NOTE ON THE KINEMATICS OF PLANE VISCOUS MOTION*

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In 1911 G. Hamel (Göttinger Nachrichten, Math.-Phys. Kl. 1911, 261-270) obtained an interesting result, which may be stated as follows:

Let $R$ be a finite plane region with boundary $B$. Then the equation $\Delta \psi = F$ possesses a solution $\psi$ for which both $\psi$ and $\partial \psi / \partial n$ vanish on $B$ if and only if $F$ satisfies

$$\int F U \, dx \, dy = 0,$$

$U$ being an arbitrary harmonic function.

In other words, for the existence of a solution with this double boundary condition, it is necessary and sufficient that the function $F$ be orthogonal to the linear space of harmonic functions.

The hydrodynamical interpretation of Hamel's theorem is as follows. For an incompressible fluid moving in the plane with vorticity $\omega$, we have

$$u_x + v_y = 0, \quad v_x - u_y = 2\omega,$$

and there is a stream-function $\psi$ such that

$$u = -\psi_y, \quad v = \psi_x, \quad \Delta \psi = 2\omega.$$

Thus Hamel's theorem tells us that in order that a given distribution of vorticity may be consistent with vanishing velocity on the boundary $B$ (the usual boundary condition for a viscous fluid in a fixed container), it is necessary and sufficient that

$$\int \omega U \, dx \, dy = 0,$$

$U$ being an arbitrary harmonic function.

However, inspection of Hamel's proof (loc. cit. p. 266) shows that he made use of a Green's function of the second type, i.e. a harmonic function $G_2$ with a singularity $\log r$ and making $\partial G_2 / \partial n = 0$ on $B$. There is, of course, no such function for Laplace's equation, since this singularity and this boundary condition are inconsistent.

Not knowing of Hamel's work, I obtained Hamel's result in 1935 in a rather special case (Proc. London Math. Soc. 40 (1935), 23-36) in a different way.** In the present note the theorem is extended to include compressibility.

Theorem: A compressible viscous fluid moves inside a fixed connected boundary $B$, on which the velocity vanishes. An expansion $\theta(x,y)$ and a vorticity $\omega(x,y)$ are consistent with this boundary condition if, and only if,

$$\int (2\omega U - \theta V) \, dx \, dy = 0,$$

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**Footnote added in proof (Feb. 20, 1950): The result (4) has recently been proved by J. Kampé de Fériet (Math. Mag. 21, 71-79(1947); Ann. Soc. Sci. Bruxelles (I) 62, 11-18(1948).
where $U$ is an arbitrary harmonic function and $V$ the conjugate harmonic function, such that

$$U_x = V_y, \quad U_y = -V_x. \quad (6)$$

In purely mathematical language, equation (5) is a necessary and sufficient condition for the consistency of the equations

$$u_x + v_y = \theta, \quad v_x - u_y = 2\omega, \quad (u)_B = 0, \quad (v)_B = 0. \quad (7)$$

**Proof:** Let $l, m$ be the direction cosines of the outward normal to $B$. Let $\theta$ and $\omega$ be arbitrarily assigned. Let $u', v'$ satisfy

$$u'_x + v'_y = \theta, \quad v'_x - u'_y = 2\omega, \quad (lu' + mv')_B = 0. \quad (8)$$

It is well known that the solution $(u', v')$ is unique, since the two partial differential equations define $(u', v')$ to within the gradient of a harmonic function, and the normal derivative of the latter on $B$ is then given by the last of (8).

Denoting the integral in (5) by $I$, we have

$$I = \int (2\omega U - \theta V) \, dx \, dy$$

$$= \int [U(v'_x - u'_y) - V(u'_x + v'_y)] \, dx \, dy$$

$$= \int [u'(U_y + V_x) + v'(V_v - U_x)] \, dx \, dy$$

$$+ \int_B [U(lu' - mu') - V(lu' + mv')] \, ds$$

or, by (6) and (8),

$$I = \int_B U(lu' - mu') \, ds. \quad (10)$$

Now if $I = 0$ for arbitrary harmonic $U$, it follows that

$$(lu' - mu')_B = 0, \quad (11)$$

since the values of $U$ on $B$ may be arbitrarily assigned. Combining (8) with (11) we get $(u')_B = 0, (v')_B = 0$; thus the condition $I = 0$ is sufficient.

On the other hand, if (7) are consistent, then $(u')_B = 0, (v')_B = 0$, and so, by (10), $I = 0$; thus the condition $I = 0$ is necessary.

We get Hamel's theorem on putting $\theta = 0$. 
