THE SPECTRUM OF FREQUENCY-MODULATED WAVES AFTER RECEPTION IN RANDOM NOISE—II*

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Part I: Introduction and Discussion

In our earlier paper [1]† expressions for the spectrum of an FM wave received in the presence of random (fluctuation) noise were obtained in the two extreme cases of no limiting and "super"-limiting.** It is the purpose here to develop the completely general theory for the demodulated wave, taking into account the effects of *arbitrary* amounts of limiting on the noise and signal spectra and power of the low-frequency output of the receiver. We are interested in the spectrum because the signal-to-noise ratio determined at the output depends noticeably on the spectral distribution after demodulation when broad-band FM (in which the IF filter width is large compared with the audio response) is used. In the case of narrow-band FM (where the IF width is comparable with the audio) we are concerned mainly with the *integrated* spectrum or power output, so that spectral shape is not so significant (cf. [2]). The general theory is outlined in Part II, including an examination of important special cases, and a detailed study of noise alone is given in Part III; the Gaussian filter amplitude response is assumed. Remarks on signal and noise are contained in Part IV; applications to evaluation of the signal-to-noise ratio are discussed more fully in [2].

A satisfactory analytical model of the actual nonlinear elements in the receiver—the limiter followed by the discriminator—can be constructed if we assume: (1) that the physical discriminator is replaced by an "ideal" one which responds everywhere linearly with frequency; accordingly, the output current (or voltage) is directly proportional to the instantaneous difference frequency between the wave and the central or resonant frequency of the (symmetrical) IF, limiter, and discriminator bands; (2) that the filter response of the limiter is taken to be wide enough to pass the IF portion of the limited signal and noise without distortion due to frequency selection. In practise this means that a limiter band width several times the IF spread is needed; it can easily be obtained, since the limiter circuit is of necessity a low Q device. (If it were not, filtering would restore randomness to the noise, and the limiting would be nullified.) The case of the discriminator's response is more critical, but if the linear portion of the actual characteristic is at least twice the r-m-s frequency deviation, distortion will not be serious, and our idealized model is then a satisfactory substitute (cf. [3, Chs. 4 and 5] for a treatment of an actual discriminator when there is no limiting).

†Numbers in brackets refer to the bibliography at the end of the paper.

In [1] is listed previous work on this problem, of which the papers by Stumpers [Proc. I.R.E. **36, 1080 (1948)], Blachman [3], and Rice [9] are particularly to be noted.

The narrow-band wave leaving the IF and entering the limiter may be represented as

$$V(t) = R(t) \cos [\omega t + \theta(t)], \tag{1.1}$$

where R is the envelope and θ a phase angle. Now $f(=\omega_0/2\pi)$ is the central frequency of the IF, limiter, and discriminator; both R and θ are slowly varying functions of the time compared with $\omega_0 t$. Letting $f(iz)$ be the Fourier transform of the limiter's dynamic characteristic $g(V)$, we may write the output of the limiter as (cf. [1]),

$$V_0(t) = \frac{1}{2\pi} \int_{\mathbf{C}} f(iz) \, dz \, \exp [izR \cos (\omega_0 t + \theta)]$$

$$= \sum_{n=0}^{\infty} B_n(R) \cos [n(\omega_0 t + \theta)], \tag{1.2}$$

where

$$B_n(R) = \frac{i^n \epsilon_n}{2\pi} \int_{\mathbf{C}} f(iz) J_n(Rz) \, dz, \tag{1.3}$$

$$(n = 0, 1, 2, \cdots).$$

The contour $\mathbf{C}$ extends along the real axis from $-\infty$ to $+\infty$ and is indented downward about a possible singularity at $z = 0$. The various $B_n(R)$ are the envelopes of the n spectral zones produced in the limiter by its nonlinear action [4, Sec. 3], while the $n\theta$ ($n \geq 0$) are the respective phases. Only the band concentrated about $f_0(n = 1)$ is passed to enter the discriminator. One may show [1, Eqs. (1.7)-(1.9)] that the low-frequency output of the discriminator is

$$E_0(t) = B_1(R)\dot{\theta} = \frac{i\dot{\theta}}{\pi} \int_{\mathbf{C}} f(iz) J_1(Rz) \, dz. \tag{1.4}$$

The transform of the characteristic of our idealized limiter [1, Fig. 1] is

$$f(iz) = 2\beta[1 - \exp (-i\mathbf{R}_0 z)]/(iz)^2, \tag{1.5}$$

in which β is a tube constant and $\mathbf{R}_0$ is the level at which limiting takes place; the factor 2 arises because both positive and negative portions of the wave contribute to the envelope. Explicitly, $E_0(t)$ is found for our particular choice of $f(iz)$ [or $g(V)$] to be

$$E_0(t) = \beta \kappa \dot{\theta} \begin{cases} R, \quad 0 \leq R \leq \mathbf{R}_0 , \\[2mm] \dfrac{4\mathbf{R}_0}{\pi} \, {}_2F_1(-1/2, 1/2; 3/2; \mathbf{R}_0^2/R^2) \\[4mm] = \dfrac{2R}{\pi}\left\{\dfrac{\mathbf{R}_0}{R} (1 - \mathbf{R}_0^2/R^2)^{1/2} + \sin^{-1} \dfrac{\mathbf{R}_0}{R}\right\}, \quad \mathbf{R}_0 \leq R. \end{cases} \tag{1.6}$$

The details of the analysis for the correlation function, the spectrum, and power follow in Parts II and III, and examples are illustrated in Figs. 1-6.

At this point it is convenient to list the principal parameters that appear throughout the text and in the figures: b_0 = mean input noise power, at the IF output, A_0 = peak

carrier amplitude, $\mathbf{R}_0$ = amplitude at which limiting takes place, R = envelope of the wave leaving the IF stage, $p \equiv A_0^2/2b_0$ = ratio of mean carrier to mean noise power at IF output, $\mathbf{r}_0 \equiv \mathbf{R}_0/(2b_0)^{1/2}$ = ratio of the r-m-s clipping level to the r-m-s noise level, ω_d = (angular) frequency displacement from exact tuning, ω_b = (angular) frequency proportional to the width of the IF spectrum (see Eq. (3.3a)), $\Omega = \omega/\omega_b = f/f_b$ = a normalized frequency, measured in terms of the IF spectral width, w_0 = maximum spectral intensity of the noise at the IF output, β, κ = limiter and discriminator circuit constants, $D_0(t)$ = (angular) frequency modulation, $R(t)$, $r(t)$ = correlation functions, $\eta(t)$ = phase of the modulated carrier wave.

A number of general observations can be made. First, as shown in Figs. 1 and 2, the total power output drops with increased limiting, since less of the original wave is then passed. When there is no carrier but only fluctuation noise, we see from Fig. 3 a similar behavior, noting that in all cases when the limiting threshold ($\mathbf{R}_0$) is very large, the output power remains constant for a given input noise voltage and varies directly with the amount of signal power, if a carrier is present. The signal is always suppressed when it is weak relative to the noise (cf. Eq. (2.10)).

Second, as in any clipping or saturation process restricting the instantaneous amplitude of the disturbance, limiting spreads the spectrum; for noise alone, the precise extent of the spread is shown in Figs. 4 and 5. Figure 4 compares absolute values of the spectral intensity for varying amounts of limiting, and Fig. 5 gives a comparison of the same spectra, all now normalized to unity at $f = 0$, the point of maximum intensity. The reason for the spread lies in the fact that clipping produces an (infinitely) large number of *new* harmonics of the original wave, and the difference or beat frequencies between the original (adjacent) components, all relatively close to the resonant frequency f_0, are the source of the added intensity in the low-frequency part of the spectrum. The effect, of course, is much more pronounced the heavier the clipping ($\mathbf{R}_0 \to 0$). For no limiting, the spectral intensity falls off $\frown$ exp $[-(f/f_b)^2]$, while in the instance of "super"-clipping the decay goes as $(f/f_b)^{-1}$. The "tails" of the spectrum well away from zero frequency are quite extended (cf. Figs. 4 and 5 once more). One finds also, as earlier [1], that limiting, when the carrier is much greater than the noise, yields a vanishingly small noise spectrum at or near $f = 0$; if there is no limiting, however, this is no longer true. Herein lies the superiority of broad-band FM over AM and narrow-band FM for large carrier amplitudes. The present paper, in conjunction with the results of [1] and [2], completes our theoretical discussion of FM reception in fluctuation noise.

Part II: General Theory of Arbitrary Limiting

As before (cf. [1]) the correlation function of the low-frequency output of the discriminator is*

$$R_0(t) = [\overline{E(t_0)E(t_0 + t)}]_{\mathrm{av}} = \kappa^2[\overline{B_1(R_1)B_1(R_2)\dot\theta_1\dot\theta_2}]_{\mathrm{av}} . \qquad (2.1)$$

The bar denotes the statistical average over the random variables describing the noise,

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*Throughout this paper we write the correlation function as $R_0(t)$, and its normalized version as $r_0(t)$, to facilitate the distinction between the correlation function and the envelope R, and clipping level $\mathbf{R}_0$.

while [ ]$_{av}$ indicates the average to be taken over the phases of the modulation, if any. By a well-known theorem (cf. [5], [6]*) the mean power spectrum is the cosine Fourier transform of the correlation function:

$$W_0(f) = 4 \int_0^\infty R_0(t) \cos \omega t \, dt, \qquad (\omega = 2\pi f),$$

and conversely

$$(2.2)$$

$$R_0(t) = \int_0^\infty W_0(f) \cos \omega t \, df.$$

The narrow-band input voltage $V_0(t)$ of the limiter-discriminator element and the final low-frequency output $E_0(t)$ of the discriminator are represented precisely as in the earlier paper [1, Eqs. (2.3)-(2.11)]. The present analysis proceeds in the same fashion also, except that instead of the integrals K_1 and K_2 [1, Eqs. (2.20)] one encounters a more general version

$$K_1 = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \; x \frac{J_1(\xi[x^2 + y^2]^{1/2})}{x^2 + y^2} \exp(-ixz - iyz'),$$

$$(2.3)$$

$$K_2 = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \; y \frac{J_1(\xi[x^2 + y^2]^{1/2})}{x^2 + y^2} \exp(-ixz - iyz'),$$

which it is convenient to reduce to a single integral. Here ξ is a variable corresponding to the variable of integration in (1.4) which is introduced to account for the general dynamic characteristic [whose Fourier transform is $f(i\xi)$] considered here. As before, the following important quantities are needed:

$$b_n = \int_0^\infty (\omega - \omega_0)^n w(f) \, df; \qquad \phi_n(t) = \frac{\partial^n}{\partial t^n} \int_0^\infty w(f) \cos(\omega - \omega_0)t \, df;$$

$$(2.4)$$

$$r_n(t) = \phi_n(t)/\phi_0(0).$$

Here $w(f)$ is the mean input power spectrum of the noise, essentially determined by the IF filter response, which is assumed to be symmetrical about f_0, the resonant frequency. The quantity b_0 is accordingly the mean input noise power.

