If we denote by x_2 the distance from the nose,

$$p_2 = 9.56 \frac{\alpha\rho c^2}{2} (x_1/x_2)^{1/4}. \tag{13}$$

The drag for a length l is

$$D = 2\alpha \int_0^l p_2 \, dx_2 = 25.5\alpha^2(x_1/l)^{1/4} \frac{\rho c^2}{2} l. \tag{14}$$

The drag coefficient is

$$c_D = 25.5\alpha^2(x_1/l)^{1/4} = 25.5\alpha^2(c^2/2\gamma l)^{1/4}. \tag{15}$$

The drag depends on the ratio of the length of the body to the distance traveled from rest to reach the speed of sound. The presence of an acceleration causes a finite drag at the speed of sound in contrast to the infinite value in the steady case. As the value of the acceleration decreases the drag tends to infinity as the inverse fourth root of the acceleration. Another difference with the steady case is the concentration of infinite pressure at the nose. In this connection it may be concluded that the lift distribution on an accelerated wing will introduce a stalling moment in going through the speed of sound. It may be seen from formula (15) that extremely high values of the acceleration are needed for usual body sizes before the effect becomes appreciable.

It must be added that the methods presented in this paper are not restricted to the acceleration of a wedge. By superposition of positive and negative wedges the method solves the problem for a symmetric body of arbitrary shapes with constant acceleration. Furthermore, it will be noted that expressions (6), (7) and (8) may easily be generalized to cover not only the symmetric body of arbitrary shape but also the case of completely arbitrary motion. The present paper indicates how the pressure distribution in such cases may be completely expressed by quadratures.

A GENERALIZATION OF THE WIENER-HOPF TECHNIQUE*

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1. Introduction. Many of the problems of mathematical physics require the solution of an integral equation of the type

$$u(x) = \varphi(x) + \int_a^\infty K(x, x_0) f(x_0) \, dx_0 ,$$

where $u(x) = 0$ when $x > a$, and $f(x) = 0$ when $x < a$. When $K(x, x_0) = K(x - x_0)$ the equation is a Wiener-Hopf integral equation and the technique by which $f(x)$ may be found is well-known (cf. [1]). However, in many of the problems which arise K is not a function of $(x - x_0)$ and thus it seems desirable to generalize the Wiener-Hopf technique to include a more general family of kernels. In this paper, we shall concern ourselves with kernels which arise as the Green's functions of a certain family of partial differential equations. Although we shall choose as a basic problem a certain boundary

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value problem associated with such differential equations, it will be evident that the technique will be of use in questions which do not arise in this manner.

Specifically we shall consider the solutions of

$$\varphi_{yy} + y^m(\varphi_{xx} + k^2\varphi) = 0$$

which are composed of an "incident simple wave"[1] and a diffracted wave such that $\varphi(y, 0)$ vanishes when $y > b > 0$. We shall then include general remarks concerning other interesting cases.

2. The Basic Problem. Let us consider the boundary value problem associated with the equation

$$\varphi_{yy} + y^m(\varphi_{xx} + k^2\varphi) = 0 \tag{1}$$

where m is an integer greater than zero. It is required here that $\varphi(y, x)$ consist of a given "incident wave" $\varphi_0(y, x)$ plus a "diffracted wave", which together obey the requirement $\varphi(y, 0) \equiv 0$ when $y > b > 0$, and that $\varphi(y, x)$ be differentiable throughout $y > 0$ except on the cut mentioned above. In the region $y < 0$, we shall be content to accept the continuation of any solution which satisfies the foregoing conditions.

A natural way in which to formulate this problem is to write

$$\varphi = \varphi_0 + \int_a^\infty K(x, 0, y, y_0)f(y_0)\, dy_0 \tag{2}$$

where $K(x, x_0, y, y_0)$ is a Green's function[2] associated with Eq. (1). We must, then, find the Green's function, and find the function $f(y)$ associated with a given φ_0. To do these things we shall need certain transform identities which are quite simple and which, in fact, can be constructed as linear combinations of identities in conventional Hankel transforms. The essential facts are as follows. Let the H-transform of $f(s)$ be defined to be

$$\bar{f}(z) = \int_0^\infty z^\nu s^{1-\nu} H_\nu^{(2)}(sz) f(s)\, ds \tag{3}$$

Then

$$f(p) = \frac{1}{4}\int_{-\infty}^\infty z^{1-\nu} p^\nu H_\nu^{(1)}(pz)\bar{f}(z)\, dz \tag{4}$$

where the integration path is to pass above the origin and where negative z has argument $+\pi$. In particular, we shall need to know that the function $f_1(p)$ associated with $\bar{f}_1(z) = z^\nu/(z - \gamma)$ is (via Eq. (4)) $f_1(p) = \gamma p^\nu H_\nu^{(1)}(p\gamma)$.

