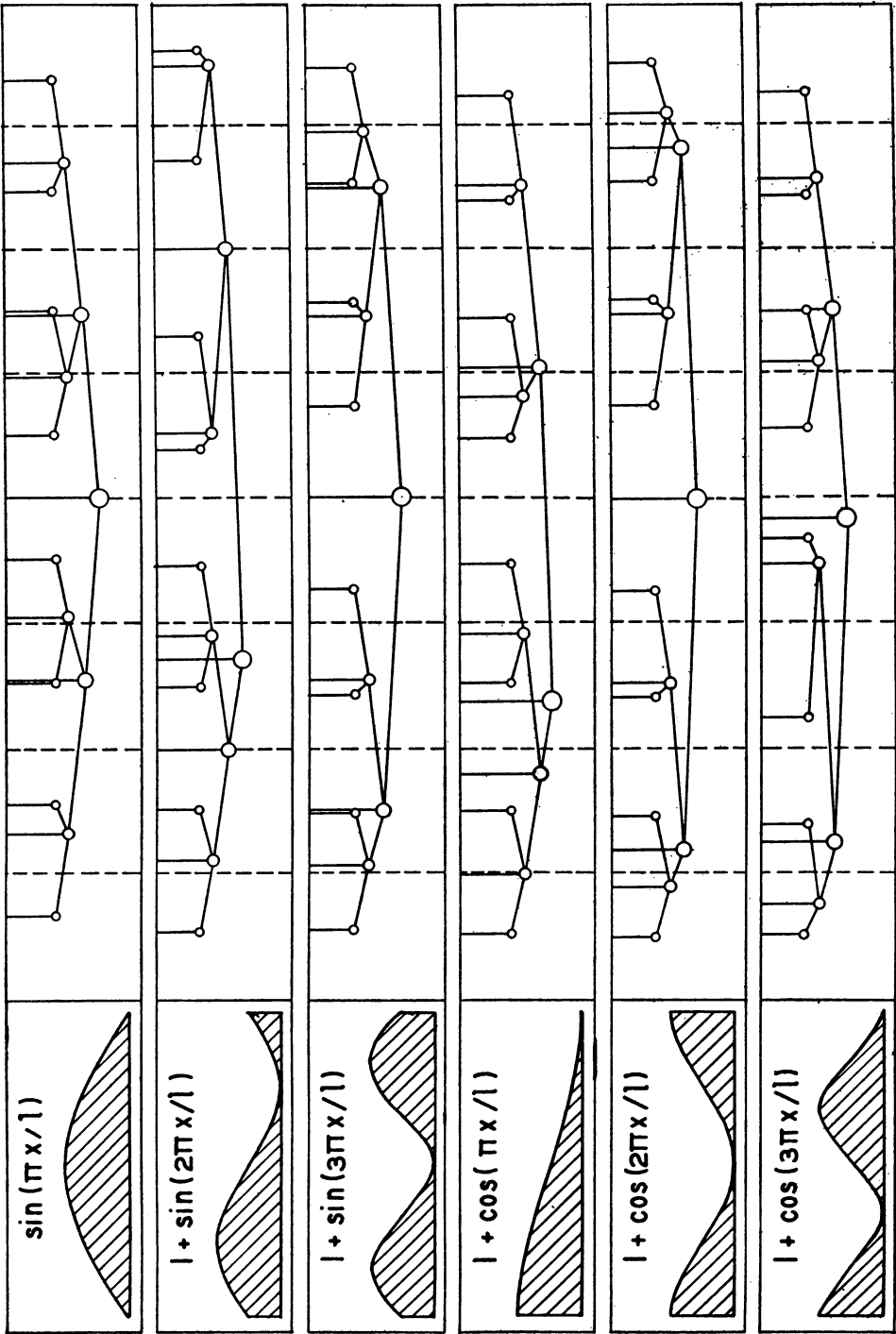

FIG. 5.

A. A. POPOFF

FIG. 6.

(c) A scale is constructed for $\phi_l = x\phi_k$. This also can be done graphically by means of the scales in Fig. 5.

(d) The foci F_{ik}, F_{il} are found following the procedure outlined in Section 2. The straight line through F_{ik}, F_{il} is the required line.

6. Graphical harmonic analysis. Orthogonality foci can be used to obtain the Fourier series expansion of a function which is given either analytically or graphically. If $\phi(x)$ denotes the function, its expansion into a Fourier series of sines or cosines will involve the integrals

$$
\left.
\begin{aligned}
a_n &= \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} \, dx \qquad (n = 1, 2, \cdots), \\
b_n &= \frac{2}{l} \int_0^l \phi(x) \cos \frac{n\pi x}{l} \, dx \qquad (n = 0, 1, \cdots),
\end{aligned}
\right\} \tag{6.1}
$$

where a_n, b_n are Fourier coefficients.