A more detailed discussion of the principal steps in the analysis is given in [1]. The final result is the low-frequency correlation function

$$R_0(t)_N = \sum_{k=0}^\infty \sum_{m=0}^\infty r_0(t)^{2m+k}\left\{-\phi_2(t) \frac{\epsilon_k}{2} \left[\mathcal{K}^2_{k,2m,k+1} \cos(k+1)(\eta_2 - \eta_1)\right.\right.$$

$$+ \mathcal{K}^2_{k,2m,|k-1|} \cos(k-1)(\eta_2 - \eta_1)]$$

$$+ \phi_1^2(t) \frac{\epsilon_k}{2} \left[\mathcal{K}^2_{k,2m+1,k} \cos k(\eta_2 - \eta_1) - \frac{1}{2} \mathcal{K}^2_{k,2m+1,k+2} \cos(k+2)(\eta_2 - \eta_1)\right.$$

$$\left.\left.- \frac{1}{2} \mathcal{K}^2_{k,2m+1,|k-2|} \cos(k-2)(\eta_2 - \eta_1)\right]\right\}$$

$$(2.5)$$

*See also S. O. Rice, Bell System Tech. J. **23**, 282 (1944) and [4] for further references.

$$+ (2b_0p)^{1/2}\phi_1(t)\frac{(\dot{\eta}_1 + \dot{\eta}_2)}{2}[(1 - \delta_0^k)\mathfrak{IC}_{k,2m,k+1}(\mathfrak{IC}_{k,2m+1,k+2}$$

$$- \mathfrak{IC}_{k,2m+1,k})\sin(k+1)(\eta_2 - \eta_1)$$

$$- \mathfrak{IC}_{k,2m,|k-1|}(\mathfrak{IC}_{k,2m+1,k} - \mathfrak{IC}_{k,2m+1,|k-2|})\sin|k-1|(\eta_2 - \eta_1)]$$

$$+ 2b_0p\dot{\eta}_1\dot{\eta}_2\cos k(\eta_2 - \eta_1)\left[\delta_0^k\mathfrak{IC}_{k,2m,k+1}^2 + \frac{1}{2}(\mathfrak{IC}_{k,2m,k+1} - \mathfrak{IC}_{k,2m,|k-1|})^2\right]\Big\},$$

in which

$$\eta_{1,2} = \omega_d t + \Psi_{1,2} \text{ and } \dot{\eta}_{1,2} = \omega_d + \dot{\Psi}_{1,2} = \omega_d + D_0(t_{1,2});$$

$$p \equiv A_0^2/2b_0 \text{ and } \Psi \equiv \int^t D_0(t)\,dt. \tag{2.5a}$$

The result (2.5) is formally identical with our earlier expression for $R_0(t)_N$ (cf. [1, Eq. (2.28)]) where only the cases of no limiting ($\mathbf{R}_0 \to \infty$) and extreme limiting ($\mathbf{R}_0 \to 0$) were considered. However, the amplitude functions $\mathfrak{IC}$ now take the more complex form

$$\mathfrak{IC}_{k,2m+n,q}(p;\mathbf{r}_0) = \frac{b_0^{(2m+k)/2}}{2^{(2m+k)/2}[m!(m+1)!]^{1/2}}\frac{\kappa}{\pi}\int_C f(i\xi)\,d\xi \int_0^\infty J_1(r\xi)\,dr$$

$$\times \int_0^\infty \rho^{\mu+1}J_1(r\rho)J_q(A_0\rho)\exp(-\rho^2 b_0/2)\,d\rho, \tag{2.6}$$

$$(\mu = 2m + n + k = 1, 3, 5, \cdots).$$

For the linear limiter considered specifically here, $\mathfrak{IC}$ becomes (cf. Appendix I)

$$\mathfrak{IC}_{k,2m-n,q} = \frac{b_0^{(2m+k)/2}}{2^{(2m+k)/2}[m!(m+k)!]^{1/2}}\Gamma_\mu = \frac{\kappa\beta p^{q/2}(2/b_0)^{n/2}}{q!\pi^{1/2}[m!(m+k)!]^{1/2}}$$

$$\times \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)\Gamma(1-s)\mathbf{r}_0^{2s+1}\Gamma(\nu+s)}{(s+1/2)\Gamma(s+1)\Gamma(3/2-s)}{}_1F_1(\nu+s;q+1;-p)\,ds, \tag{2.7}$$

where $\mathbf{r}_0 = \mathbf{R}_0/(2b_0)^{1/2}$, $\nu = (\mu + q + 1)/2 = 1, 2, 3, \cdots$, Γ_μ is given explicitly in (A1.15). [Similar results may be obtained in the same fashion for other types of limiter response, i.e. choices of $f(i\xi)$ other than Eq. (1.5)].

As before, the complete correlation function $R_0(t)$ is obtained after the average over the phases of the modulation has been performed. The mean power spectrum follows at once from the Fourier transform of $R_0(t)$, according to (2.2). We observe that the output, as in amplitude modulation (cf. [7]), consists of three classes of modulation product: (1) ($n \times n$) noise, produced by the beating of noise components with one another; (2) ($s \times n$) noise, which is the result of signal and noise beating together; and (3) ($s \times s$) signal harmonics. In (2.5) only the terms in the first bracket [ ] for which $q = |k - 1|, k, |k - 2|$, etc. $= 0$ represent ($n \times n$) noise; the remaining terms of the first and all those in the second and third brackets [ ] are ($s \times n$) noise, except in the latter when $m = k = 0$, in which case we have the signal components.

1. Signal Output. This follows immediately from (2.5) and is

$$R_0(t)_{(s \times s)} = A_0^2 [\dot{\eta}_1 \dot{\eta}_2]_{\mathrm{av}} \mathfrak{K}_{001}^2 \tag{2.8a}$$

$$= \frac{16}{\pi^2} \beta^2 \kappa^2 \mathbf{R}_0^2 \{ [\omega_d + D_0(t)][\omega_d + D_0(t_0 + t)] \}_{\mathrm{av}} p^2 \left\{ \frac{1 - \exp(-p)}{p} \right.$$

$$+ \sum_{l=1}^{\infty} \left(-\frac{1}{2} \right)_l (-\mathbf{r}_0^2)^l \frac{2l}{2l+1} \left(\left[\psi(l) + \psi(l+1)/2 - \psi(3/2 - l)/2 - \log \mathbf{r}_0 \right. \right. \tag{2.8b}$$

$$\left. - \frac{1}{2l(2l+1)} \right]_1 F_1(l+1; 2; -p) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(l+1)_n \psi(l+n+1)(-p)^n}{(2)_n n!} \right) \bigg\}^2$$
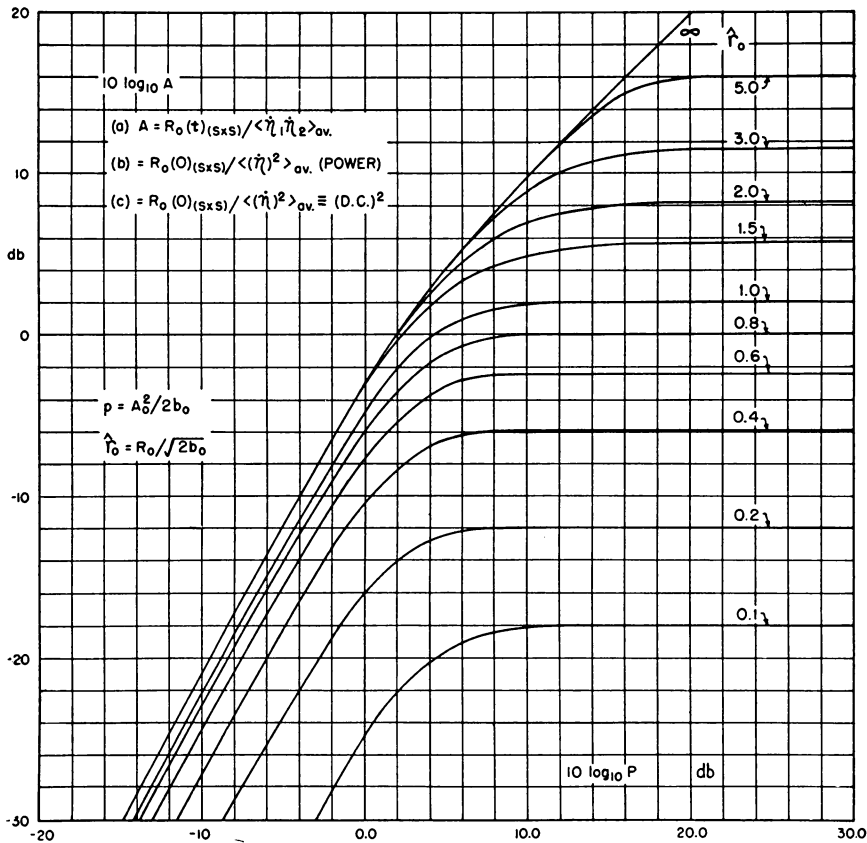

Fig. 1. Discriminator output—(a) Correlation function for signal components, (b) Mean signal power output, (c) Mean square d. c. output—as a function of carrier strength for various degrees of limiting. Here the abscissa is $10 \log_{10} p$, where $p = $ (mean carrier power from IF)/(mean noise power from IF); $\dot{\eta}$ is the modulation, including any deviation (ω_d) from exact tuning $(f_0 = f_c)$.