In a previous paper [2], conventional transform methods have been used to find the Green's function associated with an equation similar to Eq. (1). Without giving the details, it may be stated that the same techniques (using the transform pairs given above) will give the Green's function of Eq. (1) as

[1]This simple wave can be thought of as the analog to the plane wave associated with the equation $\Delta\varphi + k^2\varphi = 0$.

[2]Actually there are two linearly independent Green's functions which vanish as $y \to \pm\infty$, $x \to \infty$ and are outgoing, but which differ in their boundary conditions along $y = 0$. We shall obviously use the convenient one.

$K(x, x_0, y, y_0) =$

$$\frac{1}{4}\int_{-\infty}^{\infty} y^{1/2} y_0^{\alpha-1/2} \frac{\eta H_\nu^{(1)}(2\nu\eta y^\alpha) H_\nu^{(2)}(2\nu\eta y_0^\alpha) \exp\{-|x - x_0|(\eta^2 - k^2)^{1/2}\}}{(\eta^2 - k^2)^{1/2}} d\eta. \tag{5}$$

where $\nu = (2 + m)^{-1}$, $\alpha = 1 + m/2$. For the case $k = 0$, this reduces to a Legendre function of the second kind, but for $k \neq 0$, it is not easy to evaluate.

The choice of φ_0 will not be arbitrary, of course, in an explicit physical problem; but here, where the problem is merely to exhibit the technique, we may choose φ_0 with an eye to simplifying the manipulations. Accordingly, we choose a function whose H-transform is simple. Such a function [which is a solution of Eq. (1)] is

$$\psi_0(y, x) = \left(\frac{k \cos\beta}{4}\right) y^{1/2} H_\nu^{(1)}(2\nu y^\alpha k \cos\beta) \exp\{ikx \sin\beta\}. \tag{6}$$

This choice is not as artificial as it may seem at first glance. Actually, almost any φ_0 which one would wish to associate with Eq. (2) can be built up of a linear combination of those above by writing

$$\varphi_0 = \int \psi_0(y, x) h(\beta)\, d\beta. \tag{7}$$

In any event, we shall use ψ_0 in the present problem.

It is convenient to define $s = 2\nu y^\alpha$ so that

$$\psi_0 = \left(\frac{k \cos\beta}{4}\right) s^\nu H_\nu^{(1)}(ks \cos\beta) \exp\{ikx \sin\beta\}, \tag{8}$$

$$K = \frac{1}{4}\int_{-\infty}^{\infty} s^\nu s_0^{1-\nu} \frac{\eta H_\nu^{(1)}(\eta s) H_\nu^{(2)}(\eta s_0) \exp\{-|x - x_0|(\eta^2 - k^2)^{1/2}\}}{(\eta^2 - k^2)^{1/2}} d\eta. \tag{9}$$

Thus Eq. (2) can be written

$$\varphi = \psi_0 + \int_a^\infty K g(s_0)\, ds_0 . \tag{10}$$

If we now let $x = 0$, the integral equation has its final form and we are ready to apply the generalized Wiener-Hopf technique. We operate on Eq. (10) as in Eq. (3) to obtain (recalling that the boundary condition is applied at $x = 0$)

$$\bar{\varphi}(z)\Big]_{x=0} = \frac{z^\nu}{z - k \cos\beta} + \frac{\bar{g}(z)}{(z^2 - k^2)^{1/2}}. \tag{11}$$

It is now convenient to define $z = \zeta^{1/\nu}$ and to substitute for z in Eq. (11). When this is done and when we let[3] $\epsilon = \text{Im } k > 0$, we can observe the following facts:

First:

$$\bar{g}(\zeta^{1/\nu}) = \int_a^\infty \zeta s^{1-\nu} H_\nu^{(2)}(s\zeta^{1/\nu}) g(s)\, ds$$

is a function of ζ which is analytic throughout[4] the region R, which contains the sectors

[3]When the answer has been obtained, we can, of course, let $\epsilon \to 0$.

[4]This includes the implication that $\bar{g}$ is bounded as $|\zeta|$ tends to ∞ in R.

$(2n - 1)\pi \leq \arg \zeta^{1/\nu} \leq 2n\pi$. In fact, this statement is true for $\bar{g} \exp (ia\zeta^{1/\nu})$. This follows from the fact that $f(s) \sim e^{-\epsilon s}$ as $s \to \infty$, and that $H_\nu^{(2)}(\alpha) \sim e^{-i\alpha}$ as $|\alpha| \to \infty$.