Difficulties are encountered if the method of orthogonal foci is applied directly to the integrals in (6.1). These difficulties are avoided if we write (6.1) in the form

$$
\left.
\begin{aligned}
a_1 &= \frac{2}{l} \int_0^l \phi(x)\phi_{s1}(x)dx, \\
a_n &= \frac{2}{l} \int_0^l \phi(x)\phi_{sn}(x)dx - \frac{2}{l} \int_0^l \phi(x)dx \qquad (n = 2, 3, \cdots), \\
b_n &= \frac{2}{l} \int_0^l \phi(x)\phi_{cn}(x)dx - \frac{2}{l} \int_0^l \phi(x)dx \qquad (n = 0, 1, \cdots),
\end{aligned}
\right\} \tag{6.2}
$$

where

$$
\left.
\begin{aligned}
\phi_{s1}(x) &= \sin \frac{\pi x}{l}, \qquad \phi_{sn}(x) = 1 + \sin \frac{n\pi x}{l} \qquad (n = 2, 3, \cdots), \\
\phi_{cn}(x) &= \cos \frac{n\pi x}{l} \qquad (n = 0, 1, \cdots).
\end{aligned}
\right\} \tag{6.3}
$$

By the use of the scale of $\phi_0(z) = 1$ (Fig. 5) and the scales of the functions in (6.3) (Fig. 6), the ordinates of the orthogonality foci of $\phi(x)$ with these functions can be found graphically by the procedure of Section 2. If we denote these ordinates by f_0, f_{sn}, f_{cn}, respectively, then (6.2) takes the form

$$
\left.
\begin{aligned}
a_1 &= \frac{2}{l} f_{s1}\Omega_{s1}, \qquad a_n = \frac{2}{l}(f_{sn}\Omega_{sn} - f_0\Omega_0) \qquad (n = 2, 3, \cdots), \\
b_n &= \frac{2}{l}(f_{cn}\Omega_{cn} - f_0\Omega_0) \qquad (n = 0, 1, \cdots),
\end{aligned}
\right\} \tag{6.4}
$$

where Ω_0, Ω_{sn}, Ω_{cn} are respectively the areas under the curve $\phi_0(x) = 1$ and the curves in (6.3) for the interval $(0, l)$. Now

$$
\Omega_{s1} = \frac{2l}{\pi}, \qquad \Omega_{sn} = l\left(1 + \frac{1 - \cos n\pi}{n\pi}\right) \qquad (n = 2, 3, \cdots),
$$

$$
\Omega_0 = l, \qquad \Omega_{cn} = l,
$$

whence (6.4) becomes

$$a_1 = \frac{4}{\pi} f_{s1}, \qquad a_n = 2\left[f_{sn}\left(1 + \frac{1 - \cos n\pi}{n\pi}\right) - f_0\right] \qquad (n = 2, 3, \cdots),$$

$$b_n = 2(f_{cn} - f_0) \qquad (n = 0, 1, \cdots).$$

7. Graphical evaluation of Mohr integrals. In the theory of structures, the determination of deflections in bending often requires the evaluation of so-called Mohr
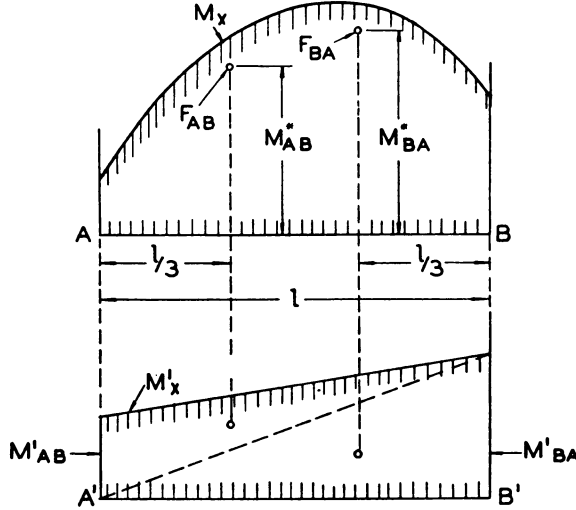

FIG. 7.

integrals, which have the form $T = \int_0^l M_x M_x' \, dx$, where M_x is a function of x. given graphically and M_x' is a linear function of x (Fig. 7).

From Fig. 7, we see that

$$M_x' = M_{AB}'\frac{l - x}{l} + M_{BA}'\frac{x}{l}. \tag{7.1}$$

Thus

$$T \doteq \int_0^l M_x \frac{M_{AB}'}{l}(l - x)dx + \int_0^l M_x \frac{M_{BA}'}{l}x\,dx. \tag{7.2}$$

By use of the scale of x given in Fig. 5, for both of these integrals the orthogonality foci F_{AB} and F_{BA} can be determined graphically. If M_{AB}^* and M_{BA}^* denote the ordinates of these foci, then

$$T = M_{AB}^*(\tfrac{1}{2}M_{AB}'l) + M_{BA}^*(\tfrac{1}{2}M_{BA}'l),$$

or

$$T = \tfrac{1}{2}l(M_{AB}^*M_{AB}' + M_{BA}^*M_{BA}'). \tag{7.3}$$

M_{AB}^* and M_{BA}^* are called the focal moments.