from (2.5a) and (A1.15); ψ is the logarithmic derivative of the Gamma-function. Note that because of our assumption of an idealized, linear discriminator, the modulation is received undistorted. Figure 1 illustrates $R_0(t)_{(s \times s)}$ as a function of p for various de-

grees of limiting. When there is no limiting at all ($\mathbf{r}_0 \to \infty$), or when the limiting is extreme ($\mathbf{r}_0 \to 0$), we obtain (cf. Eq. (A1.18)) the expressions of our earlier analysis [1, Eqs. (2.36), (2.37)]. In general, when the carrier is very strong compared with the noise ($p \to \infty$), we find (cf. (2.8a) and (2.19)) that for all degrees of limiting such that $\mathbf{R}_0 \leq A_0$ (or $\mathbf{r}_0 \leq p^{1/2}$),

$$R_0(t)_{(s \times s)}]_{p \to \infty} \leftleftarrows \kappa^2 \beta^2 [\dot{\eta}_1 \dot{\eta}_2]_{\mathrm{av}} \frac{16}{\pi^2} \mathbf{R}_0^2 \, {}_2F_1(-1/2,\, 1/2;\, 3/2;\, \mathbf{R}_0^2/A_0^2)^2$$

$$= \frac{4}{\pi^2} \kappa^2 \beta^2 [\dot{\eta}_1 \dot{\eta}_2]_{\mathrm{av}} \mathbf{R}_0^2 \Big([1 - \mathbf{R}_0^2/A_0^2]^{1/2} + \frac{A_0}{\mathbf{R}_0} \sin^{-1} \frac{\mathbf{R}_0}{A_0} \Big)^2, \qquad (2.9)$$

$$\mathbf{R}_0 \leq A_0 .$$

Here the noise is suppressed by the signal. Compare (2.9) with the square of (1.6); the dependence on $\mathbf{R}_0$ and A_0 ($=R$ if the noise is negligible) is the same, as we would expect.

At the other extreme of carriers weak compared with the noise ($p \to 0$), (2.8) reduces to

$$R_0(t)_{(s \times s)}]_{p \to 0} \doteq \Big(\frac{4}{\pi} \beta \kappa \mathbf{R}_0 \Big)^2 [\dot{\eta}_1 \dot{\eta}_2]_{\mathrm{av}} p^2 \Big\{ 1 + \sum_{m=1}^{\infty} \frac{(-1/2)_m (-\mathbf{r}_0^2)^m}{m!^2} \Big(\frac{2m}{2m+1} \Big)$$

$$\times [\psi(m) - \psi(3/2 - m)/2 - \log \mathbf{r}_0 - 1/2m(2m+1)] \Big\}^2 . \qquad (2.10)$$

This shows at once how the stronger noise suppresses the signal: instead of the (power) output being proportional to p, it is proportional to p^2, just as in the analogous situation for the detection of AM signals by a half-wave rectifier.

2. The d-c output. The d-c output is obtained from the correlation function if we let $t \to \infty$ in (2.5). The square root of the result gives us the desired mean amplitude, which is

$$[\overline{E_0(t)}]_{\mathrm{av}} = \kappa \beta A_0 [\omega_d + D_0(t)]_{\mathrm{av}} \mathcal{K}_{001} . \qquad (2.11)$$

By a different method (cf. [2]), we may express the d-c as an integral,

$$[\overline{E_0(t)}]_{\mathrm{av}} = 2\kappa \dot{\eta} p^{1/2} e^{-\nu} \int_0^{\infty} B_1(r) I_1(2rp^{1/2}) \exp(-r^2)\, dr, \qquad (2.12)$$

where $B_1(r)$ is given by (1.3) on replacing R by r and multiplying by $(2b_0)^{1/2}$. Numerical results are also shown in Fig. 1 (for the *square* of the d.c.), which may be obtained either by summing the series in (2.8b) or by graphical methods applied to (2.12). Simple forms of $[\overline{E_0(t)}]_{\mathrm{av}}$ occur when there is either no limiting or extreme clipping, (cf. [1, Eqs. (2.36), (2.37)]).

3. Mean power output. By a well-known theorem [4, Sec. 2 and Appendix II] the mean power may be derived from the correlation function on letting $t \to 0$ (cf. (2.2)). The result in this instance is the double series given in [1, Eq. (2.33)], on replacing H

by $\mathfrak{K}$. Again, a simpler expression has been found in the form of an integral (cf. [2]), namely,

$$W_0 = \kappa^2 \exp{(-p)}\left\{p[\dot{\eta}^2]_{\mathrm{av}} \int_0^\infty B_1(r)^2 \exp{(-r^2)}[I_0(2rp^{1/2}) + I_2(2rp^{1/2})] \, dr/r \right.$$

$$\left. - \frac{\phi_2(0)}{b_0} \int_0^\infty B_1(r)^2 \exp{(-r^2)} I_0(2rp^{1/2}) \, dr/r \right\}. \tag{2.13}$$

The curves of Fig. 2 show how W_0 depends on carrier strength for various clipping levels in the important special cases $[\dot{\eta}^2]_{\mathrm{av}} = 0$ (no modulation), and $[\dot{\eta}^2]_{\mathrm{av}} = -\phi_2(0)/2b_0$.
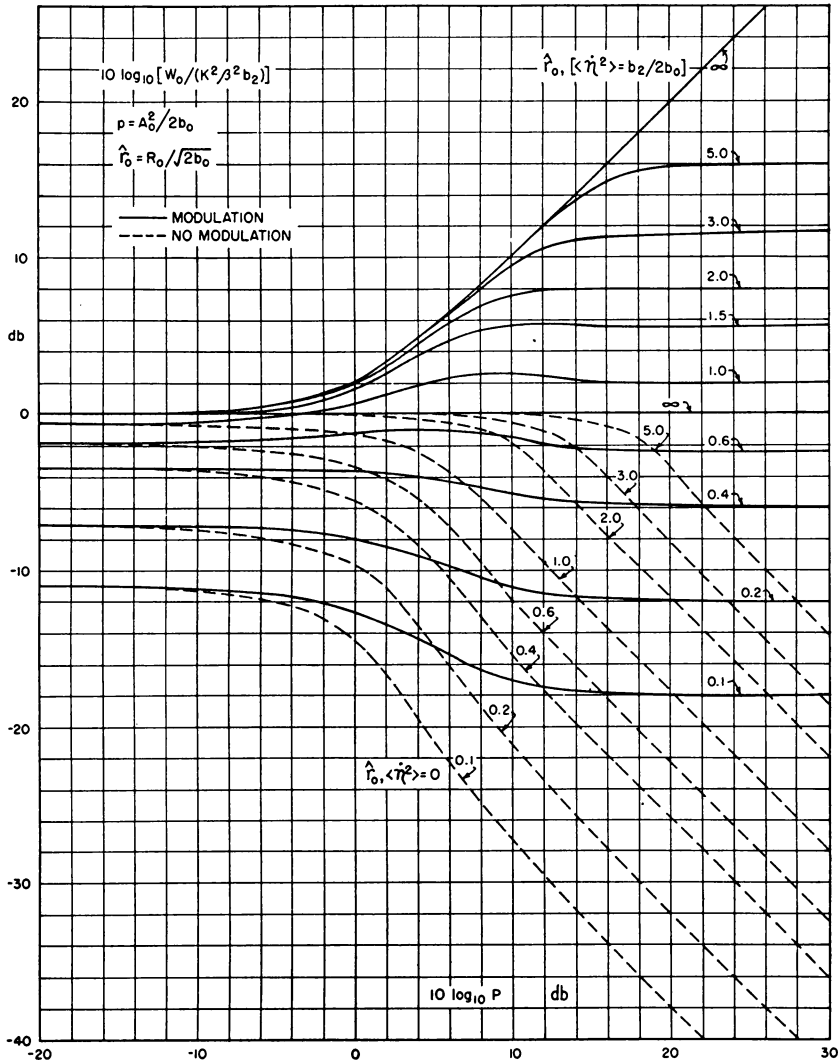

FIG. 2. Mean power output of discriminator as a function of carrier strength, with different degrees of limiting. Here the abscissa is $10 \log_{10} p$ (cf. Fig. 1), and $\dot{\eta}$ is the modulation, if any. The quantity b_2, obtained from Eq. (2.4), represents the second moment of the input noise spectrum $w(f)$.

Specific calculations were made with the help of Simpson's rule (cf. [2, Part II]). In a few special cases, however, W_0 assumes a closed form

$$\text{(no limiting: } \mathbf{r}_0 \to \infty) W_0 = 2b_0 \kappa^2 \beta^2 \left[p[\dot{\eta}^2]_{\text{av}} \left(1 - \frac{1 - \exp(-p)}{2p} \right) - \frac{\phi_2(0)}{2b_0} \right], \quad (2.14)$$

and

$$\text{(extreme limiting: } \mathbf{r}_0 \to 0) W_0 \doteq \left(\frac{4\kappa\beta}{\pi} \mathbf{r}_0 \right)^2 b_0 \exp(-p) \{ [p[\dot{\eta}^2]_{\text{av}} - \phi_2(0)/b_0]$$

$$\times [\gamma - \text{Ei}(-\mathbf{r}_0^2)]$$

$$(2.15)$$

$$+ \sum_{m=0}^{\infty} [[\dot{\eta}^2]_{\text{av}} p^{m+2} (2m+3) - \phi_2(0) p^{m+1} (m+2)/b_0]/(m+1)(m+2)! \},$$

where $\gamma = 0.2731$ and $\text{Ei}(-\mathbf{r}_0^2)$ is the exponential integral

$$- \int_{\mathbf{r}_0^2}^{\infty} e^{-y} \, dy/y.$$

When the carrier is very strong we obtain (2.9), $(t = 0)$. (Other limiting forms of the above are given in [2, Part II]).