Second:

$$\exp (ia\zeta^{1/\nu})\bar{\varphi}(z)\Big]_{x=0} = \exp (ia\zeta^{1/\nu}) \int_0^a \zeta s^{1-\nu} H_\nu^{(2)}(s\zeta^{1/\nu})\varphi(y, 0)\, ds$$

is analytic in the region R' which includes the sectors $2n\pi \leq \arg \zeta^{1/\nu} \leq (2n + 1)\pi$. Thus, the regions of analyticity of these functions actually overlap at all the boundaries of R and R' (the regions are shown for $\nu = 1/3$ in Fig. 1) and hence cover the entire plane including the point at infinity.

Finally, multiplication of Eq. (11) by $\exp (ia\zeta^{1/\nu})$ $(\zeta^{1/\nu} + k)^{1/2}$ yields (after a little algebra)

$$\bar{\varphi}(\zeta^{1/\nu} + k)^{1/2} \exp (ia\zeta^{1/\nu})$$

$$- \frac{\{(\zeta^{1/\nu} + k)^{1/2} \exp (ia\zeta^{1/\nu}) - [k(\cos \beta + 1)]^{1/2} \exp (iak \cos \beta)\}\zeta}{(\zeta^{1/\nu} - k \cos \beta)} \tag{12}$$

$$= \frac{\zeta[k(\cos \beta + 1)]^{1/2} \exp (iak \cos \beta)}{(\zeta^{1/\nu} - k \cos \beta)} + \frac{\bar{g}(\zeta^{1/\nu}) \exp (ia\zeta^{1/\nu})}{(\zeta^{1/\nu} - k)^{1/2}}.$$

The left side of this equation is regular in R' and the right side is regular in R. Thus, according to the arguments of analytic continuation, each of these functions is the con-

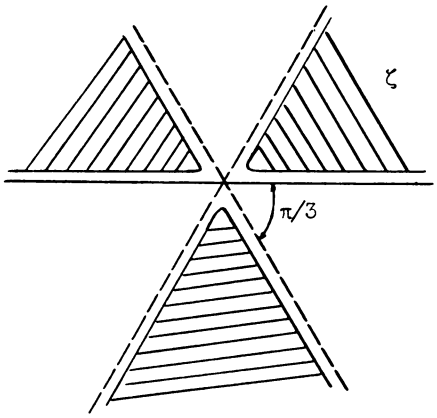

FIG. 1.

Qualitatively, the region R (that in which $\bar{g}(\zeta^3) \exp (ia\zeta^3)$ is analytic) is that excluding the shaded regions. R' is the region excluding these shaded regions after they have been rotated $\pi/3$ radians.

tinuation of the other and hence is analytic everywhere and therefore equal to a constant. Inspection of (12) together with footnote (4) immediately reveals that this constant is zero. Therefore,

$$\bar{g}(z) = \frac{[k(1 + \cos \beta)]^{1/2}(z - k)^{1/2} z^\nu \exp (-ia(z - k \cos \beta)}{(z - k \cos \beta)}, \tag{13}$$

and the desired result is given by Eq. (4). Thus, in principle, we have an answer. In a given physical problem, of course, one wishes to know the behavior of φ in certain regions. In many such cases conventional methods of interpreting the transforms will give asymptotic and local behaviors.

3. The General Problem. In this section some general but non-detailed remarks will be stated concerning the directions in which the foregoing may be readily generalized.

We have already implied that when φ_0 can be constructed as in Eq. (7), then the solution for φ can also be formulated as an integral over β of the solution we have just obtained. In the case where φ_0 has a simple transform (i.e. an easily factored transform), such an integration is obviously unnecessary.

A more general class of kernels which would be suitable in (2) can be constructed in the following way. Suppose there are a pair of functions $F_1(\xi, z)$, $F_2(\eta, z)$ such that

$$f(\xi) = \int_{-\infty}^{\infty} F_1(\xi, z) \int_a^b F_2(\eta, z) f(\eta)\, d\eta\, dz \tag{14}$$

for all f in $L^{(2)}$. Then, again in principle at least, the kernel

$$K = \int_{-\infty}^{\infty} F_1(s, z) F_2(s_0, z) h(z)\, dz \tag{15}$$

would be an appropriate one on which to use the foregoing technique, provided, of course, that the function F_2 has the sort of asymptotic behavior and the regularity which are needed if the foregoing analytic continuation arguments are to carry through. In Eq. (15), $h(z)$ is any function such that K exists and has a transform to which the arguments apply. From a practical point of view, h must produce a K whose transform can be factored in the manner used in the foregoing.

When $k = 0$, the present technique fails. However, if one solves the problem for $k \neq 0$ and then performs the limiting process $k \to 0$ (obvious in principle but messy in practice) solutions can be obtained.[5]

Finally, it seems evident that the homogeneous integral equation can also be treated with these more general transforms.

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