It is evident that by a proper superposition of the preceding results, Tranter's expressions for a finite band of pressure may be obtained immediately. A similar variation on Rankin's solution[3] for the case of a finite band of external pressure acting on a solid cylinder will yield expressions for the stresses corresponding to a semi-infinite band of external radial pressure acting on a solid cylinder.

ON A CONFORMAL MAPPING TECHNIQUE*

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1. Introduction. Many physical problems are readily reduced to the problem of finding the value of a harmonic function on a smooth closed curve corresponding to given boundary conditions and certain interior singularities. For example, it has been shown [1] that the problem of finding the flow of an incompressible non-viscous fluid past a periodic array of airfoils[1] may be replaced by that of determining the flow within a smooth closed stream-line C with a source-vortex and a sink-vortex at specified interior points.[2] Analogous problems can obviously occur in other physical situations which lead to the Laplace equation.

Here we shall develop a method of mapping a smooth closed curve C conformally onto the unit circle so as to carry two arbitrarily specified interior points into the points $\pm a$. The mapping is continuous within and on C.

Since the solution requires the integration of a single non-homogeneous Fredholm integral equation, we also present a numerical procedure which is useful in solving such equations. The over-all procedure seems preferable, in general, to Theodorsen's method of mapping "nearly circular regions" [2], [3], since the equation used here has a non-singular kernel in contrast to those in his two simultaneous integral equations.

2. The integral equation. Let us consider the problem of finding that complex potential $F(\zeta)=\phi+i\psi$ on the closed curve C[3] (see Fig. 1) which corresponds to two equal and opposite interior logarithmic singularities at the points P and Q and let ψ vanish on C. We may write then, for points on and within C,

$$F(\zeta) = \ln\left[(\zeta - P)/(\zeta - Q)\right] + f(\zeta), \tag{1}$$

where f is analytic within C and continuous on C. In principle, no difficulty, would be encountered if the unit coefficients of these singularities were replaced by complex numbers, but the present form is sufficiently general for our purpose.

Consider the integral

[3] A. W. Rankin, *Shrink-fit stresses and deformations*, J. Appl. Mech. **11**, A-77 (1944).

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[1] The airfoil shape being arbitrary.

[2] C, in this case, is a non-self-intersecting convex curve with a continuously turning tangent. By "smooth" we shall henceforth imply such a curve.

[3] C is any closed, not self-intersecting, curve.

$$I(\zeta_0) = \oint_S \frac{F(\zeta)}{\zeta - \zeta_0} d\zeta \equiv 0, \tag{2}$$

where S is that closed contour (Fig. 1) which contains no singularities. As a conse-

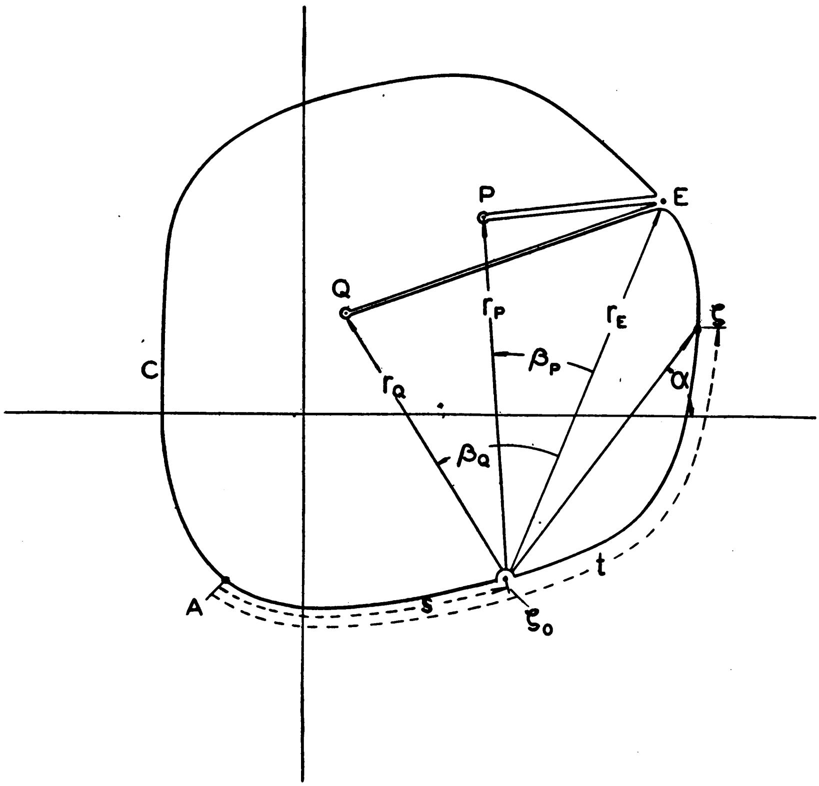

FIG. 1. A is an arbitrarily chosen origin of the coordinates s, t; the point E may also be taken arbitrarily.

quence of the fact that the logarithmic terms differ by $2\pi i$ on opposing sides of the cuts, the integral of Eq. (2) may be written in the form

$$-\pi i F(\zeta_0) + \int_{\zeta_0^+}^{\zeta_0^-} F(\zeta) d\{\ln(\zeta - \zeta_0)\} + 2\pi i \ln \frac{P - \zeta_0}{Q - \zeta_0} = 0, \tag{3}$$

provided the limiting process is performed wherein the radii of the circles about P, Q, and ζ_0, and the width of the cuts, are made to tend to zero. Since $\psi = 0$ on C, the imaginary parts of the foregoing equation lead to the integral equation

$$\phi(s) = \frac{1}{\pi} \int_C \phi(t) d\alpha + 2 \ln r_p/r_q. \tag{4}$$

The symbols not used previously are defined in Fig. 1. The angle α is obviously a function of s and t, and its increment in Eq. (4) corresponds to an increment dt. This equation has been obtained previously by a somewhat less straightforward method [1] wherein the singularities were taken with complex coefficients.[4] In this case, Eq. (4) has terms $-\Gamma_p\beta_p/\pi$ and $-\Gamma_q\beta_q/\pi$.