4. Two important limiting cases.

A. *Strong carrier with limiting* $(A_0 \to \infty; \mathbf{R}_0 \leq A_0, b_0 \neq 0)$. Here the noise and limiting levels are assumed small compared with the peak carrier amplitude, i.e. $A_0 \gg (2b_0)^{1/2}$, $A_0 \geq \mathbf{R}_0$ [however, it is *not* necessary to assume that $\mathbf{R}_0 \gg (2b_0)^{1/2}$]. Then, in our expression for the output correlation function, obtained from (2.5) we retain only terms in b_0^0 and b_0, and discard the higher powers of $\mathbf{r}_0^{-1}$ and p^{-1}. Our correlation function reduces to ($p \to \infty$ in what follows)

$$R_0(t)_N \rightleftharpoons -\mathfrak{K}_{001}^2 \phi_2(t) \cos(\eta_2 - \eta_1) + \tfrac{1}{2} A_0 \mathfrak{K}_{001} \phi_1(t) \cdot (\dot{\eta}_1 + \dot{\eta}_2)$$

$$\times (\mathfrak{K}_{012} - \mathfrak{K}_{010}) \sin(\eta_2 - \eta_1)$$

$$(2.16)$$

$$+ \frac{A_0^2}{2b_0} \cdot \phi_0(t) \dot{\eta}_1 \dot{\eta}_2 \cos(\eta_2 - \eta_1) \cdot (\mathfrak{K}_{102} - \mathfrak{K}_{100})^2 + A_0^2 \dot{\eta}_1 \dot{\eta}_2 \mathfrak{K}_{001}^2 .$$

Now we need the limiting form of $\mathfrak{K}$ when $p \to \infty$; this is most easily found from (2.7) and the asymptotic development of $_1F_1$ [4, Appendix III] to be

$$\mathfrak{K}_{k,2m+n,q}]_{p \to \infty} \rightleftharpoons \frac{\kappa\beta}{\pi} \mathbf{r}_0 \left(\frac{2}{b_0} \right)^{n/2} \frac{p^{-(\mu+1)/2}}{[m!(m+k)!]^{1/2}}$$

$$(2.17)$$

$$\times \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{-s\Gamma(-s)^2 (\mathbf{r}_0^2/p) \cdot \Gamma(\nu+s)[1 + (\nu+s)(\nu+s-q)/p + \cdots]}{(s+1/2)\Gamma(s+1)\Gamma(3/2-s)\Gamma(q+1-\nu-s)} \, ds.$$

Keeping only the leading term in the series, we easily find from (2.17) that

$$\mathfrak{K}_{012} - \mathfrak{K}_{010} = \frac{2}{A_0} \mathfrak{K}_{001}, \text{ and } \mathfrak{K}_{102} - \mathfrak{K}_{100} = \frac{(2b_0)^{1/2}}{A_0} \mathfrak{K}_{001}, \quad (2.18)$$

where

$$\mathfrak{IC}_{001}]_{p\to\infty} \backsimeq \frac{\kappa\beta\mathbf{R}_0}{A_0\pi^{1/2}} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-s)(\mathbf{R}_0/A_0)^{2s}}{(s+1/2)\Gamma(3/2-s)} \, ds,$$

$$= \frac{4\kappa\beta\mathbf{R}_0}{\pi A_0} \,_2F_1(-1/2, 1/2; 3/2; \mathbf{R}_0^2), \tag{2.19}$$

$$= \frac{2\kappa\beta}{\pi}\left[\frac{\mathbf{R}_0}{A_0}(1-\mathbf{R}_0^2/A_0^2)^{1/2} + \sin^{-1}\frac{\mathbf{R}_0}{A_0}\right], \qquad (\mathbf{R}_0 \leq A_0).$$

The general procedure for evaluating the contour integral for $\mathfrak{IC}_{001}(p \to \infty)$ is available in Appendix I. For the noise part of the output, the correlation function (2.16) is, therefore,

$$R_0(t)_{\text{Noise}} \backsimeq [-\phi_2(t) \cos(\eta_2 - \eta_1) + \phi_1(t)(\dot{\eta}_1 + \dot{\eta}_2) \sin(\eta_2 - \eta_1)$$
$$+ \phi_0(t)\dot{\eta}_1\dot{\eta}_2 \cos(\eta_2 - \eta_1)]\mathfrak{IC}_{001}^2, \qquad (p \to \infty; A_0 \geq \mathbf{R}_0). \tag{2.20}$$

We observe that the coefficient of $\mathfrak{IC}^2_{001}$ in (2.20) is precisely that given in [1, Eq. (2.40)], when $\lambda = 2$, i.e. when there is extreme limiting. Our more general result also includes the cases of moderate to negligible amounts of limiting $(\mathbf{R}_0 \to A_0 , A_0 \gg (2b_0)^{1/2})$.

In a similar fashion (cf. [1, Eqs. (2.41)-(2.47)]) we obtain the alternative representation

$$R_0(t)_{\text{Noise}} \backsimeq \mathfrak{IC}^2_{001}\left\{\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)\phi_0(t)\right.$$
$$\left. \times \int_0^{x_0} \cos\left[\omega_d t + \int_x^y D_0(t') \, dt'\right] dx/x_0\right\}, \tag{2.21}$$

in which $x_0(=2\pi/\omega_a)$ is the period of the modulation and $\partial^n\phi_0(t)/\partial t^n = \phi_n(t)$, (Eq. (2.4)). Expansion of the integrand in a (cosine) Fourier series, using the integral form of $\phi_n(t)$ and the Fourier transform relations between the correlation function and the mean power spectrum of the noise, gives us finally

$$W_0(f) \backsimeq \left(\frac{4\kappa\beta\mathbf{R}_0}{\pi A_0}\right)^2 \,_2F_1(-1/2, 1/2; 3/2; \mathbf{R}_0^2/A_0^2)$$
$$\times \sum_{n=0}^{\infty} A_n\omega^2[w(\omega_0 + n\omega_a + \omega_d + \omega) + w(\omega_0 + n\omega_a + \omega_d - \omega)], \tag{2.22}$$

$$(\mathbf{R}_0 \leq A_0).$$

Here A_n is the amplitude of the nth term in the development of $[\cos(\eta_2 - \eta_1)]_{\text{av}}$. Equation (2.22) shows that when the carrier is strong and there is limiting at or below the maximum carrier level, the noise spectrum always vanishes at $f = 0$ and is small in the vicinity of zero frequency. This accounts for the great improvement in the signal vs. the noise (cf. [2]) when broad-band FM (maximum modulation frequency small compared with the maximum carrier deviation) is used at large carrier levels with sufficient limiting.

The mean power output, including the signal, is found at once from (2.9) and (2.20) by setting $t = 0$. We have

$$(p \to \infty)W_0 = R_0(0) \backsimeq \{A_0^2 + \phi_0(0)[\dot{\eta}^2]_{\mathrm{av}} - \phi_2(0)\}\mathfrak{K}_{001}^2$$

$$= 2b_0\mathfrak{K}_{001}^2\{p[\dot{\eta}^2]_{\mathrm{av}} + \tfrac{1}{2}([\dot{\eta}^2]_{\mathrm{av}} - r_2(0))\}, \qquad (\mathbf{R}_0 \leq A_0) \tag{2.23}$$

showing how the strong carrier suppresses the noise [second term of (2.23)], as in the detection of amplitude-modulated waves.

B. *Strong carrier, little or no limiting* ($A_0 \to \infty$, $A_0 \leq \mathbf{R}_0$, $b_0 \neq 0$). In the extreme of exceedingly strong carriers, when $A_0 < \mathbf{R}_0$ the situation corresponds in the first approximation to the case of essentially no limiting, for which $\mathfrak{K}$ becomes from (A1.18),

$$\mathfrak{K}_{k,2m+n,q}]_{p\to\infty} \backsimeq \frac{\kappa\beta}{2}\left(\frac{2}{b_0}\right)^{n/2} \frac{p^{-k-(m+n)/2}\Gamma([2m + k + n + q]/2)}{\Gamma([q - 2m - k - n + 2]/2)}. \tag{2.24}$$

The correlation function (2.16) then reduces to

$$R_0(t)_N \backsimeq \kappa^2\beta^2\{-\phi_2(t) \cos(\eta_2 - \eta_1) + 2b_0p\dot{\eta}_1\dot{\eta}_2\}, \qquad (\mathbf{R}_0 > A_0) \tag{2.25}$$

and the noise spectrum associated with the low-frequency output is again precisely our earlier result [1, Eq. (2.42)], viz.:

$$W_0(f) \backsimeq \kappa^2\beta^2 \sum_{n=0}^{\infty} A_n[(n\omega_a + \omega_d + \omega)^2w(\omega_0 + n\omega_a + \omega_d + \omega)$$

$$+ (n\omega_a + \omega_d - \omega)^2w(\omega_0 + n\omega_a + \omega_b - \omega)], \tag{2.26}$$

where we note, in contrast with (2.22), that the spectral intensity at and near zero frequency does not vanish. Broad-band FM under these conditions is accordingly much less satisfactory (cf. [2]) than when limiting is used. The mean power may be obtained from (2.14) on letting $p \to \infty$.

Part III: Noise Alone

This is the simplest case to handle analytically; a complete discussion follows. Since there is no carrier, $p = \eta = \dot{\eta} = 0$. We may therefore, from (2.5), write the correlation function of the low-frequency output of our discriminator as

$$R_0(t) = \sum_{m=0}^{\infty} \left\{ -\mathfrak{K}_{1,2m,0}^2 \, \phi_2(t)r_0(t)^{2m+1} + \frac{1}{2} \, \mathfrak{K}_{0,2m+1,0}^2 \, \phi_1(t)^2r_0(t)^{2m} \right.$$

$$\left. - \frac{1}{2} \, \mathfrak{K}_{2,2m+1,0}^2 \, \phi_1(t)^2r_0(t)^{2m+2} \right\}. \tag{3.1}$$

Again, in the special instances of no limiting ($\mathbf{r}_0 \to \infty$) and super-limiting ($\mathbf{r}_0 \to 0$) (3.1) reduces to the much simpler forms of our earlier treatment.* A general expression for the mean output power W_0 is found by setting $t = 0$ in (3.1) above; however, a compact form, from which calculations are more easily made, may be derived from (2.13) on letting $p \to 0$, viz.,

*Cf. Eqs. (3.2) and (3.8), and Sec. 3(a) of [1] for details

$$W_0]_{p \to 0} = \frac{-\kappa^2 \phi_2(0)}{b_0} \int_0^\infty B_1(r)^2 \exp(-r^2) \frac{dr}{r}. \tag{3.2}$$

Figure 3 illustrates the variation of the mean noise power as the limiting threshold is changed.
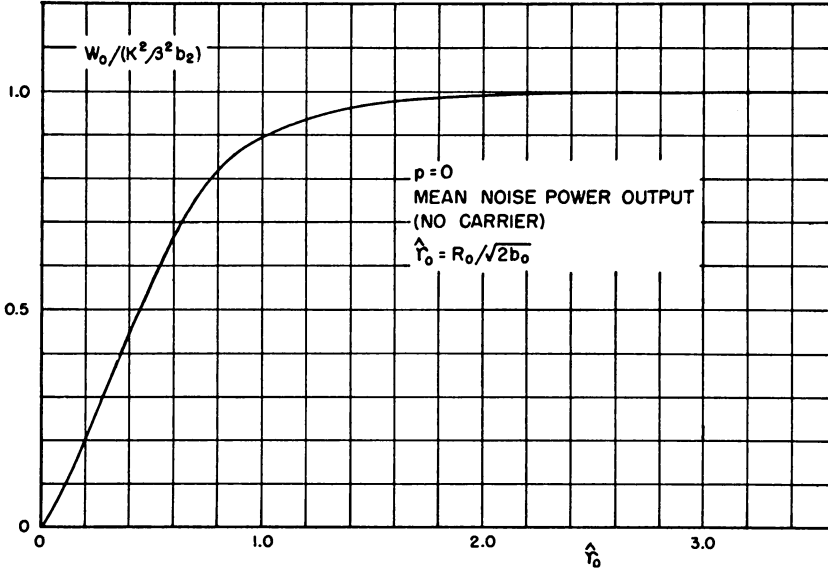

FIG. 3.   Mean low-frequency noise power output of the discriminator for different degrees of limiting when there is no carrier; b_2 is the second moment of the spectrum $w(f)$ Eq. (2.4).

To obtain the spectrum we need only take the Fourier transform of (3.1) according to (2.2). In the specific calculations, we assume a Gaussian spectral distribution for the input noise, corresponding to the composite IF-limiter-discriminator frequency response. We have

$$w(f) = w_0 \exp[-(\omega - \omega_0)^2/\omega_b^2], \quad \text{and} \quad w_0 = 2b_0\pi^{1/2}/\omega_b. \tag{3.3a}$$

From (2.4) we find also that

$$\phi_0(t) = b_0 \exp(-\omega_b^2 t^2/4), \qquad \phi_1(t) = -\tfrac{1}{2}\omega_b^2 t\, \phi_0(t)$$

$$\phi_2(t) = \tfrac{1}{2}\omega_b^2(\omega_b^2 t^2/2 - 1)\phi_0(t), \qquad r_1^2 - r_0 r_2 = \omega_b^2 r_0^2/2. \tag{3.3b}$$

The spectrum becomes finally

$$W_0(f) = 2\pi^{1/2} b_0 \omega_b \sum_{m=0}^{\infty} \left(\left\{ \mathcal{K}_{1,2m,0}^2 \left[1 - \left(1 - \frac{\Omega^2}{m+1}\right)(2m+2)^{-1} \right](2m+2)^{-1/2} \right. \right.$$

$$\left. + \frac{1}{2} b_0 \mathcal{K}_{0,2m+1,0}^2 \left(1 - \frac{\Omega^2}{m+1}\right)(2m+2)^{-3/2} \right\} \exp[-\Omega^2/(2m+2)] \tag{3.4}$$

$$\left. - \frac{1}{2} b_0 \mathcal{K}_{2,2m+1,0}^2 (2m+4)^{-3/2} \left(1 - \frac{\Omega^2}{m+2}\right) \exp[-\Omega^2/(2m+4)] \right),$$

where $\Omega \equiv \omega/\omega_b = f/f_b$. Figures 4 and 5 show the spectra for different degrees of limiting. Figure 5 is the same as Fig. 4, but now the spectra are normalized in such a way as to have the same intensity at $f = 0$. This demonstrates more effectively how limiting
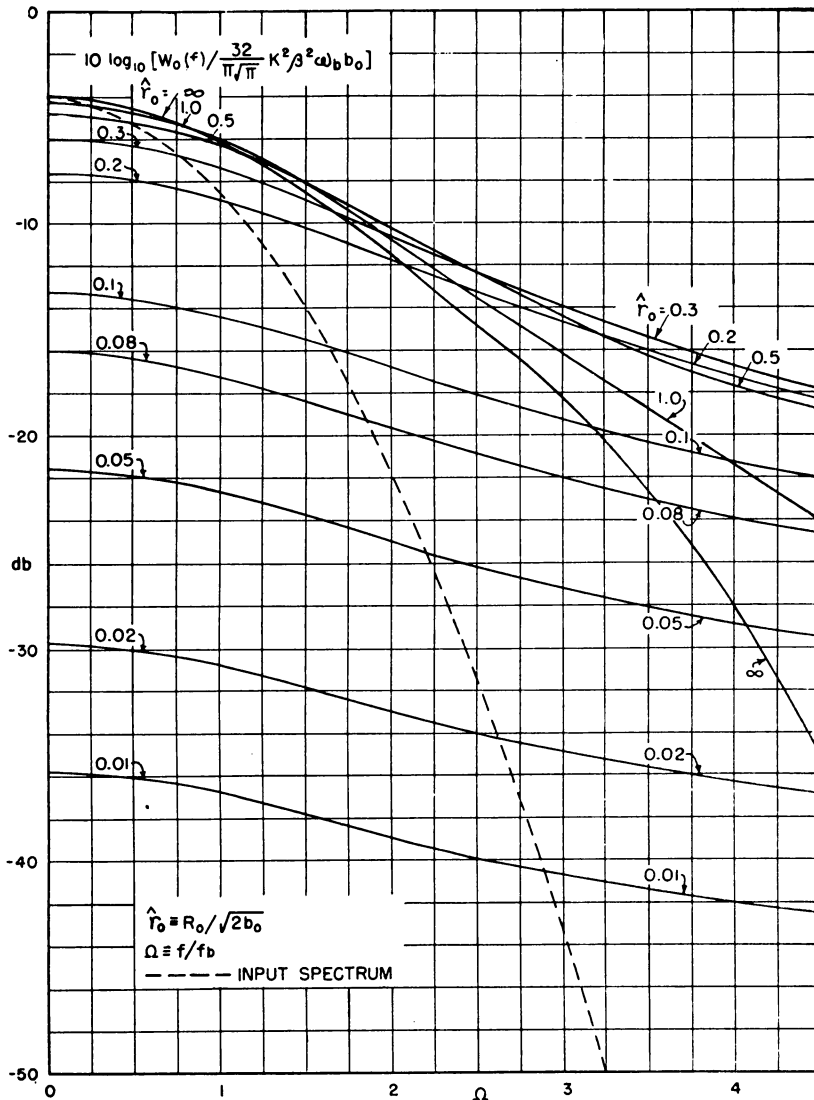

FIG. 4. The spectrum of the noise output of the discriminator for various degrees of limiting when there is no carrier ($p = 0$).

spreads the spectrum. A table of spectral intensities and a brief outline of the method of calculation involved in Figs. 4 and 5 are given in Appendix II.

With no clipping or, conversely, with very heavy clipping, the spectra are given by [1, Eqs. (3.7) and (3.9)]. These distributions are included in Figs. 4 and 5. It is interesting

to observe that well out on the "tail" of the spectrum, where $\Omega \to \infty$, the intensity falls off in the following ways:*

$$W_0(f)]_{\substack{\mathbf{r}_o \to \infty \\ \Omega \to \infty}} \simeq \pi^{3/2} \kappa^2 \beta^2 b_0 \omega_b \exp\left(-\Omega^2\right)$$

(3.5)

$$W_0(f)]_{\substack{\mathbf{r}_o \to 0 \\ \Omega \to \infty}} \simeq \frac{16}{\pi} \mathbf{R}_0^2 \kappa^2 \beta^2 \omega_b \Omega^{-1}.$$

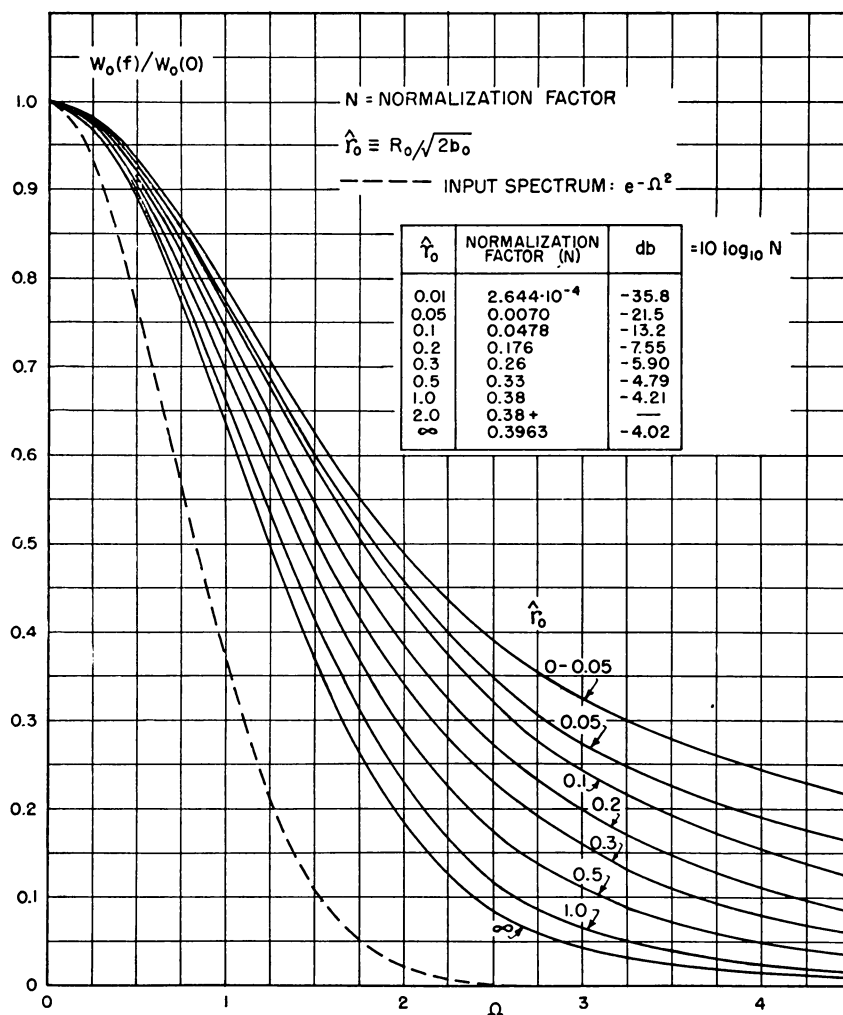| $\hat{r}_0$	NORMALIZATION FACTOR (N)	db = 10 log$_{10}$ N
0.01	2.644·10^{-4}	−35.8
0.05	0.0070	−21.5
0.1	0.0478	−13.2
0.2	0.176	−7.55
0.3	0.26	−5.90
0.5	0.33	−4.79
1.0	0.38	−4.21
2.0	0.38 +	—
∞	0.3963	−4.02

FIG. 5. Normalized spectrum of the noise output of the discriminator for various degrees of limiting when there is no carrier ($p = 0$).