A method of solving this integral equation will be presented in Sec. (4). Note that for a circle, the kernel is constant ($\partial\alpha/\partial\theta=\frac{1}{2}$, where $\zeta=e^{i\theta}$), and the solution is already obtained. A closely analogous situation was previously obtained by Prager [4].

3. The mapping problem. We now turn to the problem of mapping C on the unit circle in the z plane so that P, Q go into the points $\pm a$. The mapping is to be regular within and on C. The magnitude of the real quantity a cannot be arbitrarily chosen but will follow from the analysis. We first show that such a mapping exists. Two facts are well known: (1) there exists a mapping such that the closed curve C goes conformally into the unit circle in the z_1 plane and such that P, Q go into the (unspecified) points P', Q'; (2) there exists a bilinear transformation such that the unit circle goes into the unit circle and any two interior points P', Q', go into $\pm a$ (a being determined by the values of P', Q'). Thus the mapping exists.

Now consider that the integral equation of the foregoing section has been solved for $\phi(s)$. Denote its minimum value by $\phi_{\min}$.

Note that for the points $z=e^{i\theta}$,

$$F_2(z) = \phi_2(\theta) + iO = \ln \frac{-(z-a)\left(z-\dfrac{1}{a}\right)}{(z+a)\left(z+\dfrac{1}{a}\right)} + K(a) \tag{5}$$

is the solution of the potential problem wherein $\psi=0$ on the unit circle and unit singularities are placed at $\pm a$. The real number $K(a)$ may be chosen so that the smallest value taken by $\phi_2(\theta)$ is $\phi_{\min}$.[5]

It is evident that when the curve C is mapped into the z plane in the specified manner, the potentials ϕ, ϕ_2, in the two planes must have the same value at corresponding points, i.e. at points $z(\zeta_j)$ and ζ_j. Thus, a must be chosen such that the greatest value of ϕ is equal to the greatest value of ϕ_2. Once this has been done, the mapping function $z(\zeta)$ along C is implied by the relation

$$\phi_2(\theta) = \phi(s). \tag{6}$$

One may now quickly find $z(\zeta)$ along C, and, if it is required, find $z(\zeta)$ at any interior point by using the Cauchy formula

$$2\pi i z(\zeta_i) = \oint_C \frac{z(\zeta)}{\zeta-\zeta_i}\, d\zeta. \tag{7}$$

The physical problem may now be treated by the conventional methods used when conformal mapping is applied. In particular, for the fluid mechanics problem mentioned previously, complex singularities may be placed at $\pm a$, the imaginary coefficient of the singularity corresponding to the point infinitely far downstream in

[4] In [1], the vertex terms have the incorrect sign.

[5] Actually $K(a)$, a, are determined simultaneously by this condition and that appearing in the next paragraph.

the physical plane being determined by the Joukowski condition.[6] The velocity distribution along the vane profile is determined using only the analytic[7] transformations which led to C and the function $d\theta/ds$.

If it is only necessary to place one singular point P within the unit circle, it may be mapped into the origin. One uses doublets at P and at the origin. The strength of the doublet must be adjusted as was a in the preceding problem.

4. The solution of the integral equation. An elementary numerical procedure which resembles the relaxation process conventionally applied to differential equations [5] is useful in solving Eq. (4). We choose several points s_k (or t_k) roughly uniformly spaced along C, compute $\alpha(s_k, t_n) = \alpha_{kn}$ for each k and n, and determine the quantities $\Delta_{kn} = (\alpha_{k,n+1} - \alpha_{k,n-1})/2\pi$. We now write Eq. (4) in the form

$$\phi(s_k) \simeq \sum_n \phi(t_n)\Delta_{kn} + 2 \ln \frac{r_p(s_k)}{r_q(s_k)} \cdot \tag{8}$$

and guess the values of $\phi(t_n)$. Using any convenient arrangement, we compute $\phi(s_k)$ and compare the computed and guessed results. We then make a better guess for the quantities $\phi(t_n)$ [according to the form of the Δ_{kn} and to any experience gained in previous attempts, computing the change in $\phi(s_k)$] in such a manner that the two sets of ϕ values become essentially alike. The accuracy which can be obtained is a function of the fineness of the spacing of points in the kernel.

The usual iteration procedures which are applied to Fredholm equations, or to the systems of algebraic equations to which they may be reduced, differ from the foregoing in that no freedom of choice is allowed beyond the first approximation. In this method, the higher approximations may lead to much more rapid convergence than such iterations just as in the relaxation method in the numerical solution of differential equations.

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[6] $\partial\phi/\partial\theta$ must vanish at the point on $|z| = 1$ into which the trailing edge of the vane was mapped.

[7] By "analytic" we imply functions in closed form as contrasted with those obtained by numerical procedures.