For intermediate amounts of limiting the behavior varies between the rapid falling off [$\backsim \exp\left(-\Omega^2\right)$] for no limiting when Ω is sufficiently great and the relatively slow decay ($\backsim \Omega^{-1}$) of super-limiting. Note also that for large clipping levels ($\mathbf{r}_0 \to \infty$) the spectrum

*See Appendix II.

is independent of limiting, while for very intense clipping, the spectral intensity is proportional to $\mathbf{r}_0^2$ ($\mathbf{r}_0 \to 0$).

Part IV: Remarks on Carriers and Noise

When a carrier accompanies the noise in the receiver, our final low-frequency output is considerably modified, particularly if the carrier is intense relative to the noise. Modulation further distorts the continuous part of the output spectrum, so that a precise calculation of spectral shape becomes formidable indeed. In the more general cases including a signal we use (2.5) and average over the phases of the modulation, according to

$$R_0(t) = \frac{1}{T_0} \int_0^{T_0} dt_{0(\text{mod})} R_0(t, t_0)_N = [R_0(t)_N]_{\text{av}}, \qquad (4.1)$$

to obtain the complete correlation function. Formally, we have only to replace the amplitude functions H of our earlier analysis [1] by the more general $\mathfrak{IC}$ (cf. (2.6), (2.7)), yielding the behavior for all degrees of limiting, the most important cases of which ($\mathbf{r}_0 \to \infty$, $\mathbf{r}_0 \to 0$) have been discussed in the earlier paper [1, Secs. 3,4]. In the general instance of arbitrary limiting the results of [1, Figs. 5-11] are expected to apply qualitatively, except that as the degree of limiting decreases, there is less spreading of the spectrum for a fixed carrier power ($p = $ constant). Out on the tails of the spectrum ($\Omega \to \infty$) the intensity [for an originally Gaussian distribution of the type (3.3a)] falls off at least as exp $(-\Omega^2)$ for no limiting and at least as Ω^{-1} when there is super-limiting. The presence of a carrier ($p > 0$) increases the rate at which the intensity diminishes, especially if the carrier is strong: the noise is then largely suppressed and the spectrum is proportional to Ω^2 exp $(-\Omega^2)$ (cf. [1, Fig. 8]). On the other hand, for weak carriers the noise in turn suppresses the signal (Eq. (2.10)), and the distribution of the spectrum will follow as if noise only were present (cf. Part III and Figs. 4 and 5). The effects of modulation do not become noticeable, of course, until we attain large carrier powers; Figs. 7-11 of [1] show typical distributions in the extreme of strong carriers with simple forms of modulation ($\mathbf{r}_0 \to \infty$ or $\mathbf{r}_0 \to 0$).

APPENDIX I: The Amplitude Functions $\mathfrak{IC}_{k,2m+n,q}(p; \mathbf{R}_0)$

The quantities $\mathfrak{IC}$, which appear in (2.6) for the correlation function, become for the linear limiter used here

$$\mathfrak{IC} = \mathfrak{IC}_{k,2m+n,q}(p; \mathbf{R}_0) = \frac{b_0^{(2m+k)/2}}{2^{(2m+k)/2}[m!(m+k)!]^{1/2}} \Gamma_\mu, \qquad (A1.1)$$

where

$$\Gamma_\mu = \frac{-2i\kappa\beta}{\pi} \int_C \left(\frac{1 - \exp(-i\mathbf{R}_0 z)}{z^2}\right) dz \int_0^\infty J_1(rz) J_1(r\rho) \, dr$$

$$\times \int_0^\infty \rho^{\mu+1} J_q(A_0\rho) \exp(-b_0\rho^2/2) \, d\rho, \qquad (\mu = 2m + k + n)$$

(A1.2)

in which, from (2.5), we note that $\mu = 1, 3, 5, 7, \cdots$.

We start by integrating over r first, observing that this integral is a special case of Weber and Schafheitlin's more general expression. Using Gegenbauer's result* we obtain

$$L_1(z, \rho) = \int_0^\infty J_1(rz) J_1(r\rho) \, dr = \frac{z\rho}{2(z^2 + \rho^2)^{3/2}} \, {}_2F_1(3/4, 5/4; 2; 4z^2\rho^2/[z^2 + \rho^2]^2) \qquad \text{(A1.3)}$$

which is defined for all $z \neq \rho$. When $z = \rho$, ${}_2F_1$ is logarithmically divergent, and I_1 assumes the formal value $\Gamma(0)/2\pi z$. Since the singularity is only logarithmic, it is safely removed in the following integration over z. We have**

$$L_2(\rho) = \int_C \frac{1 - \exp(-i\mathbf{R}_0 z)}{z^2} L_1(z, \rho) \, dz$$

$$= \frac{i}{\rho^2} \cdot \sum_{n=0}^\infty \frac{(3/4)_n (5/4)_n 2^{2n}}{(2)_n n!} \int_0^\infty \frac{u^{2n-1} \sin \mathbf{R}_0 \rho u}{(u^2 + 1)^{2n+3/2}} \, du. \qquad \text{(A1.4)}$$

Next, we represent $\sin \mathbf{R}_0 \rho u$ as a contour integral involving Γ-functions, reverse the order of integration, and finally sum the series. Thus we write

$$\sin \mathbf{R}_0 \rho u = \frac{\pi^{1/2}}{2\pi i} \int_{-\infty i}^{\infty i} \left(\frac{\mathbf{R}_0 \rho u}{2} \right)^{2s+1} \frac{\Gamma(-s)}{\Gamma(s + 3/2)} \, ds, \qquad (\mathbf{R}_0 \rho \geq 0), \qquad \text{(A1.5)}$$

where the contour extends along the imaginary axis and includes the pole at the origin. With the aid of Cauchy's theorem the integral may be evaluated to give $\sin \mathbf{R}_0 \rho u$, by swinging the contour around in an infinite arc in the half of the complex plane for which $R(s) > 0$, and hence enclosing all the singularities of $\Gamma(-s)$. These singularities are simple poles at $s = m$ $(m = 0, 1, 2, \cdots)$, whose residues are $(-1)^m/m!$ The convergence of the integral and the proper vanishing of the integrand as $|s| \to \infty$ along the infinite arc are easily shown with the help of the asymptotic expansion of the Γ-function,† viz.,

$$\Gamma(\pm z + a) \leftrightharpoons \exp \{(\pm z + a - 1/2) \log(\pm z) \mp z + (\log 2\pi)/2 + O(iz)\}, \qquad \text{(A1.6)}$$

$$|\arg(\pm z + a)| < \pi, \qquad |\arg(\pm z)| < \pi, \qquad O(z) \to 0, \qquad |z| \to \infty.$$

For the integral in (A1.4) we find that

$$\int_0^\infty \frac{\sin \mathbf{R}_0 \rho u}{(1 + u^2)^{2n+3/2}} u^{2n-1} \, du$$

$$= \frac{\pi^{1/2}}{2\pi i} \int_{-\infty i}^{\infty i} \left(\frac{\mathbf{R}_0 \rho}{2} \right)^{2s+1} \cdot \frac{\Gamma(-s) \, ds}{\Gamma(s + 3/2)} \int_0^\infty \frac{u^{2n+2s}}{(u^2 + 1)^{2n+3/2}} \, du, \qquad \text{(A1.7)}$$

$$= \frac{\pi^{1/2}}{2\Gamma(2n + 3/2)} \int_{-\infty i}^{\infty i} \left(\frac{\mathbf{R}_0 \rho}{2} \right)^{2s+1} \frac{\Gamma(s)\Gamma(n + s + 1/2)\Gamma(n - s + 1)}{\Gamma(s + 3/2)} \, ds$$

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*G. N. Watson, *Theory of Bessel functions*, 2nd ed., Cambridge University Press, 1945. See p. 407, Eq. (1), which is incorrectly given; a factor $\Gamma(\lambda/2 + 1/2)$ should be inserted in the denominator.

**Termwise integration is allowed, since the series is uniformly convergent, except at $z = \rho$, where, however,

$$\int_{(z=\rho)} |g(z)| \, dz \text{ exists.}$$

See Titchmarch, *Theory of functions*, 2nd ed., Oxford University Press, 1939, p. 42.

†See E. T. Whittaker and G. N. Watson, *Modern analysis*, Cambridge University Press, 1940, p. 279.

on evaluating the Beta-function (which is defined, since $R(s) < 2n + 1/2$). Inasmuch as $\Gamma(n + s + 1/2)\Gamma(n - s + 1) = \Gamma(s + 1/2)\Gamma(1 - s)(s + 1/2)_n(1 - s)_n$, and since $(3/4)_n(5/4)_n 2^{2n-1} = \Gamma(2n + 3/2)/\pi^{1/2}$, from the duplication formula for the Gamma-function, we see that the series in (A1.4) becomes

$$\sum_{n=0}^{\infty} \frac{(s + 1/2)_n(1 - s)_n}{(2)_n n!} = {}_2F_1(s + 1/2, 1 - s; 2; 1) = \frac{\pi^{1/2}}{\Gamma(s + 1)\Gamma(3/2 - s)} \qquad (A1.8)$$

by the well-known summation formula for ${}_2F_1$. The integral $L_2(\rho)$ reduces finally to

$$L_2(\rho) = \frac{i\pi^{1/2}}{\rho^2}\left\{\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{-s\Gamma(-s)^2(\mathbf{R}_0\rho/2)^{2s+1}}{(s + 1/2)\Gamma(s + 1)\Gamma(3/2 - s)}\,ds\right\}. \qquad (A1.9)$$

Integration over ρ, with the help of [4, Eq. (A3.7)], gives us Γ_μ :

$$\Gamma_\mu = \frac{-2i\kappa\beta}{\pi}\int_0^{\infty} \rho^{\mu+1}J_q(A_0\rho)L_2(\rho)\,\exp\,(-\rho^2 b_0/2)\,d\rho,$$

$$= \frac{\kappa\beta}{\pi^{1/2}}\frac{p^{q/2}}{q!}\left(\frac{2}{b_0}\right)^{\mu/2}\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{-s\Gamma(-s)^2 \mathbf{r}_0^{2s+1}\Gamma(\nu + s)}{(s + 1/2)\Gamma(s + 1)\Gamma(3/2 - s)}$$

$$\times {}_1F_1(\nu + s; q + 1; -p)\,ds, \qquad (\nu = (\mu + q + 1)/2 = 1, 2, \cdots), \qquad (A1.10)$$

where $p \equiv A_0^2/2b_0$ and $\mathbf{r}_0 \equiv \mathbf{R}_0/(2b_0)^{1/2}$. This form is particularly useful for large values of p, as then we may replace ${}_1F_1$ by its asymptotic development before evaluating the contour integral.

We observe that the integrand of (A1.10) contains a simple pole at $s = 0$ and double poles at $s = l(l = 1, 2, \cdots)$. We need, therefore, to evaluate an expression of the form

$$L = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s)^2 f(s)\,ds, \qquad (A1.11)$$

where $f(s)$ contains no poles in the right complex half-plane $[R(s) > 0]$ and $\Gamma(-s)f(s)$ is finite at $s = 0$. The residue R_l at the double poles $s = l(l \geq 1)$ of $\Gamma(-s)^2$ is found from the Laurent expansion to be

$$R_l = \left[\frac{d}{ds}\{(-s + l)^2\Gamma(-s)^2 f(s)\}\right]_{s=l\geq1}. \qquad (A1.12)$$

Using Euler's representation of the Γ-function,

$$\Gamma(-s) = \lim_{n\to\infty} [n!n^{-s}/(-s)(-s + 1)\cdots(-s + n)],$$

we get

$$R_l = \lim_{n\to\infty} (\{-2n!^2n^{-2l}f(l)\log n + n!^2n^{-2l}f'(l)$$

$$+ 2n!^2n^{-2l}f(l)[-1/l + 1/(-l + 1)$$

$$+ \cdots + 1/-1 + 1 + 1/2 + \cdots + 1/(n - l)]\} \qquad (A1.13)$$

$$\div \{(-l)^2(-l + 1)^2\cdots(-1)^2\cdot 1^2\cdot 2^2\cdots(n - l)^2\}).$$

Since

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right)$$

$$= \gamma = 0.5772 \cdots \text{, and } \psi(m) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{m} - \gamma,$$

where γ is the Euler-Mascheroni constant and ψ is the logarithmic derivative of the Γ-function, (A1.11) becomes

$$L = \Gamma(0)f(0) + \sum_{l=1}^{\infty} \frac{1}{(l!)^2}[f'(l) - 2\psi(l)f(l)]. \tag{A1.14}$$

[The logarithmic derivative of the Γ-function is tabulated in Jahnke and Emde, *Tables of functions*, Stechert & Co., 1938, pp. 9-22; note, however, that Jahnke and Emde's definition is related to ours above by $\psi_{\text{J.E.}}(z) = \psi(z + 1)$]. The leading term of (A1.14) represents the residue at $s = 0$. If we observe that

$$\left\{\frac{d}{ds}\Gamma(s + 1)^{-1}\right\}_{s=l} = \frac{-\psi(l + 1)}{\Gamma(l + 1)}, \qquad \left\{\frac{d}{ds}\Gamma(3/2 - s)^{-1}\right\}_{s=l} = \frac{\psi(3/2 - l)}{\Gamma(3/2 - l)},$$

$$\left\{\frac{d\mathbf{r}_0^{2s+1}}{ds}\right\}_{s=l} = 2(\log \mathbf{r}_0)\mathbf{r}_0^{2l+1}$$

and that

$$\left\{\frac{d}{ds}[\Gamma(\nu + s)\,_1F_1(\nu + s; q + 1; -p)]\right\}_{s=l} = \Gamma(\nu)(\nu)_l \sum_{n=0}^{\infty}\frac{(\nu + l)_n\psi(\nu + l + n)(-p)^n}{n!(q + 1)_n},$$

$$\Gamma(3/2 - l) = (-1)^l\pi^{1/2}/2(-1/2)_l ,$$

we obtain finally for (A1.10)

$$\Gamma_\mu = \frac{4}{\pi}\beta\kappa\mathbf{r}_0\frac{p^{q/2}}{q!}\left(\frac{2}{b_0}\right)^{\mu/2}\Gamma(\nu)\left\{\,_1F_1(\nu; q + 1; -p)\right.$$

$$+ \sum_{l=1}^{\infty}\frac{(\nu)_l(-1/2)_l(-\mathbf{r}_0^2)^l}{(l!)^3}\left(\frac{2l}{2l + 1}\right)$$

$$\times\left(\left[\psi(l) + \psi(l + 1)/2 - \psi(3/2 - l)/2 - \frac{1}{2l(2l + 1)}\right.\right. \tag{A1.15}$$

$$\left. - \log \mathbf{r}_0\right]\,_1F_1(\nu + l; q + 1; -p)$$

$$\left.\left. - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(\nu + l)_n\psi(\nu + l + n)(-p)^n}{(q + 1)_n n!}\right)\right\},$$

$$(\nu = m + (k + n + q + 1)/2 = 1, 2, 3, \cdots).$$

In cases of extreme or super-limiting (A1.15) immediately gives the behavior as a function of the relative limiting level $\mathbf{r}_0$ as $\mathbf{r}_0 \to 0$; we have accordingly,

$$\lim_{\mathbf{r_0} \to 0} \mathfrak{IC}_{k,2m+n,q}(p;\mathbf{r_0}) = \frac{4\mathbf{r_0}}{\pi} \kappa\beta\left(\frac{2}{b_0}\right)^{n/2} \frac{p^{q/2}\Gamma([2m+k+n+q+1]/2)}{q![m!(m+k)!]^{1/2}}$$

$$\times {}_1F_1([2m+k+n+q+1]/2; q+1; -p)$$

$$= \frac{4}{\pi}\mathbf{R_0}\beta\kappa H_{k,2m+n,q}(p;2),$$

(A1.16)

where $H_{k,2m+n,q}$ is the amplitude function defined in an earlier paper [1]. At the other extreme of no limiting ($\mathbf{r_0} \to \infty$) we attempt to expand (A1.15) or (A1.10) in a series in $(\mathbf{r_0})^{-1}$. This is done by displacing the contour in (A1.10) to the left of the imaginary axis, a process that is valid for all $R(s) < 0$, since it is readily shown from (A1.6) that the integrand vanishes as $\exp(-\pi \mid I_m(s) \mid /2)$, when $I_m(s) \to \pm\infty$. We assume for the moment that ν is not an integer. Then the first pole for $R(s) < 0$ occurs at $s = -1/2$, and succeeding poles lie at $s = -\nu - l$, ($l = 0, 1, 2, \cdots$). Applying Cauchy's theorem to the region bounded by the displaced contour, the infinite arcs at $I_m(s) \to \pm\infty$, and the original contour, we obtain the asymptotic development

$$\Gamma_\mu \backsimeq \frac{\kappa\beta p^{q/2}}{q!}\left(\frac{2}{b_0}\right)^{\mu/2}\left\{\frac{1}{2}\Gamma(\nu - 1/2)\cdot {}_1F_1(\nu - 1/2; q+1; -p)\right.$$

(A1.17)

$$\left. + \frac{-1\cdot\Gamma(\nu)\Gamma(\nu+1)\mathbf{r_0}^{1-2\nu}}{\pi^{1/2}\Gamma(3/2+\nu)\Gamma(1-\nu)}\sum_{l=0}^{\infty}\frac{(\nu)_l^2(\nu+1)_l(\mathbf{r_0})^{-2l}}{(\nu+l-1/2)(3/2+\nu)_l l!}\, {}_1F_1(-l; q+1; -p)\right\}.$$

Now letting $\nu \to 1, 2, \cdots$ we see that the series in (A1.17) vanishes (since $\Gamma(1 - \nu) \to \pm\infty$, or more fundamentally, as a consequence of $\Gamma(\nu + s)/\Gamma(1 + s)$ not having poles when $R(s) < 0$), and we are left with

$$\lim_{\mathbf{r_0} \to \infty} \mathfrak{IC}_{k,2m+n,q}(p;\mathbf{r_0}) = \frac{\kappa\beta}{2}\left(\frac{2}{b_0}\right)^{n/2}\frac{p^{q/2}}{q!}\Gamma([2m+k+n+q]/2)$$

(A1.18)

$$\times {}_1F_1([2m+k+n+q]/2; q+1; -p) = \beta\kappa H_{k,2m+n,q}(p;1),$$

which agrees with Eq. (2.29) of our earlier work [1]. (If we note that $(\sin \mathbf{R_0}\rho u/u)]\mathbf{R_0}\to\infty = \pi\delta(u - 0)$ we obtain (A1.18) directly from (A1.4) and the integration over ρ, since only the term $n = 0$ yields a nonvanishing result). Unfortunately, the above approach does not give the succeeding terms in $\mathbf{r_0}^{-1}$, and other methods must be sought.

APPENDIX II: Calculation for the Spectrum of Noise Alone

The general spectrum of the noise output of our discriminator is given by (3.4) and the pertinent amplitude function $\mathfrak{IC}$ follow from (A1.15) when $p = 0$, namely, $\mathfrak{IC}_{1,2m,0}$, $\mathfrak{IC}_{0,2m+1,0}$, $\mathfrak{IC}_{2,2m+1,0}$. Explicitly, these quantities are

$$\mathfrak{IC}_{1,2m,0} = (m+1)^{-1/2}G_m(\mathbf{r_0}^2), \qquad \mathfrak{IC}_{0,2m+1,0} = \left(\frac{2}{b_0}\right)^{1/2}G_m(\mathbf{r_0}^2),$$

(A2.1)

$$\mathfrak{IC}_{2,2m+1,0} = \left(\frac{m+1}{m+2}\right)^{1/2}\left(\frac{2}{b_0}\right)^{1/2}G_{m+1}(\mathbf{r_0}^2),$$

where

$$G_m(\mathbf{r}_0^2) = \frac{4\kappa\beta\mathbf{r}_0}{\pi}\left\{1 + \sum_{l=1}^{\infty} \frac{(-\mathbf{r}_0^2)^l(-1/2)_l(m+1)_l}{(l!)^3}\left(\frac{2l}{2l+1}\right)\right.$$

$$\times\,[\psi(l) + \psi(l+1)/2 - \psi(3/2 - l)/2 - \psi(m+l+1)/2 \qquad \text{(A2.2)}$$

$$\left. - \frac{1}{2l(2l+1)} - \frac{1}{2}\log\mathbf{r}_0^2]\right\}.$$

The series for $G_m(\mathbf{r}_0^2)$ converges with satisfactory rapidity for all $\mathbf{r}_0$ equal to or less than unity, but unfortunately the series (3.4) for the spectrum does not converge at all speedily when $\mathbf{r}_0 \backsim 0.3$ or more. Since accuracy to within a few per cent is quite sufficient
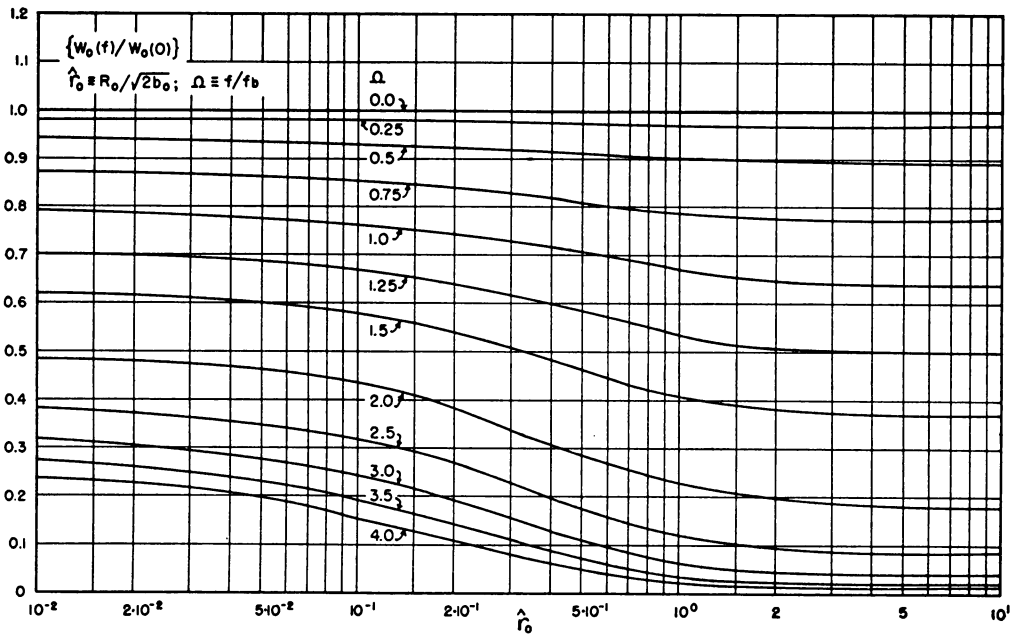

FIG. 6. The normalized spectrum of the noise output of the discriminator as a function of limiting level when there is no input carrier ($p = 0$).

for all practical purposes, it is possible to get around the convergence difficulty by interpolation. To do so we first calculate data for $W_0(f)$ in the interval $0.01 \leq \mathbf{r}_0 < 0.3$, which can be done precisely. Next, we observe that for values of $\mathbf{r}_0 \backsim 2.5$ or more the effect of limiting on the purely noise wave is slight. The probability that noise peaks 2.5 times in excess of the r-m-s value will occur is proportional to $\exp(-\mathbf{r}_0^2) \doteq e^{-6} = 0.002$, or about 0.2 per cent of the time—a negligible effect. The resulting spread in the noise spectrum due to clipping is correspondingly trivial (cf. [8]). Accordingly, we may safely use the values of the spectral intensity, when $\mathbf{r}_0 \to \infty$ [1, Eq. (3.7), when $\mathbf{r}_0 > 2.5$] leaving an interval of about one cycle over which to make the interpolation. Because we wish eventually to be able to normalize the spectra so that the maximum intensity is unity, and since $W_0(f)_{\max}$ occurs at zero frequency when there is no carrier, we need make only one interpolation, i.e. for $W_0(0)$ over the range $0.3 < \mathbf{r}_0 < 3.0$.

From this data (correct to within two per cent) we normalize the spectrum at the calculated points, obtaining the curves of Fig. 6. A short table of normalization factors N is given below:

TABLE I.

r_0	$N*$	r_0	N	r_0	N
.01	$2.644 \cdot 10^{-4}$	0.1	$4.78 \cdot 10^{-2}$	0.8	0.370
.03	$2.47 \cdot 10^{-3}$	0.2	0.176	0.9	0.375
.05	7.02 "	0.3	0.257	1.0	0.379
.06	$1.12 \cdot 10^{-2}$	0.4	0.303	1.5	0.382
.07	1.70 "	0.5	0.332	2.0	0.386
.08	2.51 "	0.6	0.350	3.0	0.389
.09	3.48 "	0.7	0.362	∞	0.3963

(*The values in bold face type are precise; N is the number by which $W_0(f)_{\text{norm.}}$ must be multiplied to give $W_0(f)_{\text{absolute}}$.)

The limiting behavior of the spectrum at large frequencies is of considerable interest. It is possible to obtain at least the leading term in the asymptotic expansion quite simply by the technique of Appendix I (cf. Eqs. (A1.16)-(A1.18)). We begin first with the simple case of super-clipping, in which we have to examine the series (3.8) of [1]. We transform the series as follows:

$$\sum_{m=0}^{\infty} \frac{\exp\left[-\Omega^2/2(m+1)\right]}{(m+1)^{3/2}} = \sum_{k=0}^{\infty} (-\Omega^2/2)^k \frac{1}{k!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k+3/2}}$$

$$= \sum_{k=0}^{\infty} \zeta(3/2 + k)(-\Omega^2/2)^k/k!, \tag{A2.3}$$

where ζ is Riemann's Zeta-function.* Next, we use a contour integral (cf. (A1.5)) involving the Gamma-function to represent the transformed series, viz:

$$\sum_{k=0}^{\infty} \zeta(3/2 + k)(-\Omega^2/2)^k/k! = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \zeta(3/2 + s)\Gamma(-s)(\Omega^2/2)^s \, ds \equiv I_1, \tag{A2.4}$$

the contour extending along the imaginary axis and including the pole at $s = 0$. Shifting the origin to the left $[R(s) < 0]$ so as to include the single, simple pole of the ζ-function, here at $s = -1/2$, we easily show by Cauchy's theorem that

$$I_1 = \left(\text{Residue at } s = -\frac{1}{2}\right) + \int_{-\infty i - c}^{\infty i - c} \zeta(3/2 + s)\Gamma(-s) \frac{ds}{2\pi i} \cdot (\Omega^2/2)^s, \tag{A2.5}$$

$$(c > 1/2),$$

since the contribution of the integrand along the infinite arcs ($\pm \infty i$ to $\pm \infty i - c$) vanishes.** Equation (A2.5) then allows us to write,

---

*Whittaker and Watson, *ibid.*, Ch. XIII.

**For we note that

$$|\Gamma(-s)(\Omega^2/2)^s \zeta(3/2 + s)| \backsimeq \exp(-\pi |v|/2) \cdot |v|^{-1}$$

as $v \to \pm \infty$, where $s = u + iv$, independently of the value of c.

at least for the leading term as $\Omega \equiv \infty$, $I_1 \backsimeq (2\pi)^{1/2}/\Omega$, a result also obtained by Rice [9] and Blachman [3]; (see also Appendix V of [1] for the other terms).

The case of no limiting ($r_0 \to \infty$) may be handled in the same way. The contour integral representation of the series (3.7) of [1] is now

$$\sum_{m=0}^{\infty} \frac{(1/2)_m^2}{(m!)^2 (m+1)^{3/2}} \exp\left[-\Omega^2/(2m+2)\right]$$

$$= \frac{1}{\pi} \int_{-\infty i}^{\infty i} \frac{\cos \pi s \, \Gamma(-s) \Gamma(s+1/2)^2}{\Gamma(s+1)(s+1)^{3/2}} \exp\left[-\Omega^2/(2s+2)\right] \frac{ds}{2\pi i}, \qquad (A2.6)$$

$$= \int_{-\infty i}^{\infty i} \frac{\Gamma(-s) \Gamma(s+1/2)}{\Gamma(1/2-s) \Gamma(s+1)(s+1)^{3/2}} \exp\left[-\Omega^2/(2s+2)\right] \frac{ds}{2\pi i}, \qquad (A2.7)$$

where we have used the relation $\sin \pi z = \pi/\Gamma(z)\Gamma(1-z)$ to transform the cosine term in (A2.6). The integral (A2.7) is convergent for all values of $s = u + iv$ whose real parts are finite, for by (A1.6) we have

$$\lim_{|v| \to \infty} \left| \frac{\Gamma(-s)\Gamma(s+1/2) \exp\left[-\Omega^2/(2s+2)\right]}{\Gamma(1/2-s)\Gamma(s+1)(s+1)^{3/2}} \right| \backsimeq \frac{1}{|v|^{5/2}}.$$

The leading term is then found as in (A2.5) from the pole at $s = -1/2$, so that

$$\lim_{\Omega \to \infty} \sum_{m=0}^{\infty} \frac{(1/2)_m^2 \exp\left[-\Omega^2/(2m+2)\right]}{(m!)^2 (m+1)^{3/2}} \backsimeq 2^{3/2} \exp\left(-\Omega^2\right). \qquad (A2.8)$$

The rapid falling-off of the spectral intensity as $\Omega \to \infty$ when there is no limiting is attributable to the equally rapid decay [$\backsim \exp\left(-\Omega^2\right)$] of the spectral intensity of the wave entering the discriminator. Roughly speaking, the number of beat-frequencies, and hence spectral density, produced in the discriminator for very large Ω decreases in the same way as the original spectral intensity.

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